Lecture 1.

The subject of Non-commutative geometry. Notions of a non-commutative geometry. Dictionary between notions from calculus and homological invariants. Hochschild Homology and Cohomology. Hochschild-Kostant-Rosenberg Theorem. Bar-resolution and the Hochschild complex. Cyclic homology (explicit definition).

1.1 The subject of Non-commutative Geometry.

It is an empirical fact that the idea of "non-commutative geometry", when seen for the first time, is met with deep scepticism (at least, this was my personal reaction for 10 years or so). Let me start these lectures with a short justification of the subject.

Back in the nineteenth century, and today in high school math, geometry was essentially set-theoretic: the subject of geometry was points, lines (sets of points of special types), and so on. This approach has been inherited by early algebraic geometry – instead of lines we maybe consider curves of higher degree, or higher-dimensional algebraic varieties, but we still think of them as sets of points with some additional structure.

However, starting from mid-twentieth century, and especially in the work of Grothendieck, a new viewpoint appeared, which can be loosely termed "categorical": one thinks of an algebraic variety simply as an object of the category of algebraic varieties. The precise "inner structure" of an algebraic variety is not so important anymore – what is important is how it behaves with respect to other varieties, what maps to or from other varieties does it admit, and so on. "Set of points" is just one functor on the category of algebraic varieties that we can use to study them; there are other important functors, such as, for instance, various cohomology theories.

These two "dual" approaches to algebraic geometry are not mutually exclusive, but rather complementary, and somewhat competing. To give you a non-trivial example, let us consider the Minimal Model Program. Here two methods of studying an algebraic variety X proved to be very successful. One is to study rational curves on X, their families, subvarieties they span etc. The other is to treat X as a whole and obtain results by considering its cohomology with various coefficients and using Vanishing Theorems. For example, the Cone Theorem claims that a certain part of the ample cone of X is polyhedral, with faces dual to certain classes in $H_2(X)$ called "extremal rays". If X is smooth, the Theorem can be proved by the "bend-and-break" techniques; extremal rays emerge as fundamental classes of certain rational curves on X. On the other hand, the Cone Theorem can be proved essentially by using consistently the Kawamata-Viehweg Vanishing Theorem; this only gives extremal rays as cohomology classes, with no generating rational curves, but it works in larger generality (for instance, for a singular X).

Now, the idea of "non-commutative" geometry is, in a nutshell, to try to replace the notion of an affine algebraic variety $X = \operatorname{Spec} A$ with something which would make sense for a non-commutative ring A. The desire to do so came originally from physics – one of the ways to interpret the formalism of quantum mechanic is to say that instead of the algebra of functions on a symplectic manifold M ("the phase space"), we should consider a certain non-commutative deformation of it. Mathematically, the procedure seems absurd. In order to define a spectrum $\operatorname{Spec} A$ of a ring A, you need A to be commutative, otherwise you cannot even define "points" of $\operatorname{Spec} A$ in any meaningful way. Thus the set-theoretic approach to non-commutative geometry quickly leads nowhere.

However, and this is somewhat surprising, the categorical approach does work: much more things can be generalized to the non-commutative setting than one had any right to expect beforehand. Let us list some of these things.

(i) Algebraic K-theory.

(ii) Differential forms and polyvector fields.

(iii) De Rham differential and de Rham cohomology, Lie bracket of vector fields, basic formalism of differential calculus.

(iv) Hodge theory (in its algebraic form given by Deligne).

(v) Cartier isomorphisms and Frobenius action on cristalline cohomology in positive characteristic.

Of these, the example of K-theory is the most obvious one: Quillen's definition of the K-theory of an algebraic variety $X = \operatorname{Spec} A$ involves only the abelian category $A\text{-}\mathsf{mod}$ of A-modules, and it works for a non-commutative ring A without any changes whatsoever. Before giving the non-commutative versions of the other notions on the list, however, we need to discuss more precisely what we mean by "non-commutative setting".

1.2 The notion of a non-commutative variety.

Actually, there are several levels of abstraction at which non-commutative geometry can be built. Namely, we can take as our definition of a "non-commutative variety" one of the following four.

(1) An associative ring A.

(2) A differential graded (DG) algebra $A^\bullet$.

(3) An abelian category $\mathcal{C}$.

(4) A triangulated category $\mathcal{D}$ "with some enhancement".

The relation between these levels is not linear, but rather as follows:

$$\begin{tikzcd} (1) \arrow[r] \arrow[d] & (2) \arrow[d] \\ (3) \arrow[r] & (4). \end{tikzcd} \tag{1.1}$$

Given an associative ring A, we can treat it as a DG algebra placed in degree 0 – this is the correspondence (1) $\Rightarrow$ (2). Or else, we can consider the category $A\text{-}\mathsf{mod}$ of left A-modules – this is the correspondence (1) $\Rightarrow$ (3). Given a DG algebra $A^\bullet$, we can construct the derived category $\mathcal{D}(A^\bullet)$ of left DG $A^\bullet$-modules, and given an abelian category $\mathcal{C}$, we can consider its derived category $\mathcal{D}(\mathcal{C})$ – this is (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4).

Of course, in any meaningful formalism, the usual notion of a (commutative) algebraic variety has to be included as a particular case. In the list above, (1) is the level of an affine algebraic variety $X = \operatorname{Spec} A$. Passing from (1) to (3) gives the category of A-modules, or, equivalently, the category of quasicoherent sheaves on X. This makes sense for an arbitrary, not necessarily affine scheme X – thus on level (3), we can work with any scheme X by replacing it with its category of quasicoherent sheaves. We can then pass to level (4), and take the derived category $\mathcal{D}(X)$.

What about (2)? As it turns out, an arbitrary scheme X also appears already on this level: the derived category $\mathcal{D}(X)$ of quasicoherent sheaves on X is equivalent to the derived category $\mathcal{D}(A^\bullet)$ of a certain (non-canonical) DG algebra $A^\bullet$. The rough slogan for this is that "every scheme is derived-affine".

Here are some other examples of non-commutative varieties that one would like to consider.

(i) Given a scheme X, one can consider a coherent sheaf $\mathcal{A}$ of algebras on X and the category of sheaves of $\mathcal{A}$. This is only "slightly" non-commutative, in the sense that we have an honest commutative scheme, and the non-commutative algebra sheaf is of finite rank over the commutative sheaf $\mathcal{O}_X$ (e.g. if $X = \operatorname{Spec} B$ is affine, then $\mathcal{A}$ comes from a non-commutative algebra which has B lying its center, and is of finite rank over this center). However, there are examples where this is useful. For instance, in the so-called *non-commutative resolutions* introduced by M. Van den Bergh, X is usually singular; generically over X, $\mathcal{A}$ is a sheaf of matrix algebras, so that its category of modules is equivalent to the category of coherent sheaves on X, but near the singular locus of X, $\mathcal{A}$ is no longer a matrix algebra, and it is "better behaved" than $\mathcal{O}_X$ – e.g. it has finite homological dimension.

(ii) Many interesting categories come from representation theory – representation of a Lie algebra, or of a quantum group, or versions of these in finite characteristic, and so on. These have appeared prominently, for examples, in the recent works of R. Rouquier.

(iii) In sympletic geometry, there is the so-called *Fukaya category* and its versions (e.g. the "Fukaya-Seidel category"). These only exist at level (4) above, and they are very hard to handle; still, the fully developed theory should apply to these categories, too.

Let us also mention that even if one is only interested in the usual schemes X, looking at them non-commutatively is still non-trivial, because *there are more maps between schemes X, X' when they are considered as non-commutative varieties.* E.g. on level (4), a map between triangulated categories is essentially a trinagulated functor, or maybe a pair of adjoint triangulated functors, depending on the specific formalism used – but in any approach, a Fourier-Mukai transform, for instance, gives a well-defined non-commutative map. Flips and flops in the Minimal Model Program are also expected to give non-commutative maps.

Passing to a higher level of abstraction in (1.1), we lose some information. A single abelian category can be equivalent to the category of modules for different rings A (this is known as Morita equivalence – e.g. a commutative algebra A is Morita-equivalent to its matrix algebra $M_n(A)$, for any $n \geq 2$). And a single triangulated category can appear as the derived category of quasicoherent sheaves on different schemes (e.g. related by the Fourier-Mukai transform) and the derived category of DG modules over different DG algebras (e.g. related by Koszul duality, the DG version of Morita equivalence). However, it seems that the information lost is inessential; especially if we think of various homological invariants of a non-commutative variety, they all are independent of the specifics lost when passing to (4). While this is not a self-evident first principle but rather an empirical observation, it seems to hold – again as a rough slogan, "non-commutative geometry is derived Morita-invariant". Thus it would be highly desirable to develop the theory directly on level (4) and not bother with inrelevant data.

However, at present it is not possible to do this. The reason is the well-known fact that the notion of triangulated category is "too weak". Here are some instances of this.

(i) "Cones are not functorial". Thus for a triangulated category $\mathcal{D}$, the category of functors $\operatorname{Fun}(I, \mathcal{D})$ for even the simplest diagrams I – e.g. the category of arrows in $\mathcal{D}$ – is not triangulated.

(ii) Triangulated categories do not patch together well. For instance, if we are given two triangulated categories $\mathcal{D}_1$, $\mathcal{D}_2$ equipped with triangulated functors to a triangulated category $\mathcal{D}$, the fibered product $\mathcal{D}_1 \times_{\mathcal{D}} \mathcal{D}_2$ is not triangulated.

(iii) Given two triangulated categories $\mathcal{D}_1$, $\mathcal{D}_2$, the category of triangulated functors $\operatorname{Fun}_{tr}(\mathcal{D}_1, \mathcal{D}_2)$ is not triangulated.

It is the consensus of all people working in the field that the correct notion is that of a triangulated category with some additional structure, called "enhancement"; however, there is no consensus as to what a convenient enhancement might be, exactly. Popular candidates are "DG-enhancement", "A_∞-enhancement" and "derivator enhancement". Within the framework of these lectures, let me just say that the only sufficiently developed notion of enhancement seems to be the DG approach, but using it is not much different from simply working in the context of DG algebras, that is, on our level (2).

Thus is the present course, we will not attempt to work in the full generality of (4) – we will start at (1), and then maybe go to (2) and/or (3).

However, it is important to keep in mind that (4) is the correct level. In particular, everything should and will be "derived-Morita-invariant" – DG algebras or abelian categories that have equivalent derived categories are indistinguishable from the non-commutative point of view.

1.3 A dictionary.

Let us now give a brief dictionary between some notions of algebraic geometry and their non-commutative counterparts. We will only do it in the affine case (level (1)). For convenience, we have summarized it in table form.

An affine scheme $X = \operatorname{Spec} A$	An associative algebra A
X is smooth	A has finite homological dimension
Differential forms $\Omega^\bullet(X)$	Hochschild homology classes $HH_\bullet(A)$
Polyvector fields $\Lambda^\bullet\mathcal{T}(X)$	Hochschild cohomology classes $HH^\bullet(A)$
De Rham differential d	Connes' differential B
De Rham cohomology $H^\bullet_{DR}(X)$	Cyclic homology $HC_\bullet(A)$, $HP_\bullet(A)$
Schouten bracket	Gerstenhaber bracket
Hodge-to-de Rham spectral sequence	Hochschild-to-cyclic spectral sequence
Cartier isomorphisms	A non-commutative version thereof

Here are some comments on the table.

(i) Polyvector fields are sections of the exterior algebra $\Lambda^\bullet\mathcal{T}(X)$ generated by the tangent bundle $\mathcal{T}(X)$, and Schouten bracket is a generalization of the Lie bracket of vector field to polyvector fields. It seems that in non-commutative geometry, it is not possible to just consider vector fields – all polyvector fields appear together as a package.

(ii) Similarly, *multiplication* in de Rham cohomology seems to be a purely commutative phenomenon – in the general non-commutative setting, it does not exist.

(iii) The first line corresponds to a famous theorem of Serre which claims that the category of coherent sheaves on a scheme X has finite homological dimension if and only if X is regular. In the literature, some alternative notions of smoothness for non-commutative varieties are discussed; however, we will not use them.

(iv) The last line takes place in positive characteristic, that is, for schemes and algebras defined over a field k with $p = \operatorname{char} k > 0$.

All the items in the left column are probably very familiar (expect for maybe the last line, which we will explain in due course). The notions in the right column probably are not familiar. In the first few lectures of this course, we will explain them. We start with Hochschild Homology and Cohomology.

1.4 Hochschild Homology and Cohomology.

Assume given an associative unital algebra A over a field k.

Definition 1.1. *Hochschild homology $HH_\bullet(A)$ of the algebra A is given by*

$$HH_\bullet(A) = \operatorname{Tor}_\bullet^{A^{opp}\otimes A}(A, A). \tag{1.2}$$

Hochschild cohomology $HH^\bullet(A)$ of the algebra A is given by

$$HH^\bullet(A) = \operatorname{Ext}^\bullet_{A^{opp}\otimes A}(A, A). \tag{1.3}$$

Here A^{opp} is the opposite algebra to A – the same algebra with multiplication written in the opposite direction (if A is commutative, then $A^{opp} \cong A$, but in general they might be different). Left modules over $A^{opp} \otimes A$ are the same as bimodules over A, and A has a natural structure of A-bimodule, called *the diagonal bimodule* – this is the meaning of A in (1.3) and in the right-hand side of $\operatorname{Tor}_\bullet(-,-)$ in (1.2). However, A also has a natural structure of a *right* module over $A^{opp}\otimes A$ – and this is what we use in the left-hand side of $\operatorname{Tor}_\bullet(-,-)$ in (1.2).

We note that by definition $HH^\bullet(A)$ is an algebra (take the composition of $\operatorname{Ext}^\bullet$-s), and $HH_\bullet(A)$ has a natural structure of a right module over $HH^\bullet(A)$. In general, neither of them has a structure of an A-module.

Given an A-bimodule M, we can also define Hochschild homology and cohomology with coefficients in M by setting

$$HH_\bullet(A, M) = \operatorname{Tor}_\bullet^{A^{opp}\otimes A}(A, M), \qquad HH^\bullet(A, M) = \operatorname{Ext}^\bullet_{A^{opp}\otimes A}(A, M).$$

In particular, $HH_\bullet(A, -)$ is the derived functor of the left-exact functor $A\text{-}\mathsf{bimod} \to k\text{-Vect}$ from A-bimodules to k-vector spaces given by $M \mapsto A \otimes_{A^{opp}\otimes A} M$. Equivalently, this functor can be defined as follows:

$$M \mapsto M/\{am - ma \mid a \in A, m \in M\}.$$

The reason Hochschild homology and cohomology is interesting – and indeed, the starting point for the whole brand of non-commutative geometry which we discuss in these lecture – is the following classic theorem.

Theorem 1.2 (Hochschild-Kostant-Rosenberg, 1962). *Assume that A is commutative, and that $X = \operatorname{Spec} A$ is a smooth algebraic variety of finite type over k. Then there exist isomorphisms*

$$HH_\bullet(A) \cong \Omega^\bullet(X), \qquad HH^\bullet(A) \cong \Lambda^\bullet\mathcal{T}(X),$$

where $\Omega^\bullet(A)$ are the spaces of differential forms on the affine variery X, and $\Lambda^\bullet\mathcal{T}(A)$ are the spaces of polyvector fields – the sections of the exterior powers of the tangent sheaf $\mathcal{T}(X)$.

Proof. To compute $HH_\bullet(A)$ and $HH^\bullet(A)$, we need to find a convenient projective resolution of the diagonal bimodule A. Since A is commutative, we can identify A and A^{opp}, so that A-bimodules are the same as $A \otimes A$-modules. Let $I \subset A \otimes A$ be the kernel of the natural surjective map $m : A \otimes A \to A$, $m(a_1 \otimes a_2) = a_1a_2$. Then I is an ideal in $A \otimes A$, and by definition, the module $\Omega^1(A)$ of 1-forms on A is equal to the quotient I/I^2. Thus we have a canonical surjective map

$$\eta : I \to \Omega^1(A).$$

Since $X = \operatorname{Spec} A$ is smooth of finite type, $\Omega^1(A)$ is a projective A-module. Therefore, if consider the A-bimodule I as an A-module by restriction to one of the factors in $A \otimes A$ – say the second one – then the map η admits a splitting map $\Omega^1(A) \to I$, which extends to a map

$$s : A \otimes \Omega^1(A) \to I$$

of A-bimodules. But the A-bimodule $A \otimes \Omega^1(A)$ is projective; thus we can let $P_0 = A \otimes A$, $P_1 = A \otimes \Omega^1(A)$, and we have a start of a projective resolution

$$P_1 \xrightarrow{s} P_0 \xrightarrow{m} A$$

of the diagonal bimodule A. Extend it to a "Koszul complex" $P_\bullet$ by setting $P_i = \Lambda^i_{A\otimes A}(P_1)$, $i \geq 0$, and extending s to a derivation $d : P_{\bullet+1} \to P_\bullet$ of this exterior algebra. This gives a certain complex $P_\bullet$, and it well-know that

> $P_\bullet$ is a resolution of A outside of a certain Zariski-closed subset $Z \subset X \times X$ which does not intersect the diagonal.

Therefore the complex $P_\bullet$ can be used to compute $HH_\bullet(A)$ and $HH^\bullet(A)$; doing this gives the desired isomorphism. □

Exercise 1.1. *Show that the isomorphisms in Theorem* 1.2 *are canonical.*

We note that this proof does not need any assumptions on characteristic (the original proof of Hochschild-Kostant-Rosenberg was slightly different, and it only worked in characteristic 0).

1.5 The bar resolution and the Hochschild complex.

The Koszul resolution is very convenient, but it only exists for a smooth commutative algebra A. We will now introduce another resolution for the diagonal bimodule called *the bar resulution* which is much bigger, but exists in full generality. This gives a certain large but canonical complex for computing $HH_\bullet(A)$ and $HH^\bullet(A)$.

The bar resolution $C_\bullet(A)$ starts with the same free A-bimodule $C_0(A) = A \otimes A$ as the Koszul resolution. Since we want the resolution to exist for any A, there is not much we can build upon to proceed to higher degrees – we have to use A itself. Thus for any $n \geq 1$, we let

$$C_n(A) = A^{\otimes(n+2)} = A \otimes A^{\otimes n} \otimes A,$$

the free A-bimodule generated by the k-vector space A. The differential $C_{n+1}(A) \to C_n(A)$ is denoted b' for historical reasons, and it is given by

$$b' = \sum_{i=1}^{n+2} (-1)^i \, \mathsf{id}^{\otimes i} \otimes m \otimes \mathsf{id}^{\otimes n+2-i}, \tag{1.4}$$

where, as before, $m : A \otimes A \to A$ is the multiplication map. We note that b' is obviously an A-bimodule map.

There is also a version with coefficients: assume given an A-bimodule M, and denote the A-action maps $A \otimes M \to M$, $M \otimes A \to M$ by the same letter m. Then we let $C_n(A, M) = A^{\otimes(n+1)} \otimes M$, $n \geq 0$, and we define the map $b' : C_{n+1}(A, M) \to C_n(A, M)$ by the same formula (1.4).

Lemma 1.3. *For any A, M, the complex $\langle C_\bullet(A, M), b' \rangle$ is a resolution of the bimodule M.*

Proof. The fact that b' squares to 0 is a standard computation which we leave as an exersize (it also has an explanation in terms of simplicial sets which we will give later). To prove that $C_\bullet(A, M)$ is a resolution, extend it to a complex $C'_\bullet(A, M)$ by shifting the degree by 1 and adding the term A – that is, we let

$$C'_n(A, M) = A^{\otimes n} \otimes M$$

for $n \geq 0$, with the differential b' given by the same formula (1.4). Then we have to prove that $C'_\bullet(A, M)$ is acyclic. But indeed, the map $h : C'_\bullet(A, M) \to C'_{\bullet+1}(A, M)$ given by

$$h(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n,$$

obviously satisfies $h \circ b' + b' \circ h = \mathsf{id}$, thus gives a contracting homotopy for $C'_\bullet(A, M)$. $\square$

Exercise 1.2. *Show that for any A-bimodule M, the bimodule $A \otimes M$ is acyclic for the Hochschild homology functor (that is, $HH_i(A, A \otimes M) = 0$ for $i \geq 1$). Hint: compute $HH_i(A, A \otimes M)$ by using the bar resolution for the right $A^{opp} \otimes A$-module A in the left-hand side of $\operatorname{Tor}_\bullet^{A^{opp} \otimes A}(A, A \otimes M)$.*

By virtue of Exercise 1.2, the resolution $C_\bullet(A, M)$ can be used for the computation of the Hochschild homology groups $HH_\bullet(A, M)$. This gives a complex whose terms are also given by $A^{\otimes n} \otimes M$, $n \geq 0$, but the differential is given by

$$b = b' + (-1)^{n+1} t, \tag{1.5}$$

with the correction term t being equal to

$$t(a_0 \otimes \cdots \otimes a_{n+1} \otimes m) = a_1 \otimes \cdots \otimes a_{n+1} \otimes m a_0$$

for any $a_0, \dots, a_{n+1} \in A$, $m \in M$. This is known as the *Hochschild homology complex.*

Geometrically, one can think of the components $a_0, \dots, a_{n-1}, m$ of some tensor in $A^{\otimes n} \otimes M$ as having been placed at $n + 1$ points on the unit interval $[0, 1]$, including the egde points $0, 1 \in [0, 1]$; then each of the terms in the differential b' corresponds to contracting an interval between two neighboring points and multiplying the components sitting at its endpoints. To visualize the differential b in a similar way, one has to take $n + 1$ points placed on the unit circle S^1 instead of the unit interval, including the point $1 \in S^1$, where we put the component m.

1.6 Cyclic homology – explicit definition.

In the case $M = A$, the terms in the Hochschild homology complex are just $A^{\otimes n+1}$, $n \geq 0$, and they acquire an additional symmetry: we let $\tau : A^{\otimes n+1} \to A^{\otimes n+1}$ to be the cyclic permutation multiplied by $(-1)^n$. Note that in spite of the sign change, we have $\tau^{n+1} = \mathsf{id}$, so that it generates an action of the cyclic group $\mathbb{Z}/(n + 1)\mathbb{Z}$ on every $A^{\otimes n+1}$. The fundamental fact here is the following.

Lemma 1.4. *For any n, we have*

$$(\mathsf{id} - \tau) \circ b' = -b \circ (\mathsf{id} - \tau),$$
$$(\mathsf{id} + \tau + \cdots + \tau^{n-1}) \circ b = -b' \circ (\mathsf{id} + \tau + \cdots + \tau^n)$$

as maps from $A^{\otimes n+1}$ to $A^{\otimes n}$.

Proof. Denote $m_i = \mathsf{id}^i \otimes m \otimes \mathsf{id}^{n-i} : A^{\otimes n+1} \to A^{\otimes n}$, $0 \leq i \leq n-1$, so that $b' = m_0 - m_1 + \cdots + (-1)^{n-1} m_{n-1}$, and let $m_n = t = (-1)^n (b - b')$. Then we obviously have

$$m_{i+1} \circ \tau = \tau \circ m_i$$

for $0 \leq i \leq n-1$, and $m_0 \circ \tau = (-1)^n m_n$. Formally applying these identities, we conclude that

$$\begin{aligned} \sum_{0 \leq i \leq n} (-1)^i m_i \circ (\mathsf{id} - \tau) &= \sum_{0 \leq i \leq n} (-1)^i m_i - m_0 - \sum_{1 \leq i \leq n} (-1)^i \tau \circ m_{i-1} \\ &= -(\mathsf{id} - \tau) \circ \sum_{0 \leq i \leq n-1} (-1)^i m_i, \end{aligned} \tag{1.6}$$

$$
\begin{aligned}
b' \circ (\mathsf{id} + \tau + \dots + \tau^n) &= \sum_{0 \le i \le n-1} \sum_{0 \le j \le n} (-1)^i m_i \circ \tau^j \\
&= \sum_{0 \le j \le i \le n-1} (-1)^i \tau^j \circ m_{i-j} + \sum_{1 \le i \le j \le n} (-1)^{i+n} \tau^{j-1} \circ m_{n+i-j} \\
&= -(\mathsf{id} + \tau + \dots + \tau^{n-1}) \circ b,
\end{aligned} \tag{1.7}
$$

which proves the claim. □

As a corollary, the following diagram is in fact a bicomplex.

$$
\begin{tikzcd}
\dots \arrow[r] & A \arrow[r, "\mathsf{id}"] & A \arrow[r, "0"] & A \\
& \arrow[u, "b"] & \arrow[u, "b'"] & \arrow[u, "b"] \\
\dots \arrow[r] & A \otimes A \arrow[r, "\mathsf{id} + \tau"] & A \otimes A \arrow[r, "\mathsf{id} - \tau"] & A \otimes A \\
& \arrow[u, "b"] & \arrow[u, "b'"] & \arrow[u, "b"] \\
\dots & \dots & \dots & \dots \\
& \arrow[u, "b"] & \arrow[u, "b'"] & \arrow[u, "b"] \\
\dots \arrow[r] & A^{\otimes n} \arrow[r, "\mathsf{id} + \tau + \dots + \tau^{n-1}"] & A^{\otimes n} \arrow[r, "\mathsf{id} - \tau"] & A^{\otimes n} \\
& \arrow[u, "b"] & \arrow[u, "b'"] & \arrow[u, "b"]
\end{tikzcd} \tag{1.8}
$$

Here it is understood that the whole thing extends indefinitely to the left, all the even-numbered columns are the same, all odd-numbered columns are the same, and the bicomplex is invariant with respect to the horizontal shift by 2 columns.

Definition 1.5. The total homology of the bicomplex (1.8) is called the *cyclic homology* of the algebra A, and denoted by $HC_\bullet(A)$.

We see right away that the first, the third, and so on column when counting from the right is the Hochschild homology complex computing $HH_\bullet(A)$, and the second, the fourth, and so on column is the acyclic complex $C'_\bullet(A)$. (the top term is A, and the rest is the bar resolution for A). Thus the spectral sequence for this bicomplex has the form

$$HH_\bullet(A)[u^{-1}] \Rightarrow HC_\bullet(A), \tag{1.9}$$

where u is a formal parameter of cohomological degree 2, and $HH_\bullet(A)[u^{-1}]$ is shorthand for "polynomials in u^{-1} with coefficients in $HH_\bullet(A)$". This is known as *Hochschild-to-cyclic*, or *Hodge-to-de Rham* spectral sequence (we will see in the next lecture that it reduces to the usual Hodge-to-de Rham spectral sequence in the smooth commutative case).

Shifting (1.8) to the right by 2 columns gives the *periodicity map* $u : HC_{\bullet+2}(A) \to HC_\bullet(A)$, which fits into an exact triangle

$$HH_{\bullet+2} \longrightarrow HC_{\bullet+2}(A) \longrightarrow HC_\bullet(A) \longrightarrow, \tag{1.10}$$

known as the *Connes' exact sequence.* One can also invert the periodicity map – in other words, extend the bicomplex (1.8) not only to the left, but also to the right. This gives the *periodic cyclic homology* $HP_\bullet(A)$. Since the bicomplex for $HP_\bullet(A)$ is infinite in both directions, there is a choice involved in taking the total complex: we can take either the product, or the sum of the terms. We take the product.

Remark 1.6. The n-th row of the complex (1.8) is the standard complex which computes the homology $H_\bullet(\mathbb{Z}/n\mathbb{Z}, A^{\otimes n})$ of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. In the periodic version, we have the so-called *Tate homology* instead of the usual homology. It is known that, $Z/n\mathbb{Z}$ being finite, Tate homology is always trivial over a base field of characteristic 0. Were we to take the sum of terms of the periodic bicomplex instead of the product in the definition of $HP_\bullet(A)$, the corresponding spectral sequence would have converged, and the resulting total complex would have been *acyclic*. This is the first instance of an important feature of the theory of cyclic homology: convergence or non-convergence of various spectral sequences is often not automatic, and, far from being just a technical nuissance, has a real meaning.