

A computation of invariants of a rational self-map

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Abstract

I prove the algebraic stability and compute the dynamical degrees of C. Voisin's rational self-map of the variety of lines on a cubic fourfold.

Je démontre la stabilité algébrique et calcule les degrés dynamiques de l'auto-application rationnelle (construite par C. Voisin) de la variété des droites sur une cubique dans $\mathbb{P}^5$.

Let V be a smooth cubic in $\mathbb{P}^5$ and let $X = \mathcal{F}(V)$ be the variety of lines on V . Thus X is a smooth four-dimensional subvariety of the grassmanian $G(1, 5)$, more precisely, the zero locus of a section of S^3U^* , where U is the tautological rank-two bundle over $G(1, 5)$. It is immediate from this description that the canonical class of X is trivial. Let $\mathcal{F} \subset V \times X$ be the universal family of lines on V , and let $p : \mathcal{F} \rightarrow V$, $q : \mathcal{F} \rightarrow X$ be the projections. Beauville and Donagi prove in [BD] that the Abel-Jacobi map $AJ : q_*p^* : H^4(V, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ is an isomorphism, at least after tensoring up with $\mathbb{Q}$; since this is also a morphism of Hodge structures, we obtain, by the Noether-Lefschetz theorem for V , that $Pic(X) = \mathbb{Z}$ for a sufficiently general X .

Lines on cubics have been studied in [CG]. It is shown there that the normal bundle of a general line on a smooth 4-dimensional cubic is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ (l is then called "a line of the first kind"), and that some special lines ("lines of the second kind") have normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. The lines of the second kind form a two-parameter subfamily $S \subset X$, and all lines on V are either of the first or of the second kind.

An alternative description of lines on a smooth cubic in $\mathbb{P}^n$ from [CG] is as follows: let $F(x_0, \dots, x_n) = 0$ define a smooth cubic V in $\mathbb{P}^n$ and consider the Gauss map

$$D_V : \mathbb{P}^n \rightarrow (\mathbb{P}^n)^* : x \mapsto \left(\frac{\partial F}{\partial x_0}(x) : \dots : \frac{\partial F}{\partial x_n}(x) \right)$$

(so $D_V(x)$ is the tangent hyperplane to V at x). For a line $l \subset V$ there are only two possibilities: either $D_V|_l$ maps l bijectively onto a plane conic, or $D_V|_l$ is two-to-one

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onto a line in $(\mathbb{P}^n)^*$. In the first case, l is of the first kind and the intersection of the hyperplanes $\cap_{x \in l} D_V(x) \subset \mathbb{P}^n$ is a subspace of dimension $n - 3$, tangent to V along l ; in the other case, l is of the second kind and the linear subspace $Q_l = \cap_{x \in l} D_V(x) \subset \mathbb{P}^n$, which is of course still tangent to V along l , is of greater dimension $n - 2$.

As remarked in [V], X always admits a rational self-map. This map is constructed as follows: from the description of the normal bundle, one sees that for a general line $l \subset V$, there is a unique plane $P_l \subset \mathbb{P}^5$ which is tangent to V along l . So $P_l \cap V$ is the union of l and another line l' , and one defines $f : X \dashrightarrow X$ by sending l to l' . This is only a rational map: indeed, if l is of the second kind, we have a one-parameter family, more precisely, a pencil of planes $\{P : l \subset P \subset Q_l\}$ tangent to V along l . So the surface S is the indeterminacy locus of the map f . Blowing S up, one obtains a resolution of singularities of f ; it introduces the exceptional divisor E which is a $\mathbb{P}^1$ -bundle over S , each $\mathbb{P}^1$ corresponding to the pencil of planes tangent to V along l . That is, the "image" by f of every exceptional point is a (possibly singular) rational curve on X .

In [V], it is shown that the degree of f is 16. The proof is very short: the space $h^{2,0}(X)$ is generated by a nowhere-degenerate form σ ([BD]); C.Voisin then uses a Mumford-style argument (saying that a family of rationally equivalent cycles induces the zero map on certain differential forms) to show that $f^*\sigma = -2\sigma$. As $(\sigma\bar{\sigma})^2$ is a volume form, the result follows.

In holomorphic dynamics, one considers the *dynamical degrees* of a rational self-map (the definition below seems to appear first in [RS]). Those are defined as follows: let X be a Kähler manifold of dimension k . Fix a Kähler class ω . Define

$$\delta_l(f, \omega) = [f^*\omega^l] \cdot [\omega^{k-l}]$$

(the product of cohomology classes on X), and the l th *dynamical degree*

$$\lambda_l(f) = \limsup(\delta_l(f^n, \omega))^{1/n}$$

(this does not depend on ω). It turns out that this is the same as

$$\rho_l(f) = \limsup(r_l(f^n))^{1/n},$$

where $r_l(f^n)$ is the spectral radius of the action of f^n on $H^{l,l}(X)$. If X is projective, one can restrict oneself to the subspace of algebraic classes, replacing ρ_l and r_l by the correspondent spectral radii r_l^{alg} and ρ_l^{alg} . Moreover, Dinh and Sibony [DS] recently proved that the limsup in the definition of $\lambda_l(f)$ is actually a limit, and also the invariance of the dynamical degrees by birational conjugation.

The map f is said to be *cohomologically hyperbolic* if there is a dynamical degree which is strictly greater than all the others. In fact the dynamical degrees satisfy a convexity property which says that if $\lambda_l(f)$ is such a maximal degree, then λ_i grows together with i until i reaches l , and decreases thereafter. It is conjectured that the cohomological hyperbolicity implies certain important equidistribution properties

for f , and this is proved in the works of Briend-Duval [BrDv] and Guedj [G] when the maximal dynamical degree is the topological degree λ_k .

In general it is very difficult to compute the dynamical degrees of a rational self-map, because for rational maps it is not always true that $(f^n)^* = (f^*)^n$ (where f^* denotes the transformation induced by f in the cohomology.). In fact few examples of such computations are known.

If $(f^n)^* = (f^*)^n$, the map f is called *algebraically stable*. The computation of its dynamical degrees is then much easier: one only has to study the linear map f^* , and it is not necessary to look at the iterations of f .

The purpose of this note is to show that the example f considered in the beginning is algebraically stable and to compute its dynamical degrees. The map f turns out to be cohomologically hyperbolic, and the dominating dynamical degree is in this case λ_2 .

I am very much indebted to A. Kuznetsov and C. Voisin for crucially helpful discussions.

1. Algebraic stability of the self-map

From now on let $X = \mathcal{F}(V)$ and f be as described in the beginning. Let S be the indeterminacy locus $I(f)$, that is, the surface of lines of the second kind.

We assume that V is *sufficiently general*. The first goal is to prove that f is algebraically stable.

Lemma 1: *For a general V , the surface S is smooth.*

Proof: Let T be the parameter space of cubics in $\mathbb{P}^5$. Consider the incidence variety $\mathcal{I} \subset G(1, 5) \times T$: $\mathcal{I} = \{(l, V) : l \text{ is of the 2nd kind on } V\}$. It is enough to show that $\mathcal{I}$ is smooth in codimension two. By homogeneity, it is enough to verify that the fiber $I_l = pr_1^{-1}l$ is smooth in codimension two for some $l \in G(1, 5)$. Choose coordinates $(x_0 : \dots : x_5)$ on $\mathbb{P}^5$ so that l is given by $x_2 = x_3 = x_4 = x_5 = 0$; then $l \subset V$ means that four coefficients (those of x_0^3 , $x_0^2x_1$, $x_0x_1^2$ and x_1^3) of the equation $F = 0$ of V vanish, and l being of the second type means that the 4 quadratic forms $\partial F / \partial x_i|_l$, $i = 2, 3, 4, 5$ in two variables x_0, x_1 , span a linear space of dimension 2, rather than 3. This in turn means that a 4×3 matrix M , whose coefficients are (different) coordinates on $T' \subset T$, the projective space of cubics containing l , has rang 2. It follows from general facts about determinantal varieties that $Sing(I_l)$ is the locus where $rg(M) \leq 1$, and it has codimension four in I_l (in fact, $Sing(I_l)$ is also contained in the locus of singular cubics, [CG]).

Let us now look at the map $f : X \dashrightarrow X$. The first thing to observe is as follows: $K_X = 0$ means that f cannot contract a divisor. Indeed, let $\pi : Y \rightarrow X$, $g : Y \rightarrow X$ be the resolution of the indeterminacies of f . Then the ramification divisor (i.e. the vanishing locus of the Jacobian determinant) of π is equal to that of g , so g can only contract a divisor which is already π -exceptional. By the same token, any subset

contracted by g must lie in the exceptional divisor of π , in other words, anything contracted by f is in S .

Lemma 2: *The map g does not contract a surface (neither to a point nor to a curve), unless possibly some surface already contracted by π .*

Proof (C. Voisin): If it does, this surface Z projects onto S . Recall that X is holomorphic symplectic, with symplectic form $\sigma = AJ(\eta)$, where η generates $H^{3,1}$ of the cubic. Notice that S is not lagrangian: $\sigma|_S \neq 0$. This is because, as we shall see in the next section, the cohomology class of S is $5(H^2 - \Delta)$, where H is the hyperplane section class for the Plücker embedding, and Δ is the Schubert cycle of lines contained in a hyperplane. We have $[\sigma] \cdot \Delta = 0$ in the cohomologies, by projection formula (because $[\eta]$ is a primitive cohomology class). But $[\sigma] \cdot H^2$ is non-zero, because $[\sigma\bar{\sigma}] \cdot H^2$ is the Bogomolov square of an ample H . Thus so is $\sigma|_S$.

Now recall ([V]) that $f^*\sigma = -2\sigma$, that is, $g^*\sigma = -2\pi^*\sigma$. This is a contradiction, since $g^*\sigma|_Z = 0$, whereas $\pi^*\sigma|_Z \neq 0$.

Remark: The same argument with σ shows that g cannot contract the π -exceptional divisor (even not to a surface). A somewhat more elaborate version of it shows that g does not contract any surface at all; but we won't need this to prove the algebraic stability. See also the remark after Theorem 3.

Theorem 3: *The self-map f of $X = \mathcal{F}(V)$ described above is algebraically stable, that is, we have $(f^n)^* = (f^*)^n$, where the upper star denotes the action on the cohomologies, and n is a positive integer.*

Proof: The transformation induced by f on the cohomologies is given by the class of the graph $\Gamma(f) \in H^8(X \times X)$. In general, $(hf)^* = h^*f^*$ is implied by $\Gamma(f) \circ \Gamma(h)$ having only one 4-dimensional component (which is then, of course, $\Gamma(hf)$). Recall that the correspondence $\Gamma(f) \circ \Gamma(h)$ is

$$\{(x, z) | \exists y : (x, y) \in \Gamma(f), (y, z) \in \Gamma(h)\}.$$

Let $h = f^n$. Suppose that $\Gamma(f) \circ \Gamma(h)$ has an extra 4-dimensional component M . There are three possibilities for its image under the first projection:

1) It is a divisor D . Then f is defined at a general point $d \in D$; but fibers of M over D must be at least one-dimensional. That is, D is contracted by f (to the indeterminacy locus of h). This is impossible.

2) It is a surface Z ; fibers of M over Z must be two-dimensional. But if $Z \not\subset I(f)$, there is only one y corresponding to generic $x \in Z$, and furthermore, since Z is not contracted by f , $h(y)$ is at most a curve. Indeed, the locus $I_2(h) = \{y : \dim(h(y)) \geq 2\}$ is of codimension at least three in X . Thus a fiber of M over a generic $x \in Z$ cannot be two-dimensional. If $Z \subset I(f)$, one concludes in a similar way: for any $x \in Z$, the corresponding y are parametrized by a rational curve D_x ; for a general x , the curve D_x is not the indeterminacy locus of h , and, moreover, since no surface is contracted, D_x intersects $I(h)$ in a finite number of points, all outside $I_2(h)$. Thus again the general fiber of $\Gamma(f) \circ \Gamma(h)$ over T is at most one-dimensional.

3) It is a curve C . By a simple dimension count as in the previous case, we see that then C must be contracted by f to a point y such that $\dim(h(y)) = 3$. Recall that $h = f^n$. We may assume that n is the smallest number with the property $\dim(h(y)) = 3$. As for any $x \in X$, $f(x)$ is either a point or a rational curve, S must be contained in $f^{n-1}(y)$ as a component. This implies that S is covered by rational curves. But, as we shall see in the next section, K_S is ample; so this case is also impossible.

Remark: In fact one can show that g is finite (and this obviously implies the algebraic stability). Indeed, by the argument of Lemma 2, if $g(C)$ is a point, then $\sigma|_S$ vanishes on $\pi(C)$ and so $\pi(C)$ is a component of a canonical divisor on S . On the other hand, all lines on V corresponding to the points of $\pi(C)$ intersect the line corresponding to the point $g(C)$. This means that $\pi(C)$ is contained in a hyperplane section of S . But it turns out (see the remark in the end of next section) that K_S is even ample than the hyperplane section class H_S . To get a contradiction, it remains to show that the vanishing locus of $\sigma|_S$ is smooth and irreducible for generic V . C. Voisin indicates that this is not very difficult to calculate explicitly using the Griffiths' description of the primitive cohomology of hypersurfaces (here cubics) in terms of residues.

2. Some cohomology classes on X

Recall that $X \subset G(1, 5)$ is the zero-set of a section of the bundle S^3U^* (of rang 4). We shall describe the action of f^* on those cohomology classes which are restrictions of classes on $G(1, 5)$.

So let $H = c_1(U^*)|_X = [\mathcal{O}_{G(1,5)}(1)]|_X$; $\Delta = c_2(U^*)|_X$ (so Δ is the class of the subset of points $x \in X$ such that the corresponding lines $l_x \subset \mathbb{P}^5$ lie in a given hyperplane).

Lemma 4: $H^4 = 108$; $H^2\Delta = 45$; $\Delta^2 = 27$.

Proof: $[X] = c_4(S^3U^*)$ and

$$c_4(S^3U^*) = 9c_2(U^*)(2c_1^2(U^*) + c_2(U^*)).$$

Let $\sigma_i = c_i(U^*)$; those are Schubert classes on $G(1, 5)$: $\sigma_1 = \{\text{lines intersecting a given } \mathbb{P}^3\}$, $\sigma_2 = \{\text{lines lying in a given hyperplane}\}$.

Notice that $c_1(U^*)$ is the hyperplane section class in the Plücker imbedding and restricts as such onto a "sub-grassmanian" of lines lying in a linear subspace. So

$$H^4 = 9\sigma_2\sigma_1^4(2\sigma_1^2 + \sigma_2) = 18\deg(G(1, 4)) + 9\deg(G(1, 3)) = 108.$$

The other equalities are proved similarly..

For generic X , the class H generates $H_{alg}^{1,1}$ over $\mathbb{Z}$, and H^3 generates $H_{alg}^{3,3}$ over $\mathbb{Q}$.

One can also show that H^2 and Δ generate $H_{alg}^{1,1}$ over $\mathbb{Q}$; we do not need this for our computation, but see the remark in the end of the last section.

The next step is to compute the indeterminacy surface S .

Let A denote the underlying vector space for $\mathbb{P}^5 \supset V$, and let $L \subset A$ be the underlying vector space for a line $l \subset V$. Write the Gauss map associated to F defining V as $D_F : A \rightarrow S^2 A^*$. The condition $l \subset V$ implies that this map descends to $\phi : A/L \rightarrow S^2 L^*$:

$$\begin{array}{ccc} A & \rightarrow & S^2 A^* \\ \downarrow & & \downarrow \\ A/L & \rightarrow & S^2 L^* \end{array}$$

For a general L , the map ϕ is surjective with one-dimensional kernel; and it has two-dimensional kernel exactly when L is the underlying vector space of a line of the second kind. Now L is the fiber over a point $x_l \in X$ of the restriction U_X of U to X ; so, globalizing and dualizing, we get the following resolution for S :

$$0 \rightarrow S^2 U_X \rightarrow Q_X^* \rightarrow M \rightarrow M \otimes \mathcal{O}_S \rightarrow 0,$$

where Q_X^* is the restriction to X of the universal quotient bundle over $G(1, 5)$, and M is a line bundle. In fact

$$[M] = \det(Q_X^*) - \det(S^2 U_X) = -H + 3H = 2H.$$

So one has a resolution for the ideal sheaf $\mathcal{I}_S$:

$$0 \rightarrow S^2 U_X(-2H) \rightarrow Q_X^*(-2H) \rightarrow \mathcal{I}_S \rightarrow 0.$$

The cohomology class $[S]$ can be computed by the Thom-Porteous formula: a partial case of this formula identifies the cohomology class of the degeneracy locus $D_{e-1}(\phi)$, where a vector bundle morphism $\phi : E \rightarrow F$ ($rg(E) = e \leq rg(F) = f$) is not of maximal rang, as $c_{f-e+1}(F - E)$, where we put formally

$$c(F - E) = 1 + c_1(F - E) + c_2(F - E) + \dots = c(F)/c(E).$$

So

$$\begin{aligned} [S] &= c_2(Q_X^* - S^2 U_X) = c_2(Q_X^*) - c_2(S^2 U_X) + 2Hc_1(S^2 U_X) = \\ &= H^2 - \Delta - 4\Delta - 2H^2 + 6H^2 = 5(H^2 - \Delta). \end{aligned}$$

One can also make a direct calculation using the equality

$$[S] = -c_2(\mathcal{O}_S) = c_2(\mathcal{I}_S)$$

(see e.g. [F], chapter 15.3). Later, we shall need its extension

$$i_* c_1(N_{S,X}) = i_* K_S = c_3(\mathcal{I}_S)$$

where $i : S \rightarrow X$ is the usual embedding. From the resolution of the ideal sheaf, one computes

$$c_3(\mathcal{I}_S) = c_3(Q_X^*(-2H)) - c_3(S^2 U_X(-2H)) - c_2(\mathcal{I}_S)c_1(S^2 U_X(-2H)) = 20H^3 - 27H\Delta.$$

Thus we have obtained the following

Proposition 5: *The Chern classes of $\mathcal{I}_S$ are: $c_1(\mathcal{I}_S) = 0$; $c_2(\mathcal{I}_S) = 5(H^2 - \Delta)$; $c_3(\mathcal{I}_S) = 20H^3 - 27H\Delta = \frac{35}{4}H^3$.*

The last equality is obtained by taking the intersection with H and using lemma 4.

Remark: One can also remark that S being the degeneration locus of a map $\phi : S^2U \rightarrow Q^*$, its normal bundle is isomorphic to $(\text{Ker}(\phi|_S))^* \otimes (\text{Coker}(\phi|_S))$, so its determinant is $c_1(\text{Coker}(\phi|_S)) - 2c_1(\text{Ker}(\phi|_S)) = c_1(Q^*|_S) - c_1(S^2U|_S) - c_1(\text{Ker}(\phi|_S)) = 2H_S - c_1(\text{Ker}(\phi|_S))$, in particular it is at least as positive than $2H_S$. It is easy to see that the cohomology class of the intersection HS is $\frac{35}{12}H^3$; this suggests that the canonical class of S is equal to $3H_S$, but I have not checked this (one probably can write an analogue of the Thom-Porteous formula, but I could not find a reference, and in our particular case the direct calculation is easy).

3. Computation of the inverse images

Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of S ; this resolves the singularities of f , that is, $g = f \circ \pi : \tilde{X} \rightarrow X$ is holomorphic. Let E denote the exceptional divisor of π . We have $g^*H = a\pi^*H - bE$; the number a is, by algebraic stability, the first dynamical degree we are looking for. In fact the numbers a and b can be computed by a simple geometric argument; we begin by sketching this computation.

Proposition 6 $a = 7$ and $b = 3$.

Proof: Consider the two-dimensional linear sections $V \cup \mathbb{P}^3$ of V ; they form a family of dimension $\dim(G(3, 5)) = 8$. A standard dimension count shows that in this family, there is a one-parameter subfamily of cones over a plane cubic (indeed, one computes that out of ∞^{19} cubics in $\mathbb{P}^3$, ∞^{12} are such cones, so being a cone imposes only 7 conditions on a cubic surface in $\mathbb{P}^3$). Furthermore, some (a finite number) of those plane cubics are degenerate, so on V we have a finite number of cones over a plane cubic with a double point (as it was already mentioned in the first section). Thus on X , we have a one-parameter family of plane cubic curves C_t .

Another way to see this is to remark that the lines passing through a point of V sweep out a surface which is a cone over the intersection of a cubic and a quadric in $\mathbb{P}^3$ ([CG]). The quadric sometimes degenerates in the union of two planes, so the intersection becomes a union of two plane cubics. Moreover, the two plane cubics have three points in common (e.g. from the arithmetic genus count).

The resulting curves C_t on X are clearly f -invariant, and when smooth, map 4:1 to themselves (indeed, through a given point p of an elliptic curve $C \in \mathbb{P}^2$, there are 4 tangents to C , because the projection from p is a 2:1 map from C to $\mathbb{P}^1$ and so has 4 ramification points). A smooth C_t has 3 points in common with S . Let C_0 be singular; then it has at least one point x in common with S , and is the image by g of the exceptional $E_x \cong \mathbb{P}^1 \subset \tilde{X}$ over this point; g is 1:1 from E_x to C_0 .

It follows that $E_x g^* H = 3$, so $b = 3$ (recall that E restricts to itself as $\mathcal{O}_E(-1)$, hence $E_x E = -1$). Also, $C_t H = C_t E = 3$ and $C_t g^* H = 12$, from where $a = 7$.

However, I could not find such a simple way to compute the higher dynamical degrees. The following approach is suggested by A. Kuznetsov.

On $\tilde{X}$, one has a vector bundle P of rank three: its fiber over x_l is the underlying vector space of the plane tangent to V along l when l is of the first kind, and we take the obvious extension to the exceptional divisor. One has $\pi^* U_X \subset P$.

First of all, let us describe the quotient: let E denote the exceptional divisor of π . Then one can see that we obtain P as the following extension:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \pi^* U_X & \rightarrow & P & \rightarrow & M^*(E) & \rightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \pi^* U_X & \rightarrow & A \otimes \mathcal{O} & \rightarrow & \pi^* Q_X & \rightarrow & 0, \end{array}$$

where the map $M^*(E) \rightarrow \pi^* Q_X$ is the dual of the pull-back of the map $Q_X^* \rightarrow M \otimes \mathcal{I}_S$ from the resolution of S in the last section.

(Alternatively, one can compute the cokernel of the natural inclusion $\pi^* U_X \rightarrow P$ using the exact sequences from the proof of the next proposition and the knowledge of $c_1(g^* U_X^*) = 7\pi^* H - 3E$).

Proposition 7: *The bundle $g^* U_X^*$ fits into an exact sequence*

$$0 \rightarrow (M^*)^{\otimes 2}(2E) \rightarrow P^* \rightarrow g^* U_X^* \rightarrow 0.$$

Proof: Consider the projective bundle $p : \mathbb{P}_{\tilde{X}}(P) \rightarrow \tilde{X}$. It is equipped with a natural map $\psi : \mathbb{P}_{\tilde{X}}(P) \rightarrow \mathbb{P}(A)$, and the inverse image of $\mathcal{O}_{\mathbb{P}(A)}(1)$ is the tautological line bundle $\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)$, meaning that its direct image under the projection to $\tilde{X}$ is P^* . Let $h = c_1(\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1))$. We have a natural inclusion $\mathbb{P}_{\tilde{X}}(\pi^* U_X) \subset \mathbb{P}_{\tilde{X}}(P)$. Denote by d the class of $\mathbb{P}_{\tilde{X}}(\pi^* U_X)$ considered as a divisor on $\mathbb{P}_{\tilde{X}}(P)$. The subvariety of $\mathbb{P}_{\tilde{X}}(P)$ swept out by the lines $f(l)$ (where l is a fiber of $\mathbb{P}_{\tilde{X}}(\pi^* U_X)$), is a divisor; let us denote it by Σ . We clearly have $2d + [\Sigma] = 3h$. Furthermore, Σ itself is a projective bundle over $\tilde{X}$, namely, it is the projectivization of $g^* U_X$, in the sense that the vector bundle $g^* U_X^*$ is the direct image of $\mathcal{O}_{\Sigma}(1) = \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)|_{\Sigma}$ under the projection to $\tilde{X}$. On $\mathbb{P}_{\tilde{X}}(P)$, we have the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - [\Sigma]) \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1) \rightarrow \mathcal{O}_{\Sigma}(1) \rightarrow 0.$$

Also, $h - [\Sigma] = 2(d - h)$. So our proposition follows by taking the direct image of the above exact sequence, once we show that

$$\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - d) = p^* M(-E).$$

As for this last statement, it is clear that $\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - d) = p^* F$ for some line bundle F on $\tilde{X}$, because it is trivial on the fibers. We deduce that $F = M(-E)$ from the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - d) \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1) \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)|_{\mathbb{P}_{\tilde{X}}(\pi^* U_X)} \rightarrow 0 :$$

indeed, it pushes down to

$$0 \rightarrow F \rightarrow P^* \rightarrow \pi^* U_X^* \rightarrow 0.$$

We are ready to prove the main result:

Theorem 8 *We have $f^*H = 7H$, $f^*H^2 = 4H^2 + 45\Delta$, $f^*\Delta = 31\Delta$ and $f^*H^3 = 28H^3$. The dynamical degrees of f are therefore 7, 31, 28 and 16. In particular, f is cohomologically hyperbolic.*

Proof: We have already seen (proposition 6) that $f^*H = \pi_*g^*H = 7H$. Notice that $\pi_*(\pi^*HE) = 0$ and $\pi_*E^2 = -S$ (this is because $\mathcal{O}(E)$ restricts to E as $\mathcal{O}_E(-1)$, where we view E as a projective bundle over S). This gives

$$f^*H^2 = 49H^2 - 9S = 49H^2 - 45(H^2 - \Delta) = 4H^2 + 45\Delta.$$

Further, $g^*\Delta = c_2(g^*U_X)$, and we can find the latter from the exact sequences describing g^*U_X and P . From

$$0 \rightarrow \pi^*U_X \rightarrow P \rightarrow M^*(E) \rightarrow 0,$$

we get $c_2(P) = \pi^*\Delta - \pi^*H(-2\pi^*H + E) = \pi^*\Delta - \pi^*HE + 2\pi^*H^2$, and from

$$0 \rightarrow g^*U_X \rightarrow P \rightarrow M^{\otimes 2}(-2E) \rightarrow 0,$$

$c_2(g^*U_X) = c_2(P) + (4\pi^*H - 2E)(7\pi^*H - 3E) = \pi^*\Delta + 30\pi^*H^2 + 6E^2 - 27\pi^*HE$, that is,

$$f^*\Delta = \Delta + 30H^2 - 30(H^2 - \Delta) = 31\Delta.$$

Finally,

$$g^*H^3 = (7\pi^*H - 3E)^3 = 343\pi^*H^3 - 441\pi^*H^2E + 189\pi^*HE^2 - 27E^3.$$

We have $\pi_*(\pi^*H^2E) = 0$ and

$$\pi_*(\pi^*HE^2) = -5H(H^2 - \Delta) = -\frac{35}{12}H^3.$$

Let us compute π_*E^3 : this is $\pi_*\xi^2$, where ξ is the tautological class on E viewed as a projective bundle $\mathbb{P}_S(N)$ over S . Let r be the projection of E to S , i.e. the restriction of π to E . We have $\xi^2 + c_1(r^*N)\xi + c_2(r^*N) = 0$, which yields $\pi_*E^3 = -i_*c_1(N)$, i being the embedding of S into X . But the latter is just $c_3(i_*\mathcal{O}_S)$ ([F]). We have computed this class in the last section:

$$\pi_*E^3 = 27H\Delta - 20H^3 = -\frac{35}{4}H^3.$$

Putting all this together, we get

$$f^*H^3 = 343H^3 - \frac{35}{4}(63 - 27)H^3 = 28H^3.$$

This finishes the proof of Theorem 8.

Remark 9: Notice that the eigenvectors of f^* on the invariant subspace generated by H^2 and Δ are orthogonal with respect to the intersection form. There are reasons for this; moreover, at least for X generic the map f^* on the whole cohomology group $H^4(X)$ is self-adjoint with respect to the intersection form. One can see it as follows: X is a deformation of the punctual Hilbert scheme $\text{Hilb}^2(S)$ of a K^3 surface [BD]; this implies a decomposition of $H^4(X)$ into an orthogonal direct sum of Hodge substructures

$$H^4(X) = \langle H^2 \rangle \oplus (H \cdot H^2(X)^0) \oplus S^2(H^2(X)^0),$$

where $H^2(X)^0$ denotes the orthogonal to H in $H^2(X)$. For X generic, the second summand is an irreducible Hodge structure and the third one is a sum of an irreducible one and a one-dimensional subspace: this follows from the fact that $H^2(X)^0 = H^4(V)^{\text{prim}}$ (where V is the corresponding cubic fourfold and "prim" is primitive cohomologies) and a theorem of Deligne which says that the closure of the monodromy group (of a general Lefschetz pencil) acting on $H^4(V)^{\text{prim}}$ is the full orthogonal group. Therefore the decomposition rewrites as

$$H^4(X) = \langle H^2, \Delta \rangle \oplus H \cdot H^2(X)^0 \oplus V,$$

where the last two summands are simple, and so f^* must be a homothety on each of them.

It also follows from this decomposition that on X generic, the space of algebraic cycles of codimension two is only two-dimensional, and that all lagrangian surfaces must have the cohomology class proportional to Δ . In particular, since f^* and f_* must preserve the property of being lagrangian, this explains why Δ is an eigenvector of f^* and f_* .

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