Rational fibrations of certain K-trivial varieties: english summary

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In this paper we make several observations related to the search for smooth projective varieties X with an endomorphism h of degree greater than one. Such examples are very rare if we suppose that h is regular, and they are less rare if h is only a rational map. Still, certain examples should be thought of as "not quite interesting": for instance, any surface of Kodaira dimension one has a rational endomorphism of high degree, but such an endomorphism respects the elliptic fibration (the Iitaka fibration), that is, it sends its fibers to its fibers; moreover it induces an automorphism of the base, which, as one can check, is always of finite order. So in fact such endomorphisms "come from" endomorphisms elliptic curves.

It is often difficult to decide whether a given variety has a rational endomorphism of higher degree or not. For example, whereas some special $K3$ surfaces (elliptic surfaces, Kummer surfaces) do, the answer is not known for a generic $K3$ surface (say with Picard group $\mathbb{Z}$). That is why the following example from [V] is very interesting:

Consider a smooth cubic hypersurface $V \subset \mathbb{P}^5$. Let X be the variety of lines on V; it is smooth, 4-dimensional, and $K_X = 0$. In fact X is an irreducible hyperkähler variety ([BD]). For V generic, $Pic(X) = \mathbb{Z}$.

It turns out that for a general line $l \subset V$, there is a unique plane P tangent to V along l. One has thus a rational endomorphism h of X: set $h(l) = l'$, where $V \cap P = l \cup l'$. Voisin proves that $deg(h) = 16$.

It is a natural question whether this endomorphism is "interesting" in our sense; that is, is it true that it does not preserve a (meromorphic) fibration, at least for a general X? Our first result is the following

Theorem 2.1 *Let X be a smooth projective variety such that $K_X = 0$, $Pic(X) = \mathbb{Z}$. Suppose that there is a meromorphic fibration $g : X \dashrightarrow B$. Then a general fiber F of g is of general type.*

For the proof, one compares the Kodaira dimension of the (desingularized) generic fiber with its Iitaka dimension with respect to the inverse image of an ample line bundle on B.

It is well-known that varieties of general type do not have rational endomorphisms of higher degree. That is, at least in the generic case (of cyclic Picard group), no power of Voisin's endomorphism can preserve a fibration.

As X from Voisin's example is hyperkähler, it is interesting to ask what can one say about meromorphic fibrations of hyperkähler varieties (of dimension $2n$). Of course one has to suppose that the fibers are not of general type; so one is naturally led (by Iitaka fibration) to meromorphic fibrations in varieties of zero Kodaira dimension. Recall that in the holomorphic case we have Matsushita's results saying that the fibers are lagrangian tori of dimension n, and the base is "very similar" to $\mathbb{P}^n$. For $n = 2$ (that is, for four-dimensional X), we prove the following results showing that the situation is similar to Matsushita's:

Proposition 3.1 *Let L be a line bundle defining g; then $L^4 = 0$, that is, L is an isotropic vector of the Beauville-Bogomolov form.*

In particular, if the Beauville-Bogomolov form does not represent zero on $Pic(X)$, then any rational fibration of X is in varieties of general type. Proposition 3.1 also implies immediately (Proposition 3.4) that the general fiber of g is not a curve.

Proposition 3.3 *The proper transform of any $M \in Pic(B)$ is proportional to L.*

This is analogous to Matsushita's $Pic(B) = \mathbb{Z}$.

The reason that our proofs are valid only in dimension 4 is that we use the mininal model program for the fibers of g. Modulo MMP, the proofs work in arbitrary dimension.

The crucial point in the proof of 3.1 is a theorem of Huybrechts saying that any curve of negative Beauville-Bogomolov square lies in the uniruled subvariety of X. The proof of 3.3 is along Matsushita's lines.

For the following theorems we use the results of Wierzba [W] on symplectic flops in dimension 4, which in turn rely on the work of Wierzba-Wisniewski. The results of Wierzba say that L can be rendered nef by a finite sequnce of Mukai flops (in fact there is a gap in Wierzba's proof, but it concerns the termination of flops, which has been proved for all 4-folds by Matsuki [Mat]; so the result is valid). We observe that L nef implies g almost holomorphic, and deduce the

Theorem 3.5 *The fibers of g are lagrangian, birational to abelian surfaces.*

Theorem 3.6 *If L is nef, g has a holomorphic model; that is, there is a birational map $\phi : B \dashrightarrow B_0$ such that the composition $\phi \circ g$ is holomorphic.*

The main idea here is to modify X and B in such a way that one has a well-defined map from B to the Chow variety of X, mapping a point $b \in B$ to the fiber X_b. The image of this map can then be taken for B_0.

The main result of the section 4 is the following

Theorem 4.1 *Let X be an arbitrary compact Kähler variety, and $f : X \dashrightarrow X$ a dominant meromorphic map. Then there is a meromorphic fibration $g : X \dashrightarrow Y$,* *such that for x sufficiently general, the fiber of g passing through x is the Zariski closure of the iterated orbit $\{x, f(x), \dots f^k(x), \dots\}$ of x.*

Here "sufficiently general" means "in the complement of a countable union of proper subvarieties".

This implies immediately that in Voisin's example, the iterated orbit of a generic (complex!) point is Zariski-dense, at least for X generic. We observe that there exist such an X with $Pic(X) = \mathbb{Z}$ and defined over a number field; the natural question is, can one prove that the iterated orbit of a generic $\bar{\mathbb{Q}}$-point is Zariski-dense? (If yes, this would of course imply the potential density of rational points on X). Unfortunately, our method does not apply at all: while proving Theorem 4.1, we systematically discard countable unions of proper subvarieties; we cannot do that over $\bar{\mathbb{Q}}$ because of its countability.

In Section 5, we prove that if $g : X \to B$ is a meromorphic fibration with X compact hyperkähler (not necessarily projective) and B Kähler (without the loss of generality, we may suppose that B is smooth), then all covariant holomorphic tensors on B vanish. If $dim(B) = 2$, this implies the rationality of B, and if $dim(B) = 3$, its rational connexity.

Finally, in section 6, we analyse the behaviour of some natural fibrations on X (compact Kähler) with respect to a dominant meromorphic self-map $f : X \dashrightarrow X$. It turns out that the "best-behaved" fibration is the *core* $c : X \dashrightarrow C$, introduced by the second author in [C3]. Namely the following theorem holds:

Theorem *In the situation as above, f induces a rational self-map c_f of C, such that $fc = c_f f$ and $c_f^N = id_C$ for some positive integer N. In other words, a power of f commutes with c.*

In particular, if X admits an f with a Zariski-dense iterated orbit, then X must be special in the sense of [C3].

References

[BD] A. Beauville, R. Donagi, La variété des droites d'une hypersurface cubique de dimension 4, C. R. Acad. Sci. Paris Sér. I Math., 301 (1985), no. 14, 703–706.

[C3] F. Campana: Orbifolds, special varieties and classification theory, Ann. Inst. Fourier

[M] D. Matsushita, On fibre space structures of a projective irreducible symplectic manifold, Topology 38 (1999), no. 1, 79–83.

[Mat] K. Matsuki, Termination of flops for 4-folds, Amer. J. Math. 113 (1991), no. 5, 835-859.

[V] C. Voisin, Intrinsic pseudovolume forms and K-correspondences. The Fano Conference, 761–792, Univ. Torino, Turin, 2004.

[W] J.Wierzba, Birational geometry of symplectic 4-folds, unpublished,
http://www.dpmms.cam.ac.uk/ jw227/publications.html.