Remarks on endomorphisms and rational points

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WORK IN PROGRESS!!! VERY PRELIMINARY PARTIAL VERSION

1 Invariant neighbourhoods

Let X be a smooth projective variety of dimension n and let $f : X \dashrightarrow X$ be a rational self-map, both defined over a "sufficiently large" number field K. We assume that f has a fixed point $q \in X(K)$. This assumption is not restrictive if, for example, f is a regular polarized (that is, such that $f^*L = L^{\otimes k}$ for a certain ample line bundle L and a positive integer k) endomorphism: indeed, in this case the set of periodic points in $X(\bar{\mathbb{Q}})$ is even Zariski-dense [F], so replacing f by a power and taking a finite extension of K if necessary, we find a fixed point.

Our starting point is that, for a suitable prime ideal $\mathfrak{p} \subset \mathcal{O}_K$, we can find a "$\mathfrak{p}$-adic neighbourhood" $q \in O_{\mathfrak{p},q} \subset X(K_{\mathfrak{p}})$, on which f is defined and which is f-invariant.

More precisely, choose an affine neighbourhood $q \in U \subset X$, which is the domain of the definition of a set of local coordinates $x_1, \dots x_n$ (so that the x_i define an étale (AT q???) map from U to the affine space). Then choose a model $\mathcal{X}$ over Spec(A) where $A = \mathcal{O}_K[1/N]$ is a suitable localization of $\mathcal{O}_K$, so that $\mathcal{X}$ has good reduction everywhere. Let $\mathcal{U} \subset \mathcal{X}$ be the corresponding open subset of $\mathcal{X}$. We have

$$\mathcal{O}(\mathcal{U}) = A[x_1, \dots x_n, x_{n+1} \dots x_m]/I$$

for some regular functions $x_{n+1}, \dots x_m$ integral over $A[x_1, \dots x_n]$. By Hensel's lemma, we can write $x_{n+1}, \dots x_m$ as power series in $x_1, \dots x_n$, with coefficients in some further localisation $A[1/M]$:

$$\mathcal{O}(\mathcal{U}) \subset A[1/M][[x_1, \dots, x_n]].$$

The functions f^*x_i, $1 \le i \le n$, are power series in x_i with coefficients in K, since f^* defines an endomorphism of the ring $\mathcal{O}_{q,X}$ and of its completion. We have the following

Lemma 1.1 *Let k be a field of characteristic zero and let $f \in k[[x_1, \dots x_n]]$ be a function algebraic over $k(x_1, \dots, x_d)$. Then $f \in A[[x_1, \dots x_n]]$, where A is a finitely generated $\mathbb{Z}$-algebra.*

(THE PROOF COPIED FROM YOUR FILE. It is OK up to some details which should be verified and possibly corrected)

Proof. Let F be a minimal polynomial of f over $k[x_1, \dots, x_d]$, so $F(f) = 0$ and $F'(f) \neq 0$. Then $F'(f) \in \mathfrak{m}^s \setminus \mathfrak{m}^{s+1}$ for some $s \geq 0$, where $\mathfrak{m}$ is the maximal ideal in $k[[x_1, \dots, x_d]]$.

Denote by f_n the only polynomial of degree $< n$ congruent to f modulo $\mathfrak{m}^n$. For a polynomial Φ in x and an integer m denote by $\Phi_{(m)}$ the homogeneous part of Φ of degree m. Clearly, $F'(f_n)_{(m)}$ is independent of n for $n > m$.

We are going to show that the $\mathbb{Z}$-subalgebra in k generated by the coefficients of f (equivalently, by the coefficients of the homogeneous components of f of all degrees $n > s$) is generated, in fact, by coefficients of F (as polynomial in $d+1$ variables), by coefficients of f_{s+1} and by the inverse of a polynomial in coefficients of F and in coefficients of f_{s+1}.

This is done by induction on degree $n > s$: by definition, $F(f_n) \in \mathfrak{m}^n$ and we have to find (assuming that it exists!) a homogeneous polynomial Δ of degree n such that $F(f_n + \Delta) \in \mathfrak{m}^{n+1}$. One has $F(f_n + \Delta) \equiv F(f_n) + F'(f_n)\Delta \pmod{\Delta^2}$, so the condition is $F(f_n)_{(n+s)} + F'(f_n)_{(s)}\Delta = 0$. This is a linear system with polynomial coefficients in coefficients of F and in coefficients of f_{s+1}. As Δ is a (unique!) solution of this linear system, one can talk about the *determinant* of this linear system, which is non-zero and denoted by D. Then the coefficients of Δ are polynomials over $\mathbb{Z}$ in coefficients of F, coefficients of f_{s+1} and in D^{-1}. So the lemma is proved.

Therefore, for almost all primes $\mathfrak{p} \subset \mathcal{O}_K$, the coefficients of the power series $f^*x_1, \dots f^*x_n$ are integral in $K_\mathfrak{p}$. We choose a $\mathfrak{p}$ not dividing N and M and such that this last condition holds. Obviously,

$$\mathcal{O}(\mathcal{U}) \subset \mathcal{O}_\mathfrak{p}[[x_1, \dots, x_n]]. \quad (*)$$

Define the $\mathfrak{p}$-adic neighbourhood $O_{\mathfrak{p},q}$ of the point q as follows:

$$O_{\mathfrak{p},q} = \{t \in U(K_\mathfrak{p}) | g(t) \equiv g(q) \ mod\ \mathfrak{p}\ for\ g \in \mathcal{O}(U)\}.$$

(THE USE OF U IS NOT A PROBLEM? It is arbitrary, not f-invariant, etc)

By a subneighbourhood $O_{\mathfrak{p},q,n} \subset O_{\mathfrak{p},q}$, we shall mean, throughout the paper, the subset of points of $O_{\mathfrak{p},q}$ such that the values of regular functions at those points are congruent to the values at q modulo $\mathfrak{p}^n$.

The following properties are clear from the definition, using the observation $(*)$:

Proposition 1.2 *(1) The functions $x_1, \dots x_n$ give a bijection between $O_{\mathfrak{p},q}$ and the n-th cartesian power of $\mathfrak{p}$.*

(2) $f(O_{\mathfrak{p},q}) \subset O_{\mathfrak{p},q}$.

(3) $Indet(f)$ does not intersect $O_{\mathfrak{p},q}$.

(4) The $\bar{Q}$-points are dense in $O_{\mathfrak{p},q}$ and in $O_{\mathfrak{p},q,n}$.

(A FEW WORDS OF PROOF??)

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the tangent map Df_q. The following is a consequence of the p-adic versions of several well-known results in dynamics and number theory:

Proposition 1.3 *Assume that $\lambda_1, \dots, \lambda_n$ are multiplicatively independent. Then in some subneighbourhood $O_{\mathfrak{p},q,n}$ of $O_{\mathfrak{p},q}$, the map f is equivalent to its linear part Λ.*

Proof: Note that the eigenvalues λ_i are algebraic numbers. It is well-known that in absence of relations

$$\lambda_1^{m_1}...\lambda_n^{m_n} = \lambda_j,\ 1 \leq j \leq n,\ m = \sum m_i \geq 2,\ m_i \geq 0$$

("resonances"), there is a unique formal linearization of f, obtained by formally solving the equation $f(\phi(x)) = \phi(\Lambda(x))$; the expressions $\lambda_1^{m_1}...\lambda_n^{m_n} - \lambda_j$ appear in the denominators of the coefficients of ϕ (see for example [Arn]). The problem is of course whether ϕ has non-zero radius of convergence. By Siegel's theorem (see [HY]) for its p-adic version) this is the case as soon as the numbers λ_i satisfy the diophantine condition

$$|\lambda_1^{m_1}...\lambda_n^{m_n} - \lambda_j|_p > Cm^{-\alpha}$$

for some C, α. By [Yu] , this condition is always satisfied by algebraic numbers.

Corollary 1.4 *If $\lambda_1, \dots, \lambda_n$ are multiplicatively independent, the rational points on X are potentially dense.*

Proof: since algebraic points are dense in $O_{\mathfrak{p},q,n}$, we can find a point $x \in X(\bar{Q})$ which is contained in $O_{\mathfrak{p},q,n}$, away from the coordinate hyperplanes in the local coordinates linearizing f.

(PLEASE WRITE A KIND OF A PROOF: since everithing is analytic not algebraic we need some words)

2 Variety of lines of the cubic fourfold

Now the difficulty is that it can be hard to find an interesting example such that the eigenvalues of the tangent map at some fixed point are multiplicatively independent. For instance if f is an automorphism and X is a projective $K3$ surface, or, more generally, an irreducible holomorphic symplectic variety, then the product of the eigenvalues is always a root of unity, as noticed in [Bv].

So first of all the map f may be only "partially linearized" in the neighbourhood of q, in some sense, and after such a "partial linearization", the orbit of a general algebraic point may be contained in a relatively small analytic subvariety of the neighbourhood (of course this subvariety does not have to be algebraic, but it is unclear how to prove that it actually is not). Nevertheless, with some additional geometric information, this partial linearization can still be used to prove potential density.

In the rest of this note, we illustrate this by giving a simplified proof of the potential density of the variety of lines of a cubic fourfold, which is the main result of [AV]. The proof uses several ideas from [AV], but we think that certain aspects become more transparent thanks to the introduction of our dynamical point of view and the use of $\mathfrak{p}$-adic neighbourhoods.

We recall the setting of [AV] (the facts listed below are taken from [**?**] and [A]). Let V be a general smooth cubic in $\mathbb{P}^5$ and let $X \subset G(1,5)$ be the variety of lines on V. This is an irreducible holomorphic symplectic fourfold: $H^{2,0}(X)$ is generated by a nowhere vanishing form σ. For $l \subset V$ general, there is a unique plane P tangent to X along l (consider the Gauss map, it sends l to a conic in the dual projective space). The map f maps l to the residual line l'. It multiplies the form σ by -2; in particular, its degree is 16. The indeterminacy locus S consists of points such that the image of the corresponding line by the Gauss map is a line (and the mapping is 2:1). This is a smooth surface of general type, resolved by a single blow-up. For a general X, the Picard group is cyclic and thus the Hodge structure on $H^2(X)^{prim}$ is irreducible (thanks to $h^{2,0}(X)=1$); the space of algebraic cycles is generated by $H^2 = c_1^2(U^*)$ and $\Delta = c_2(U^*)$, where U is the restriction of $U_{G(1,5)}$, the universal rank-two bundle on $G(1,5)$. By Terasoma's theorem [T], these conditions are satisfied by a "sufficiently general" X defined over a number field, in fact even over $\mathbb{Q}$; "sufficiently general" meaning "outside of a thin subset in the parameter space". One computes that the cohomology class of S is $5(H^2-\Delta)$ to conclude that S is irreducible and non-isotropic with respect to σ.

2.1 Fixed points and linearization

The fixed point set F of our rational self-map $f : X \dashrightarrow X$ is the set of points such that along the corresponding line l, there is a tritangent plane to V. Strictly speaking, this is the closure of the fixed point set, since some of such points are in the indeterminacy locus; but for simplicity we shall use the term "fixed point set" as far as there is no danger of confusion.

Proposition 2.1 *The fixed point set F of f is an isotropic surface of general type.*

Proof: It is clear from $f^*\sigma = -2\sigma$ that F is isotropic. Let $I \subset G(1,5) \times G(2,5)$ with projections p_1, p_2 be the incidence variety $\{(l,P) | l \subset P\}$ and let $\mathcal{F} \subset I \times \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ denote the variety of triples $\{(l,P,V) | V \cup P = 3l\}$. This is a projective bundle over I, so $\mathcal{F}$ is smooth and thus its fiber F'_V over a general $V \in \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ is also smooth. This fiber clearly projects generically one-to-one on the corresponding $F = F_V$, since along a general line $l \subset V$ there is only one tangent plane, and a fortiori only one tritangent plane if any; so $F' = F'_V$ is a desingularization of F. Since $dim(I) = 11$ and since intersecting the plane P along the triple line l imposes 9 conditions on a cubic V, we conclude that F' and F are surfaces.

To compute the canonical class, remark that F' is the zero locus of a section of a globally generated vector bundle on I. This vector bundle is the quotient of $p_2^* S^3 U^*_{G(2,5)}$ (where $U_{G(2,5)}$ denotes the tautological subbundle on $G(2,5)$) by a line subbundle $\mathcal{L}_3$ whose fiber at (l, P) is the space of degree 3 homogeneous polynomials on P with zero locus l. One computes that the class of $\mathcal{L}_3$ is three times the difference of the inverse images of the Plücker hyperplane classes on $G(2,5)$ and $G(1,5)$, and it follows that the canonical class of F is $p_2^*(3c_1(U^*))$, which is ample (we omit the details since an analogous computation is given in [V], and a more detailed version of it in [P]).

Remark 2.2 *Since F is isotropic and S is not, S cannot coincide with a component of F. In fact, dimension count shows that $F \cap S$ is a curve.*

Proposition 2.3 *For a general point $q \in F$, the tangent map Df_q is diagonalized with eigenvalues* $1, 1, -2, -2$.

Proof: This follows from the fact that $f^*\sigma = -2\sigma$. and the fact that the map is the identity on the lagrangian plane $T_pF \subset T_pX$. Let e_1, e_2, e_3, e_4 be the Jordan basis with $e_1, e_2 \in T_pF$. There is no Jordan cell corresponding to the eigenvalue 1, since in this case e_4 would be an eigenvector with eigenvalue 4, but then $\sigma(e_1, e_4) = \sigma(e_2, e_4) = \sigma(e_3, e_4) = 0$, contradicting the fact that σ is non-degenerate. By the same reason, the eigenvalues at e_3 and e_4 are both equal to ± 2. Suppose that Df_q is not diagonalized, so sends e_3 to $\pm 2e_3$ and e_4 to $e_3 \pm 2e_4$. In both cases $\sigma(e_3, e_4) = 0$. If e_3 goes to $2e_3$, we immediately see that $e_3 \in Ker(\sigma)$, a contradiction. Finally, if $Df_q(e_3) = -2e_3$ and $Df_q(e_4) = e_3 - 2e_4$, we have

$$-2\sigma(e_1, e_4) = \sigma(e_1, e_3) - 2\sigma(e_1, e_4),$$

so that $\sigma(e_1, e_3) = 0$, but by the same reason $\sigma(e_2, e_3) = 0$, again a contradiction to non-degeneracy of σ.

Proposition 2.4 *(1) Let q be a general fixed point of f and let $O_{\mathfrak{p},q}$ be its $\mathfrak{p}$-adic neighbourhood for a suitable $\mathfrak{p}$, as in the previous section. We identify $O_{\mathfrak{p},q}$ with $\mathcal{O}_{\mathfrak{p}}^4$ (so that q becomes a 0). There exists a formal power series $h = h_q$ in two variables $(t_1, t_2) = t$ such that $h(-2t) = f \circ h(t)$. This series is determined uniquely by its linear part and converges on a certain neighbourhood of zero $\mathfrak{p}^n \times \mathfrak{p}^n$ for some n (which can be chosen depending only on the prime $\mathfrak{p}$). Locally, h_q can be chosen to depend analytically on q.*

(2) In the complex setting, the analogous statements are true. Moreover, the maps h_q extend to global meromorphic maps from $\mathbb{C}^2$ to X.

Proof: 1) We take the coordinates in O in which

$$Df_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

and fix the linear part $h^{(1)}(t_1, t_2) = (0, 0, t_1, t_2)$. The existence and uniqueness of the formal solution is classical and goes in the same way as for the Poincare-Dulac normal form (see for example [Arn]): finding $h^{(2)}, h^{(3)}, \dots$ is linear algebra involving division by $(-2)^2 - 1,\ (-2)^2 + 2, \dots, (-2)^m - 1,\ (-2)^m + 2,\ m \in \mathbb{N}$. Since none of those is equal to zero ("no resonances"), this works. To begin with, taking the second order terms in $h(-2t) = f \circ h(t)$ gives

$$h^{(2)}(-2t) = Df_q \cdot h(t) + f^{(2)}(0, 0, t_1, t_2)$$

and thus the coefficients of $h^{(2)}$ are obtained from those of $f^{(2)}$ by dividing by $(-2)^2 - 1$ (for the first two components h_1, h_2) or by $(-2)^2 + 2$ (for the last two). In general, comparing the terms of order m, we get that $((-2)^m - 1)h_i^{(m)}(t),\ i = 1, 2$ (resp. $((-2)^m + 2)h_i^{(m)}(t),\ i = 3, 4$) are sums of terms of type

$$f^{(l)}(h^{(i_1)}, h^{(i_2)} \dots, h^{(i_l)}),$$

where $i_1 + i_2 + \cdots + i_l = m$. Recall that we may assume that the coefficients of f are integers by 1.1. Thus, the denominators of the coefficients of $h_i^{(m)}$ are products of numbers of the form $(-2)^j - 1$ or $(-2)^j + 2$, where $2 \leq j \leq m$.

The following claim is elementary (induction by m):

Claim: Each denominator is a product of at most $m-1$ factors, and the exponents $j_1 \geq j_2 \geq \cdots \geq j_{m-r}$ satisfy $j_k \leq m - k + 1$.

To see that the radius of convergence of our formal power series is positive, we must estimate the p-adic order of the denominators and conclude that it is at most linear in m. Suppose for simplicity that all our factors are of the form $(-2)^j - 1$. Since $\mathbb{Z}_p^* \cong \mathbb{F}_p^* \times \mathbb{Z}_p$, we have

$$ord_p((-2)^j - 1) = ord_p(ja), j \equiv 0(s)$$

and equal to zero otherwise; here -2 corresponds to (x, a) under the above isomorphism and s is the order of x in $\mathbb{F}_p^*$. Thus a very rough estimate gives that the order of our denominator is at most

$$\sum_{j=2}^{m} ord_p(ja) \leq m \cdot ord_p(a) + ord_p(m!) \leq m \log a + \frac{m}{p-1}.$$

So the series converges on $\mathfrak{p}^n \times \mathfrak{p}^n$ as soon as $n \geq \log a + \frac{1}{p-1} + 1$.

In the general case with some factors of the form $(-2)^j + 2$, it suffices to double our estimate for n.

2) In the complex case, the convergence of the power series on some neighbourhood U of zero in $\mathbb{C}^2$ follows from $|-2| > 1$ as in the classical Poincare theorem. To extend the map h to $\mathbb{C}^2$, set

$$h(x) = f^k(h((-2)^r x)),$$

where $(-2)^r x \in U$; one checks that this is independent of choices.

We immediately get the following corollary (which follows from the results of [AV], but for which there was as yet no elementary proof):

Corollary 2.5 *There exist points in $X(\bar{\mathbb{Q}})$ which are not preperiodic for f.*

Proof: Indeed, $\bar{Q}$-points are dense in O. Take one in a suitable invariant subneighbourhood and use the linearization given by the proposition above.

Remark 2.6 *If f were regular, this would follow from the theory of canonical heights; but this theory does not seem to work sufficiently well for polarized rational self-maps.*

2.2 Non-preperiodicity of certain surfaces

The starting point of [AV] was the observation that X is covered by a two-parameter family $\Sigma_b, b \in B$ of birationally abelian surfaces, namely, surfaces parametrizing lines contained in a hyperplane section of V with 3 double points. On a general X, a general such surface has cyclic Neron-Severi group ([AV], ??); moreover, many of those surfaces Σ defined over a number field have the same property, as shown by an argument similar to that of Terasoma [T]. In fact, given a general X, the set of such surfaces on X whose Neron-Severi group is not cyclic, is *thin*.

In [AV], it is shown that the iterations of a suitable Σ defined over a number field and with cyclic Neron-Severi group is Zariski-dense. The first step is to prove its non-preperiodicity, that is, the fact that the number of $f^k(\Sigma)$, $k \in \mathbb{N}$, is infinite. Already at this stage the proof is highly non-trivial, using the l-adic Abel-Jacobi invariant in the continuous étale cohomology.

In this subsection, we give an elementary proof of the non-preperiodicity of a suitable Σ, which is based on 2.4. Moreover, this works without an assumption on its Néron-Severi group.

Lemma 2.7 *The surface Σ is not invariant by f.*

Proof: The surface Σ is the variety of lines contained in the intersection $Y = V \cap H$, where H is a hyperplane in $\mathbb{P}^5$ tangent to V at exactly three points. For a general line l corresponding to a point of Σ, there is a unique plane P tangent to V along l, and the map f sends l to the residual line l'. If Σ is invariant, l' and therefore P lie in H, and P is tangent to Y along l. But this means that l is "of the second type" on Y in the sense of Clemens-Griffiths (i.e. the Gauss map of $Y \subset H = \mathbb{P}^4$ sends l $2:1$ to a line in $(\mathbb{P}^4)^*$), see [CG]. At the same time it follows from the results of [CG] that a general line on a cubic threefold with double points is "of the first type" (mapped bijectively onto a conic by the Gauss map), a contradiction.

Now let us work in the p-adic setting.

Let $x \in \Sigma(K)$ be a point of the p-adic neighbourhood.

Proposition 2.8 *The Zariski closure D of the set of iterates of Σ contains the image of the line Ox.*

Since in the complex situation, everything is given by the same power series, this is also true over $\mathbb{C}$. Note that by 2.7, Σ cannot coincide with a leaf of our local fibration from 2.4. Suppose that Σ is preperiodic, that is, D is a finite union of surfaces; then from the form of f in 2.4 it is clear that each of them, in particular Σ itself, contains the germ of the line Ox. Now there are two possible cases: either the Zariski closure of this germ is the whole of Σ and then Σ is invariant, an immediate contradiction with 2.7; or the Zariski closure is a proper subvariety. In this case we remark that extending the field K if necessarily, we can construct as many of such subvarieties as we wish (since there is one through any K-point of Σ in our p-adic neighbourhood). Since these are in the intersection $\Sigma \cap f(\Sigma)$, this again means that Σ must be invariant.

To sum up, we have the following

Theorem 2.9 *The Zariski closure D is of dimension at least three. If it is of dimension three, this is an irreducible divisor which either contains the surface of fixed points F, or has a curve in common with F. In this last case, D contains correspondent "leaves" (images of $\mathbb{C}^2$ from 2.4) through the points of this curve.*

2.3 Potential density

In this subsection, we exclude the case when D is a divisor.

Our proof is a case-by-case analysis on the Kodaira dimension of D. In [AV] , we already have simple geometric arguments ruling out the cases of $\kappa(D) = -\infty$ and $\kappa(D) = 0$. The case $\kappa(D) = -\infty$ is especially simple since then the holomorphic 2-form would be coming from the rational quotient of D, but Σ obviously must dominate the rational quotient and this cannot be isotropic. The case $\kappa(D) = 0$ is less easy and uses the fact that $Pic(X) = \mathbb{Z}$ or, equivalently, that the Hodge structure $H^2_{prim}(X)$ is irreducible of rank 22. Namely, an argument using Minimal Model theory and the existence of an holomorphic 2-form on D gives that D must be rationally dominated by an abelian threefold or by a product of a K3 surface with an elliptic curve. But the second transcendental Betti number of those varieties cannot exceed 21, which contradicts the fact that D carries an irreducible Hodge substructure of rank 22; see [AV] for details.

Let us deal with the case $\kappa(D) = 2$. We need the following lemma:

Lemma 2.10 *On a general X, the points of order 3 with respect to f form a curve.*

Proof: Let l_1 be (a line corresponding to) such a point, $l_2 = f(l_1)$, $l_3 = f^2(l_1)$, so that $f(l_3) = l_1$. There are thus planes P_1, P_2, P_3, such that P_1 is tangent to V along l_2 and contains l_3, etc. Clearly, $P_1 \neq P_2 \neq P_3$. The span of the planes P_j is a projective 3-space Q. Let us denote the two-dimensional cubic, intersection of V

and Q, by W. We can choose the coordinates $(x : y : z : t)$ on Q such that l_1 is given by $y = z = 0$, etc. Then the intersection of W and P_1 is given by the equation $z^2y = 0$, etc. The only other monomial from the equation of W, up to a constant, can be xyz, since it has to be divisible by the three coordinates. Therefore W is a cone over the plane cubic given by the equation $ax^2y + by^2z + cy^2z + dxyz = 0$ in the plane at infinity. Now a standard dimension count shows that a general cubic admits a one-parameter family of two-dimensional linear sections which are cones. Each cone on V gives rise to a plane cubic on X. This cubic is invariant under f, and f acts by multiplication by -2 (for a suitable choice of zero). The points of order 3 with respect to f lie on such cubic and are their points of 9-torsion.

Remark 2.11 *In fact the lemma says slightly more: it applies to the indeterminacy points which are "of order three in the generalized sense", that is, points appearing if one replaces the condition "$f(l_1) = l_2$, $f(l_2) = l_3$, $f(l_3) = l_1$" by "$l_2 \in f(l_1)$, etc."; here by $f(l_1)$ we mean the rational curve which is the image of l_1 by the correspondence which is the graph of f (equivalently, $l_2 \in f(l_1)$ says that for some plane P_3 tangent to V along l_1, the residual line in $P_3 \cap V$ is l_2).*

By blowing-up $\tilde{D}$, we may assume that the Iitaka fibration $\tilde{D} \to B$ is regular. Its general fiber is an elliptic curve. By [NZ], the rational self-map f descends to B and induces a transformation of finite order, so the elliptic curves are invariant by a power of f. From proposition 2.4, we obtain that they are in fact invariant by f itself: indeed, locally in a neighbourhood of a fixed point, the curves invariant by f are the same as the curves invariant by its power. On a general elliptic curve, there is a finite (non-zero) number of points of order three, since f acts as multiplication by -2. We have two possibilities:

1) These are mapped to points of order three (in the "generalized sense" as in the 2.11) on X (or the surface they form is contracted to any other curve on X). Then any preimage of our surface by an iteration of f is contracted as well, but since there are infinitely many of them, this is impossible.

2) This surface dominates a component of the surface of fixed points of f. In this case, several points of order three must collapse to the same fixed point p. But then the resulting branches of each elliptic curve near the generic fixed point are interchanged by f, which contradicts the local description of f in 2.4.

This rules out the possibility $\kappa(D) = 2$.

Finally, let us consider the case $\kappa(D) = 1$. The Iitaka fibration $D \to C$ maps D to a curve C and the general fiber U is of Kodaira dimension 0. As before, by [NZ] f induces a finite order automorphism on C, and one deduces from 2.4 that this is in fact the identity. We have two possible cases:

Case 1: U is not isotropic with respect to the holomorphic 2-form σ. We use the idea from [AV] as in the case $\kappa(D) = 0$. Namely, since X is generic, the Hodge structure $H^2_{prim}(X, \mathbb{Q})$ is simple.Since the restriction of σ to D is non-zero, $H^2(\tilde{D}, \mathbb{Q})$

carries a simple Hodge substructure of rang 22. Since U is non-isotropic, the same is true for U, but a surface of Kodaira dimension zero never satisfies this property.

Case 2: U is isotropic with respect to σ. The kernel of the restriction of σ to D gives a locally free subsheaf of rank one in the tangent bundle T_D, which is in fact a subsheaf of T_U since U is isotropic. There is thus a foliation in curves on U, and this foliation has infinitely many algebraic leaves (these are intersections of U with the iterates of our original surface Σ). By Jouanolou's theorem, this is a fibration. In other words, D is fibered over a surface T in integral curves of the kernel of σ_D, and U project to curves. These cannot be rational curves since the surface T is not uniruled (indeed, the form σ_D must be a lift of a holomorphic 2-form on T). Therefore these are elliptic curves, and since $\kappa_U = 0$, so are the fibers of $\pi : D \to S$.

Recall from 2.4 that either D contains F, or it contains a curve on F; and in this last case, locally near generic such point, D is a fibration in (isotropic) two-dimensional disks over a curve; in particular, such a point is a smooth point of D. If D contains F, this is a contradiction with 2.1: indeed, F must be dominated by a union of fibers of π. If D contains a curve on F, then we look at the generic "leave" (image of $\mathbb{C}^2$) at this point p. Its intersection with the image of U is an invariant curve, that is, the image of a line through the origin. And this must be an integral curve of the kernel of σ_D and U varies in a family, this implies that σ_D is zero at p, a contradiction since σ is non-degenerate.

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