Existence of non-preperiodic algebraic points for a rational self-map of infinite order

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Let X be a smooth projective variety defined over a number field K and let $f : X \dashrightarrow X$ be a rational self-map defined over the same number field. As shown in [AC], one can attach to f a dominant rational map $g : X \dashrightarrow T$, commuting with f and such that the fiber of g through a sufficiently general complex point $x \in X(\mathbb{C})$ is the Zariski closure of its iterated orbit (or "f-orbit") $\{f^k(x),\ k \in \mathbb{N}\}$. Here "sufficiently general" means "outside a countable union of proper subvarieties", and so this theorem does not give any information on the f-orbits of algebraic points, which, apriori, can have smaller Zariski closure than general complex points.

One would of course like to show that in reality this never happens and one can always find an algebraic point whose f-orbit is "as large" as the general one. For instance, a conjecture already implicit in [AC] and formulated by Medvedev and Scanlon in [MS] (Conjecture 5.3) states that if no power of f preserves a non-trivial fibration, then there is a point $x \in X(\bar{\mathbb{Q}})$ with Zariski-dense f-orbit ; a variant of this is an earlier conjecture by S.-W. Zhang stating the same in the case when f is regular and polarized (that is, there is an ample line bundle L on X with $f^*L = qL$ for some $q > 1$).

What is certainly true in the case when f is regular and polarized is that, at least, there exist points in $X(\bar{\mathbb{Q}})$ with infinite f-orbits (that is, non-preperiodic algebraic points). The reason is that in this case, one can introduce the so-called *canonical height* $\hat{h}_L : X(\bar{\mathbb{Q}}) \to \mathbb{R}$ which is a Weil height function for L with the property $\hat{h}_L(f(x)) = q\hat{h}_L(x)$; it follows that the set of preperiodic points is a set of bounded height and therefore it cannot exhaust $X(\bar{\mathbb{Q}})$ (see [CS]). However, the theory of canonical heights does not seem to work well enough for rational self-maps.

The purpose of this note is to provide an elementary proof of the existence of non-preperiodic algebraic points for such rational self-maps (using, though, a result by E. Hrushovski which does not seem to have been treated in a very accessible way for the moment). The argument is very similar to the one used by Bell, Ghioca and Tucker to prove a version of the "dynamical Mordell-Lang conjecture" for unramified endomorphisms of quasiprojective varieties; in fact this note is directly inspired by [BGT] (and is a continuation of [ABR]). The point is that, thanks to Hrushovski's result, for some positive integer k one can find an f^k-invariant $\mathfrak{p}$-*adic neighbourhood* in X, for a suitable prime $\mathfrak{p}$ in the ring of integers of a suitable finite extension of K. Then one uses [BGT] to conclude that all preperiodic points in this neighbourhood are periodic with bounded period $\leq N$ and thus are contained in a certain proper analytic subvariety.

The result of Hrushovsky we are using is as follows:

Theorem 1 *([H], Corollary 1.2) Let U be an affine variety over a finite field $\mathbb{F}_q$ and let $S \subset U^2$ be an irreducible subvariety over $\bar{\mathbb{F}}_q$. Assume that the two projections of S to U are dominant. Denote by ϕ_q the Frobenius map. Then for any proper subvariety W of U, for large enough m, there exists $x \in U(\bar{\mathbb{F}}_q)$ with $(x, \phi_q^m(x)) \in S$ and $x \notin W$.*

In particular, let $\bar{X}$ be any irreducible variety defined over a finite field $\mathbb{F}_q$, and let $f : \bar{X} \dashrightarrow \bar{X}$ be a separable rational self-map defined over the same field. Let $Y = I \cup R$, where I is the indeterminacy locus and R is the ramification locus of f. Those are subvarieties defined over a finite extension of $\mathbb{F}_q$, and therefore they are periodic under ϕ_q: for some $k \in \mathbb{N}$ and any $l \in \mathbb{N}$, $\phi_q^l(I) = \phi_q^{l+k}(I)$ and $\phi_q^l(R) = \phi_q^{l+k}(R)$. Set $V = \bar{X} - Y \cup \phi_q(Y) \cup \cdots \cup \phi_q^k(Y)$. Let U be an affine open subset of V defined over $\mathbb{F}_q$ and let S be the intersection of $U \times U$ with the graph of f. Then by Hrushovski's theorem we have the following

Corollary 2 *In the setting as above, there is a point $x \in \bar{X}(\bar{\mathbb{F}}_q)$ such that no iterate $f^k(x)$ is an indeterminacy or ramification point of f, and x is f-periodic. Moreover such points are Zariski-dense in $\bar{X}$.*

Let now X be a variety defined over a number field K', and let $f : X \dashrightarrow X$ be a rational self-map defined over K'. We are going to use corollary 2 to find, for a suitable finite extension K of K' and for a suitable prime $\mathfrak{p} \subset \mathcal{O}_K$, a "$\mathfrak{p}$-adic neighbourhood" in X, invariant under some power of f, with good properties as in [ABR]. The procedure is almost the same as in [ABR]. Take an affine $U \subset X$ such that f is regular on U, together with a surjective K'-morphism $\pi = (x_1, \dots, x_n) : U \to \mathbb{A}^n$ (Noether normalization). Write $\mathcal{O}_U = K'[x_1, \dots, x_n, x_{n+1}, \dots, x_m]/I$, where I contains for instance the minimal polynomials P_i of x_{n+i}, $i > 0$ over $K[x_1, \dots, x_n]$ (but probably also something else), so that

$$U = Spec(K'[x_1, \dots, x_n, x_{n+1}, \dots, x_m]/I);$$

we may suppose that I is given by a system of generators with coefficients from $\mathcal{O}_{K'}$ and take a model over $\mathcal{O}_{K'}$: $\mathcal{U} = Spec(\mathcal{O}_{K'}[x_1, \dots, x_n, x_{n+1}, \dots, x_m]/I)$. By abuse of notation, we denote the rational map on the model by the same f. Consider $x_{n+1}, \dots x_m, f^*x_1, \dots f^*x_m$ as power series in $x_1, \dots, x_n$. By Lemma 2.1 of [ABR], their coefficients are $\mathfrak{p}'$-integral for almost all primes $\mathfrak{p}' \subset \mathcal{O}_{K'}$. Choose $\mathfrak{p}'$ with this property, and, moreover, such that the minimal monic polynomials P_i have $\mathfrak{p}'$-integral coefficients and the derivatives P_i' are not identically zero modulo $\mathfrak{p}'$, and such that f reduced modulo $\mathfrak{p}'$ is well-defined and separated. Consider the reduction $\bar{U}$ modulo $\mathfrak{p}'$: $\bar{U} = Spec((\mathcal{O}_{K'}/\mathfrak{p}')[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m]/\bar{I})$. It is equipped with a rational self-map $\bar{f}$, the reduction of f. By corollary 2, we can find an $\bar{f}$-periodic point $x \in \bar{U}$ over some finite extension $\mathbb{F}_q(\alpha)$ (where $\mathbb{F}_q = \mathcal{O}_{K'}/\mathfrak{p}'$), such that no iterate of x by $\bar{f}$ is in the indeterminacy or ramification, and such that the values of the derivatives P'_i calculated at x are non-zero modulo $\mathfrak{p}'$. If $\bar{U}$ is singular, let us moreover choose x in its smooth locus (however, when our original U is smooth, we can already choose the prime $\mathfrak{p}'$ in such a way that $\bar{U}$ is smooth). Let k be the period: $\bar{f}^k(x) = x$.

Let β be an algebraic number integral over $\mathcal{O}_{K'}$ such that the reduction of its monic minimal polynomial over $\mathcal{O}_{K'}$ modulo $\mathfrak{p}'$ gives the minimal polynomial of α, and let $K = K'(\beta)$. Let $\mathfrak{p}$ be some prime of K lying over $\mathfrak{p}'$. The point x lifts to a point $y \in U(K_{\mathfrak{p}})$ (by Hensel's lemma) (alternatively, we can take a slightly larger finite extension of $(K', \mathfrak{p}')$ as $(K, \mathfrak{p})$ to produce a point $y \in U(K)$ which reduces to x). Define the $\mathfrak{p}$-adic neighbourhood $O_{\mathfrak{p},y}$ of y as follows:

$$O_{\mathfrak{p},y} = \{t \in U(K_{\mathfrak{p}}) | x_i(t) \equiv x_i(y) \pmod{\mathfrak{p}} \; for \; 1 \leq i \leq m\}.$$

We may suppose that $\pi(y) = (0, \ldots 0) \in \mathbb{A}^n$. Then, exactly as in [ABR], we get the following

Proposition 3 *(1) The functions $x_1, \ldots x_n$ give a bijection between $O_{\mathfrak{p},y}$ and the n-th cartesian power of $\mathfrak{p}$.*

(2) The set $O_{\mathfrak{p},y}$ contains no indeterminacy and no ramification points of f.

(3) $f^k(O_{\mathfrak{p},y}) \subset O_{\mathfrak{p},y}$, moreover, f^k is bijective on $O_{\mathfrak{p},y}$.

(4) The $\bar{\mathbb{Q}}$-points of X are dense in $O_{\mathfrak{p},y}$.

It will be more convenient for us to identify $O_{\mathfrak{p},y}$ with a cartesian power of $\mathcal{O}_{\mathfrak{p}}$ rather than that of $\mathfrak{p}$. So, if our map f^k is given (say on the completion of the local ring at y) by the power series $H_i(x_1, \ldots x_n) = f^* x_i \in \mathcal{O}_{\mathfrak{p}}[[x_1, \ldots x_n]]$, set $F(t_1, \ldots, t_n) = \frac{1}{r} H(rx_1, \ldots rx_n)$ where r is some fixed uniformizing element in $\mathcal{O}_{\mathfrak{p}}$. In this way, we may view $O_{\mathfrak{p},y}$ as $\mathcal{O}_{\mathfrak{p}}^n$ with coordinates t_i, and the map f^k is given by the power series F_i on $O_{\mathfrak{p},y}$. Note that, as in [BGT], the F_i have integral coefficients (by construction the constant terms of H_i are divisible by r), and moreover the coefficient of $t_1^{k_1} \ldots t_n^{k_n}$ in F_i is divisible by $r^{k_1 + \cdots + k_n - 1}$ when $k_1 + \cdots + k_n \geq 1$ (remark 2.3 of [BGT]).

Write $F = (F_1, \ldots F_n) : \mathcal{O}_{\mathfrak{p}}^n \to \mathcal{O}_{\mathfrak{p}}^n$ (so F is a way to think of the restriction of f^k to the $\mathfrak{p}$-adic neighbourhood) and consider F modulo $\mathfrak{p}$. Again as in [BGT], we have the following

Proposition 4 *There is a positive integer l such that for every $z \in \mathcal{O}_{\mathfrak{p}}^n$, $F^l(z) = z$ (mod $\mathfrak{p}$).*

Proof: Modulo $\mathfrak{p}$, F is an affine transformation of the linear space $(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})^n$. Its linear part L is invertible by Proposition 2.4 of [BGT] since no f-iterate of the smooth point x into which our $\mathfrak{p}$-adic neighbourhood reduces modulo $\mathfrak{p}$ is ramification or indeterminacy, and so $\bar{f}^k$ is unramified at x. Therefore F modulo $\mathfrak{p}$ is an automorphism of a finite-dimensional affine space over a finite field, and some power of it is the identity.

In Section 3 of [BGT], the similar situation is considered; the only difference is that the coefficients of all power series are in $\mathbb{Z}_p$ rather than in an extension $\mathcal{O}_{\mathfrak{p}}$. The authors prove:

Theorem 5 *([BGT], Theorem 3.3) Let $\phi_1, \dots, \phi_n \in \mathbb{Z}_p[[x_1, \dots, x_n]]$ be convergent power series such that $\phi_i(x) = x \pmod p$ and the coefficient of $x_1^{k_1} \dots x_n^{k_n}$ in the series ϕ_i is divisible by $p^{k_1+\dots+k_n-1}$ for $k_1 + \dots + k_n > 1$. Let $(\omega_1, \dots \omega_n) \in \mathbb{Z}_p^n$. If $p > 3$, there exist p-adic analytic functions $g_1, \dots g_n \in \mathbb{Q}_p[[z]]$, convergent on $\mathbb{Z}_p$, such that $g_i(\mathbb{Z}_p) \subset \mathbb{Z}_p$, $g_i(0) = \omega_i$ and $g_i(z+1) = \phi_i(g_1(z), \dots, g_n(z))$.*

They construct $g_i(z)$ "by approximation", as a Mahler series

$$g_i(z) = \omega_i + \sum_{k=1}^{\infty} b_{ik} \binom{z}{k},$$

with b_{ik} of the form $\sum_{j=(k+1)/2}^{\infty} p^j c_{ijk}$, $c_{ijk} \in \mathbb{Z}_p$. Since $|b_{ik}|_p \to 0$ when $k \to \infty$, these Mahler series define continuous functions on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p$. To show that these functions are in fact analytic on $\mathbb{Z}_p$, one needs to check ([R], Theorem 4.7 of ChapterVI) that $|b_{ik}|_p/|k!|_p \to 0$ when $k \to \infty$, and this is true for $p > 3$ since $|b_{ik}|_p \le p^{-(k+1)/2}$ and $1/|k!|_p < p^{k/(p-1)}$.

In the situation when $\mathbb{Z}_p$ (not as the domain of definition of g_i but as the domain where the g_i take their values) is replaced by an extension $\mathcal{O}_{\mathfrak{p}}$, their argument goes through almost verbatim, replacing p with a uniformizing element r where appropriate: indeed the theory of Mahler series applies to $\mathcal{O}_{\mathfrak{p}}$-valued functions on $\mathbb{Z}_p$ as well ([R], chapters IV.2.3,VI.4.7). The only exception is the last step concerning the analyticity of the $\mathcal{O}_{\mathfrak{p}}$- valued map g_i on $\mathbb{Z}_p$: instead of being analytic on the whole of $\mathbb{Z}_p$, it is going to be analytic on a certain neighbourhood $p^l\mathbb{Z}_p$. This is because in the expression $b_{ik} = \sum_{j=(k+1)/2}^{\infty} p^j c_{ijk}$, we have to replace p by the uniformizing element r, which can be of smaller p-adic order $1/e$; so that $|b_{ik}|_p \le p^{-(k+1)/2e}$ and we need the condition $p > 2(e+1)$ in order to guarantee $|b_{ik}|_p/|k!|_p \to 0$. But one always has convergence on some $p^l\mathbb{Z}_p$ (the condition for this convergence being $|p^{k\frac{p^l-1}{(p-1)p^l}} b_{ik}/k!|_p \to 0$).

So the theorem of [BGT] becomes

Theorem 6 *Let $\phi_1, \dots, \phi_n \in \mathcal{O}_{\mathfrak{p}}[[x_1, \dots, x_n]]$ be convergent power series such that $\phi_i(x) = x \pmod{\mathfrak{p}}$ and the coefficient of $x_1^{k_1} \dots x_n^{k_n}$ in the series ϕ_i is divisible by $r^{k_1+\dots+k_n-1}$ for $k_1 + \dots + k_n > 1$ for a uniformizing element r. Let $(\omega_1, \dots \omega_n) \in \mathcal{O}_{\mathfrak{p}}^n$. Then there exist functions $g_1, \dots g_n$, continious on $\mathbb{Z}_p$ and analytic on $p^l\mathbb{Z}_p$ for a certain positive integer l, such that $g_i(\mathbb{Z}_p) \subset \mathcal{O}_{\mathfrak{p}}$, $g_i(0) = \omega_i$ and $g_i(z+1) = \phi_i(g_1(z), \dots, g_n(z))$.*

Here is an immediate corollary of Theorem 6:

Corollary 7 *Let X, $f : X \dashrightarrow X$ be a variety and a rational self-map defined over a number field. Let $O_{\mathfrak{p},y}$ be an f^k-invariant $\mathfrak{p}$-adic neighbourhood constructed in proposition 3 (formed by $K_{\mathfrak{p}}$-points for a suitable extension K of our number field and a suitable prime $\mathfrak{p}$). Then there exists a positive integer N such that any preperiodic point in this neighbourhood is periodic of bounded period $\leq N$.*

Proof Indeed, the neighbourhood is $F = f^k$-invariant, and some further power F^l of f satisfies the conditions for ϕ in theorem 6. Let ω be a preperiodic point. From theorem 6, its orbit under a still larger power $\psi = \phi^s = f^N$, depending only on the neighbourhood itself (in particular, on the ramification index of $K_{\mathfrak{p}}$ over $\mathbb{Q}_p$), is encoded by an analytic map $g : p^l\mathbb{Z}_p \to O_{\mathfrak{p},y}$, satisfying $g(p^l i) = \psi^i(\omega)$ for $i \in \mathbb{N}$. But an analytic function taking only finitely many values on $p^l\mathbb{Z}$ must be constant, so in fact $\psi(\omega) = \omega$ and so all points with a finite orbit in the neighbourhood are N-periodic.

From this, we immediately draw the conclusion announced in the beginning:

Corollary 8 *Unless if f is of finite order, there exist points in $X(\bar{\mathbb{Q}})$ which are not preperiodic with respect to f.*

Proof: Indeed, points of bounded period must be contained in a proper analytic subvariety of $O_{\mathfrak{p},y}$, whereas algebraic points are dense in $O_{\mathfrak{p},y}$.

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