RATIONAL CURVES AND UNIRULED VARIETIES

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Unless otherwise stated, we work over an algebraically closed field of caracteristic zero (mostly over $\mathbb{C}$).

A *rational curve* is a curve C such that its normalization is $\mathbb{P}^1$, that is, a curve of geometric genus zero: $g(C) = 0$.

Rational curves naturally appear in the structure theory of algebraic varieties. For instance, blowing up a smooth point on a surface introduces a $\mathbb{P}^1$ which replaces this point; more generally, blowing up a subvariety in a smooth variety introduces an exceptional divisor covered by rational curves. So, for instance, if $f : X \dashrightarrow Y$ is a rational map which is not regular everywhere (that is, has points of indeterminacy), then Y must contain rational curves.

Since Mori's fundamental work, we have a good understanding of the way in which the rational curves on X are related to the positivity properties of the canonical line bundle $K_X = \Lambda^n T_X^*$ (here $n = dim(X)$):

Theorem 1 ([Mo]**):** *Let X be a smooth projective variety. Assume that there is a curve $D \subset X$ such that the intersection number $K_X D < 0$. Then through any point $d \in D$, there is a rational curve C_d. Moreover, one can choose C_d such that $dim(X) + 1 \geq -K_X C_d$.*

Remarks: 1) If H is an ample divisor on X and $a > 0$ is a number such that $-K_X D \geq aHD$, one can arrange $-K_X C_d \geq aHC_d$; so the rational curves of the theorem have bounded degree.

2) In particular, if $-K_X$ is ample, so its intersection number with any curve is positive, we obtain that through *every* point of X there is a rational curve C satisfying $dim(X) + 1 \geq -K_X C$. Manifolds with ample anticanonical class are called *Fano varieties*. They have many special properties, for instance, Fano varieties are simply connected. Fano varieties of dimension one are, of course, just rational curves. A Fano variety of dimension 2 is called a *del Pezzo surface* and is either $\mathbb{P}^2$, or $\mathbb{P}^1 \times \mathbb{P}^1$, or a blow-up of $\mathbb{P}^2$ at d points (with $1 \leq d \leq 8$) in general position. We shall say more about Fano threefolds in the article on Fano varieties.

To prove this theorem, one applies the *Mori's bend-and-break method* to a reduction of X modulo a prime p. This gives rational curves on the reduction; then one shows that they can be lifted to X. See the article on the bend-and-break method.

A proper variety X is called *uniruled* if there is a dominant rational map $\phi : Y \times \mathbb{P}^1 \dashrightarrow X$, where Y is a variety of dimension $dim(X) - 1$. X uniruled obviously implies that X is covered by rational curves; over an uncountable field this is, in fact, equivalent (whereas over a countable field, the curves covering X could all come from different families, because there is a countable number of such families; so, at least formally, the uniruledness is a stronger notion. I do not know of any examples actually confirming this).

For smooth X, there is another characterization of uniruledness in terms of rational curves ([Ko]). Let $f : \mathbb{P}^1 \to X$ be a morphism; let us say that f is *free* (or, by abuse of terminology, a *free rational curve*) if f^*T_X is generated by the global sections. From deformation theory, the deformations of a free rational curve cover X. The following proposition is rather straightforward:

Proposition 1: *X is uniruled if and only if X admits a free rational curve.*

Remark: This remains true in positive characteristics if one replaces "uniruled" by "*separably* uniruled", that is, requires that the morphism ϕ from the definition of uniruledness is separable (we need a $\mathbb{P}^1$ such that ϕ is unramified at its general point).

If X is smooth and uniruled, then for any m, $H^0(X, K_X^{\otimes m}) = 0$: on X, there cannot exist any pluricanonical form. Indeed, such a form would pull back to $Y \times \mathbb{P}^1$, because the indeterminacy locus I of ϕ is of codimension at least 2, and so the pull-back to $(Y \times \mathbb{P}^1) - I$ extends to the whole of $Y \times \mathbb{P}^1$ by Hartogs' theorem. But, obviously, the canonical line bundle of $Y \times \mathbb{P}^1$ has no sections.

Alternatively, one can observe that any pluricanonical form must be zero in restriction to a free rational curve: in fact, let $f : \mathbb{P}^1 \to X$ be free. As f^*T_X is globally generated and there is an injection from $\mathcal{O}_{\mathbb{P}^1}(2) = T_{\mathbb{P}^1}$ to f^*T_X, one deduces that $deg(f^*T_X) \geq 2$; therefore $K_X \cdot f(\mathbb{P}^1) < 0$, which implies the assertion. But free rational curves cover an open subset of X, so all pluricanonical forms vanish.

Remark: The previous remark about the positive characteristics applies here as well.

Conjecture 1: *The converse is also true: if all pluricanonical forms vanish on X, then X is uniruled.*

Remark: In birational geometry, one introduces the Kodaira dimension $\kappa(X)$ of X, as, roughly, the rate of growth of the dimension of $H^0(X, K_X^{\otimes m})$ together with m: $H^0(X, K_X^{\otimes m}) \sim m^{\kappa(X)}$, and if there are no sections of $K_X^{\otimes m}$ at all, one puts $\kappa(X) = -\infty$. The conjecture thus says that X is uniruled if and only if $\kappa(X) = -\infty$.

Conjecture 1 is classically known in dimension 2. Indeed, the Castelnuovo's criterion asserts that a surface S is rational if and only if $q(S) = p_{12}(S) = 0$: here $q(S)$ is the irregularity $h^0(S, \Omega_S^1)$ and $p_m(S) = h^0(S, K_S^{\otimes m})$. Using Albanese map, it is not very difficult to show that $p_{12}(S) = 0$ and $q(S) > 0$ means that S is birational to a ruled surface.

In dimension three, this is a consequence of a deep result by Y.Miyaoka ([Mi]) and the minimal model theory. The minimal model program provides the following alternative: either X is covered by rational curves, or X is birational to a variety X' with mild ("terminal") singularities and such that $K_{X'}$ is numerically effective. This is completely worked out in dimension 3 (due to the efforts of many people; see for example [CKM] for a reasonably short and clear exposition). The result of Miyaoka, also valid in dimension 3, is that $K_{X'}$ numerically effective implies

$K_{X'}^{\otimes m}$ effective for some positive integer m (this is in fact a partial case of the so-called *abundance conjecture*, crucial for the proof of that conjecture in dimension 3). Finally, the "mild" singularities are (by definition) so mild that they do not influence the plurigenera.

In arbitrary dimension, the conjecture is still open. In this direction, there is an interesting recent work by Boucksom, Demailly, Peternell and Paun [BDPP]. Consider the cone spanned by the classes of effective divisors in $H^{1,1}_{\mathbb{R}}(X)$ (the space of real (1,1)-classes on X). Let $\mathcal{E}$ be its closure. Let us call a line bundle pseudoeffective, if its class is contained in $\mathcal{E}$.

Theorem 2 ([BDPP]) *A line bundle L on a projective manifold X is pseudoeffective if and only if $L \cdot C \geq 0$ for every curve C which moves in a family covering X.*

Corollary: *If K_X is not pseudoeffective, then X is uniruled.*

(this obviously follows from the theorem by Mori's result stated in the beginning).

To deduce the conjecture, one would have to prove that K_X pseudoeffective implies $\kappa(X) \geq 0$. In [BDPP], there is the following partial result in dimension 4: suppose moreover that there is a covering family C_t with $K_X \cdot C_t = 0$; then $\kappa(X) \geq 0$.

So, at least conjecturally, X uniruled is equivalent to the Kodaira dimension of X being $-\infty$. On the other hand, again conjecturally (this is a form of Lang's conjectures), $\kappa(X) = dim(X) = n$ should be equivalent to the existence of a proper subvariety $Y \subset X$, containing all rational and elliptic curves of X. Here, one must remark that the two conjectures have very different status: the characterization of uniruled varieties, though undoubtedly difficult, seems to be subject to a significant progress now, whereas for Lang's conjectures, there is still very few evidence in dimension > 2. We shall discuss some results on the latter towards the end of this short survey.

Another important notion is that of *rational connectedness*. X is said to be *rationally connected* (RC) if any two sufficiently general points of X can be joined by a rational curve, and *chain rationally connected* (CRC), if any two sufficiently general points of X (or, equivalently, any two points of X) can be joined by a chain of rational curves. For smooth varieties, RC is equivalent to CRC; for singular varieties, being RC is stronger: so, for example, a cone over an elliptic curve is CRC but not RC. For certain classes of singularities (in particular, among those appearing in the minimal model theory) the two notions coincide (see for example [HMK], Corollary 1.8 and similar statements).

We have the following parallel with the uniruled situation ([Ko]): let us say that $f : \mathbb{P}^1 \to X$ is a *very free rational curve*, if f^*T_X is ample, that is, $f^*T_X = \sum \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i > 0$. By deformation theory, this is the same as to say that the deformations of this rational curve with one point fixed cover X.

Proposition 2: *X is rationally connected if and only if X admits a very free rational curve.*

It follows that on a rationally connected varieties, all contravariant holomorphic tensors vanish: $H^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for all $m > 0$. Indeed, such a tensor must vanish along a very free rational curve; but such curves cover a Zariski-open subset of X.

As before, in positive characteristic one must replace here rational connectedness by "separable rational connectedness"; we shall not go into the details.

Conjecture 2 *Conversely, $H^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for all $m > 0$ implies rational connectedness.*

In fact this is implied by Conjecture 1 (see the article on Graber-Harris-Starr theorem).

Besides the vanishing of holomorphic tensors, we have the following general property of rationally connected varieties:

Proposition 3 ([Ca1]) *Rationally connected varieties are simply connected.*

Let us give a sketch of Campana's argument. Roughly speaking, X rationally connected means that the deformations of some rational curve with one point $p \in X$ fixed cover X. Let T be the parameter space of those curves and $Z \subset T \times X$ be the universal family, so $p_1 : Z \to T$ is a $\mathbb{P}^1$-bundle over an open subset of T, with a section S coming from the fixed point p. The second projection $p_2 : Z \to X$ contracts S to a point a. One may suppose that all our varieties are normal. From the fact that p_1 is generically a $\mathbb{P}^1$-bundle, one deduces that $\pi_1(S)$ surjects onto $\pi_1(Z)$. Now $(p_2)_*\pi_1(Z)$ is of finite index in $\pi_1(X)$ (this holds in general for a proper surjection of a normal Z onto a smooth X). But the image of composition $\pi_1(S) \to \pi_1(Z) \to \pi_1(X)$ is trivial, because S is contracted to a point; so $\pi_1(X)$ must be finite.

Consider the universal covering $\sigma : \tilde{X} \to X$. As $\pi_1(X)$ is finite, $\tilde{X}$ is compact, and it is easy to see that it is rationally connected. Therefore (by Hodge theory and the vanishing of holomorphic tensors) $h^i(X, \mathcal{O}_X) = h^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. So the Euler characteristics $\chi(X, \mathcal{O}_X) = \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$; but a finite unramified covering multiplies the Euler characteristic by its degree, so σ is an isomorphism and X is simply connected.

The following construction is also due to F. Campana (in fact, his setting is much more general, see [Ca2] for a detailed exposition), and to Kollar-Miyaoka-Mori:

Definition-Theorem 3([Ca], [KMM])*Let X be a normal and proper variety. Then there is a rational map $\pi : X \dashrightarrow Z$, unique up to birational equivalence, such that fibers X_z are CRC, and for a general $z \in Z$, any rational curve on X meeting X_z lies in X_z.*

Thus, a general fiber X_z consists of all points of X which can be joined to a certain point x_z by a chain of rational curves.

An important property of the map π (and other maps associated to good families of cycles, see [Ca2]) is that it is *almost regular*, that is, its general fiber does not meet the indeterminacy locus.

In [KMM],[Ko], the map π is called the *MRCC-fibration* (maximally rationally chain connected fibration), or the *MRC-fibration* (maximally rationally connected fibration) in the case when X is smooth (then we can of course assume that the fibers are rationnally connected). The variety Z from the theorem 4 (defined up to birational equivalence) is called the *rational quotient* of X according to Campana's terminology ([Ca]).

The following is a consequence of a theorem by Graber, Harris and Starr [GHS] (see the article on Graber-Harris-Starr theorem for more details):

Theorem 4 ([GHS]) *The rational quotient is not uniruled.*

By Mori's bend-and-break, Fano varieties are uniruled; in fact a stronger result holds:

Theorem 5 ([Ca], [KMM2]) *Fano varieties are rationnally connected.*

Corollary: *Fano varieties are simply connected.*

There are in fact several proofs of the simple-connectedness of Fano manifolds. Historically, the first approach uses the L^2-cohomology and Atiyah's L^2-index theorem ([A]). The argument is based on the "covering trick" which we described in the proof of Proposition 3; if one works with L^2-cohomology instead of the usual cohomology of the structure sheaf, it produces an L^2-holomorphic function on the universal covering, which must thus be compact. S. Takayama develops this approach in [T] to show that if one allows certain ("log-terminal") singularities, Fano varieties still are simply-connected.

Let us also mention some "Mori-type" (that is, relating rational curves to the negativity properties of canonical or cotangent bundle) results for algebraic foliations. Here, the starting point is the following semipositivity theorem by Miyaoka:

Theorem 6 ([Mi2])**:** *Let $\mathcal{F}$ be a foliation on an algebraic surface. If $\mathcal{F}$ is not a meromorphic fibration by rational curves, then the canonical line bundle $K_{\mathcal{F}}$ is pseudoeffective.*

(If one views a foliation as a subsheaf of T_X, then its *canonical bundle*, by definition, is just the dual of its determinant).

The following result was first obtained by Bogomolov and McQuillan [BM], then Kebekus, Sola Conde and Toma [KSCT] gave a simpler proof:

Theorem 7: *Let X be normal projective, $C \subset X$ a curve contained in its smooth locus and $\mathcal{F} \subset T_X$ a foliation regular along C. If $\mathcal{F}|_C$ is ample, the leaf through any point $c \in C$ is algebraic, and its closure is rationally connected for a general c. If $\mathcal{F}$ is regular, all leaves are rationally connected submanifolds.*

In conclusion, let us discuss rational curves on some non-uniruled varieties. For instance, it is known (see [MM], where the theorem is attributed to Bogomolov and Mumford) that there is an infinite number of such curves on a K3-surface (on a generic one, all those curves must be singular, because its Picard number is one). So rational curves on a K3-surface are Zariski-dense. It has been conjectured that they are dense in the analytic topology, and even that through any rational point of a K3-surface defined over $\mathbb{Q}$, there is a rational curve; but so far, this is neither proved nor disproved.

On the contrary, it is expected that neither rational nor elliptic curves can be Zariski-denses on a variety X of general type. Moreover, in the complex case, even the *entire* curves (that is, images of nonconstant holomorphic maps $\mathbb{C} \to X$) should be contained in a proper subvariety: this is a variant of Lang's conjecture. In other words, a Zariski-open subset X must be *Brody hyperbolic* (see for example [L] for the basic definitions and results on hyperbolicity).

In this direction, probably the first major result is due to Bogomolov [B]:

Theorem 8 *Let X be a surface of general type, such that $c_1^2(X) > c_2(X)$. Then for any g, the curves of geometric genus g on X form a bounded family.*

A bounded family is, roughly, a family having only a finite number of irreducible components. Since X, being of general type, is not covered by rational or elliptic curves, those curves cannot deform. Bogomolov's theorem thus says that there is only a finite number of rational and elliptic curves on X.

McQuillan in [MQ] handles the case of entire curves, under the same numerical condition on X:

Theorem 9 *Let X be a surface of general type, such that $c_1^2(X) > c_2(X)$. Then any entire curve on X is contained in a rational or elliptic curve (in other words, a holomorphic map $f : \mathbb{C} \to X$ cannot have Zariski-dense image).*

The condition $c_1^2 > c_2$ is quite restrictive, for example, it is never satisfied by a hypersurface in $\mathbb{P}^3$. For general hypersurfaces in projective space $\mathbb{P}^{n+1}$, a conjecture of Kobayashi predicts the hyperbolicity as soon as the degree is high enough (here, "general" means "outside a countable union of proper subvarieties in the parameter space"). This is known for $n = 2$ by the work of Demailly and El Goul [DEG]:

Theorem 10 *There are no entire curves on a general hypersurface of degree at least 21 in $\mathbb{P}^3$.*

(M. Paun lowered the bound to 18; conjecturally, it should be 5.)

If one is interested only in algebraic curves, there are results in any dimension by H. Clemens, L. Ein, C. Voisin, G. Xu and others. For example, the following is partly proved by H. Clemens and partly by C. Voisin; we refer to [V] for the proof and for some generalizations and precisions:

Theorem 11 *A general hypersurface of degree d in $\mathbb{P}^{n+1}$ contains no rational curves for $d \geq 2n + 1$. If $n \geq 3$, this is true for $d \geq 2n$.*

The key idea behind these results, except for the last one, is that the entire curves must satisfy certain algebraic differential equations and thus lift to special subvarieties of the so-called "jet bundles". The scope of this article does not permit to give more details here; we refer the reader to the work of Green and Griffiths [GG] which develops the modern approach to this, or to Demailly's lecture notes [D]. In fact, Bogomolov's theorem slightly preceeds [GG] and has been one of its sources of inspiration.

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