On morphisms onto quadrics

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The purpose of this note is to prove the following result:

Theorem *Let X be a smooth projective variety of dimension n with cyclic Néron-Severi group, and let Q denote an n-dimensional quadric ($n \leq 3$). The degree of a morphism $f : X \to Q$ is bounded in terms of the discrete invariants of X.*

More precisely, let L denote an ample generator of $NS(X)$, and let k, r be such that $K_X \equiv kL$ and rL is very ample. If $f^*\mathcal{O}_Q(1) \equiv mL$, then $m < 2k + (2n+5)r$ when $n \geq 4$, and $m < 3k + 16r$ for $n = 3$.

The existence of such a result was briefly mentioned in [A], Remark 4.1.2; however, the proof (very much analogous to that of [A], Theorem 4.1.1) was never made explicit. Recently, J.-M. Hwang, to whom I am grateful for his interest in that work, suggested that I should write down this proof, which I am glad to do here.

Let $f : X \to Q$, $NS(X) \cong \mathbb{Z}$, be a morphism; f is finite if non-trivial. Take l a sufficiently general line on Q, and let $S \subset Q$ be a general 2-dimensional linear section of $Q \subset \mathbb{P}^{n+1}$ containing l (so S is a two-dimensional quadric). Write $S = Q \cap H_1 \cap H_2 \cap \ldots \cap H_{n-2}$, with H_i sufficiently general hyperplanes.

One has: $N_{l,S} = \mathcal{O}_{\mathbb{P}^1}$ and $N_{S,Q}|_l = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-2}$ That is, the exact sequence of normal bundles

$$0 \to N_{l,S} \to N_{l,Q} \to N_{S,Q}|_l \to 0$$

splits. Let $C = f^{-1}l$ and $M = f^{-1}S$. As l and S are chosen to be sufficiently general, C and M are smooth by Kleiman's version of Bertini theorem ([H], chapter III, section 10). The exact sequence of normal bundles

$$0 \to N_{C,M} \to N_{C,X} \to N_{M,X}|_C \to 0$$

is just the pull-back of the corresponding sequence on Q, that is, it splits. So the induced map $\alpha : H^0(C, N_{C,X}) \to H^0(C, N_{M,X}|_C)$ is surjective.

In particular, $Im(\alpha)$ contains the image of the restriction map $\beta : H^0(M, N_{M,X} \to H^0(C, N_{M,X}|_C)$. This is interpreted as follows: any infinitesimal deformation of M in X contains an infinitesimal deformation of C. That is, the class of C in $H^{1,1}(M)$ is "infinitesimally fixed" according to the terminology of [CGGH].

Now there are various results of "infinitesimal Noether-Lefschetz" type, which state that if M is sufficiently ample on X, then all infinitesimally fixed classes in $H^{1,1}(M)$ are restrictions of (1,1)-classes on X. Our class $[C]$ obviously cannot be such a restriction: indeed, $N_{C,M} = \mathcal{O}_C$, that is, $[C]^2 = 0$; but, thanks to the hypothesis that $NS(X) = \mathbb{Z}$, any (1,1)-class which has zero square on M and is a restriction of a (1,1)-class on X, must be numerically trivial, which is obviously not the case for $[C]$. So such results give the non-existence of f as soon as M is sufficiently ample, that is, $deg(f)$ is sufficiently high.

In order to make this more precise, we shall prove the following proposition, which is a very direct generalization of Proposition 3.4 from [EL]:

Proposition *Let X be a smooth projective n-fold and Y a smooth complete intersection of ample divisors $D_1, D_2, \ldots, D_{n-2}$ on X (so that dim(Y)=2). Suppose that for some very ample A, one has $D_i \in |K_X + (n+1)A + B_i|$ and $D = D_1 + D_2 + \ldots + D_{n-2} \in |nK_X + (n+1)^2 A + E|$ with B_i and E nef. Then the infinitesimal Noether-Lefschetz theorem holds for $Y \subset X$, that is, any (1,1)-class on Y which remains of type (1,1) under all infinitesimal deformations of Y in X, is a restriction of a class on X.*

Proof: We follow the exposition of [EL]. Let $N = \oplus_i \mathcal{O}_Y(D_i)$ be the normal bundle of Y in X. bundle According to the theory of [CGGH], the space of infinitesimally fixed classes is the kernel of the map

$$\alpha : H^1(Y, \Omega^1_Y) \stackrel{\gamma}{\longrightarrow} H^2(Y, N^*) \stackrel{\beta}{\longrightarrow} Hom(H^0(Y, N), H^2(Y, \mathcal{O}_Y),$$

γ being the usual coboundary map, and β coming from the multiplication map $\mu : H^0(Y, N) \otimes H^0(Y, K_Y) \to H^0(Y, N \otimes K_Y)$ by Serre duality. We shall be done if we check that

1) β is injective, that is, μ is surjective;

2) the restriction map $H^1(X, \Omega^1_X) \to H^1(Y, \Omega^1_X|_Y)$ is surjective.

To check the surjectivity of μ, it is enough to check the surjectivity of each "component"

$$\mu_i : H^0(Y, \mathcal{O}_Y(D_i)) \otimes H^0(Y, K_Y) \to H^0(Y, \mathcal{O}_Y(D_i) \otimes K_Y).$$

From the resolution

$$(*)\ 0 \to \mathcal{O}_X(-D) \to \ldots \to \oplus_{i<j} \mathcal{O}_X(-D_i - D_j) \to \oplus_i \mathcal{O}_X(-D_i) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

one sees that $H^0(X, \mathcal{O}_X(D_i) \otimes K_X \otimes \mathcal{O}_X(D))$ surjects onto $H^0(Y, \mathcal{O}(D_i) \otimes K_Y)$ (indeed, the D_i are ample, so this is implied by the Kodaira vanishing). That is, the surjectivity of μ_i would, by restriction, follow from the surjectivity of the multiplication map on X:

$$H^0(X, \mathcal{O}_X(D_i)) \otimes H^0(X, \mathcal{O}_X(D) \otimes K_X) \to H^0(X, \mathcal{O}_X(D) \otimes K_X \otimes \mathcal{O}_X(D_i)).$$

This, in turn, follows from the case $p = 0$ of [EL], Theorem 2, by taking (in the notations of [EL]) $L_d = \mathcal{O}_X(D_i)$, $W = H^0(X, L_d)$, $N_f = K_X + D$. Indeed, $N_f = K_X + (n+1)A + n(K_X + (n+1)A) + E$, and $n(K_X + (n+1)A)$ is nef; and D_i is globally generated because it's of the form $K_X + (n+1)A + B_i$ with A very ample.

As for the surjectivity of the restriction map, one sees from the resolution $(*)$ and the Kodaira vanishing that it is enough to check that $H^{n-1}(X, \Omega^1_X(-D)) = 0$, or, dually, $H^1(X, \Omega^{n-1}_X(D)) = 0$. Now the bundle $\Omega^{n-1}_X(nA)$ is globally generated, so one can apply Griffiths' vanishing: $H^i(X, E \otimes F) = 0$ for $i > 0$, where E is a globally generated vector bundle and F is a line bundle such that $F \otimes (det(E) \otimes K_X)^{-1}$ is ample. As $det(\Omega^{n-1}_X(nA)) = n^2A + (n-1)K_X$, the required vanishing follows.

Let us return to our $M \subset X$, $M = D_1 \cap D_2 \cap \ldots \cap D_{n-2}$ with $D_i \in |f^*\mathcal{O}_Q(1)|$, violating the infinitesimal Noether-Lefschetz condition. It is clear that if $f^*\mathcal{O}_Q(1)$ is sufficiently ample, then the D_i shall satisfy the two conditions of our proposition, leading to a contradiction. More precisely, suppose that D_i is of the form $2K_X + (2n+5)A$ for some very ample A; an easy calculation (using the fact that $K_X + (n+1)A$ is nef) shows that the conditions of the proposition are satisfied provided that $n \geq 4$. (For $n = 3$, these hold when $D = D_1 = 3K_X + 16A$; this is already in [EL]). So D_i cannot get that ample; in other words, if m is such that $f^*\mathcal{O}_Q(1) \equiv mL$, then $m < 2k + (2n+5)r$ for $n \geq 4$ and $m < 3k + 16r$ for $n = 3$.

Remark: This can be slightly improved by remarking that X is not $\mathbb{P}^n$, and if X is a quadric, then f is an isomorphism; so one may suppose that X is neither $\mathbb{P}^n$ nor a quadric, in which case already $K_X + (n-1)A$ is nef.

References

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