Non-commutative Cartier operator and Hodge-to-de Rham degeneration

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Introduction

When one thinks about differential forms and the de Rham complex in the context of homological algebra and algebraic geometry, one usually considers de Rham cohomology and treats it as one more cohomology theory for algebraic varieties, on a par with Betti cohomology, étale cohomology, crystalline cohomology in positive characteristic, and so on. However, there is an alternative point of view which has been developping slowly over the years and recently became quite prominent.

Since the pioneering paper [HKR], it has been known that a differential form on a smooth commutative algebra can be treated as its Hochschild homology class. In 1983, it was discovered, simultaneously and independently in [C], [FT1], [LQ], that the de Rham differential also has such an algebraic interpretation. Moreover, and most surprisingly, the whole formalism makes sense for an associative, not necessarily commutative algebra A. The de Rham cohomology appears in this setting as *additive K-theory* ([FT2]) or *cyclic homology* ([C]). Hochschild and cyclic homology are related by a spectral sequence analogous to (and reducing to) the commutative Hodge to de Rham spectral sequence. For an introduction to the subject, we refer the reader to [L] (and also to [FT2], which contains much material not covered in [L]).

A natural next step in this direction would be to study the Hodge theory – since the standard spectral sequence relating Hodge cohomology and de Rham cohomology makes sense for a non-commutative algebra A, under what assumptions does this spectral sequence degenerate? Recently there has been much interest in this topic – most notably, in the work of M. Kontsevich, who has stated and popularized a conjectural non-commutative analog on the standard degeneration theorem for smooth projective algebraic varieties. He also gave some very beautiful applications, and formulated a set of very precise finiteness assumptions which are necessary for the degeneration and play the role of both properness and smoothness in the non-commutative setting.

To the best of our knowledge, so far there has been very little progress in proving Kontsevich's conjecture. This is perhaps not very surprising, since in the non-commutative setting, the usual analytic approach to Hodge theory makes no sense. Fortunately, there is an alternative algebraic approach. It was discovered by P. Deligne and L. Illusie [DL] – incidentally, in the same year 1983 that the cyclic homolgy was discovered. The argument uses reduction to positive characteristic and the Cartier isomorphism between de Rham cohomology spaces and spaces of forms for smooth varieties over a field of positive characteristic.

The goal of the present paper is to apply the Deligne-Illusie method in the non-commuative setting and to prove that the non-commutative Hodge to de Rham spectral sequence also degenerates, provided some natural finiteness conditions are met. We do this by giving a version of the Cartier isomorphism valid without any assumptions of commutativity. The final result is a direct generalization of [DL]. In particular, it reduces to [DL] for smooth proper commutative algebraic varieties (except that our general construction of the Cartier isomorphism is somewhat more canonical and does not use any explicit formulas at all).

Our approach generally follows that of Kontsevich, and differs from his in only one respect. Kontsevich gave his conjecture in characteristic 0, and he used the language of A_∞-algebras. Our method is reduction to positive characteristic, and the notion of A_∞-algebra is not very convenient there (because it uses the "naive", not simplicial tensor structure on the category of complexes of vector spaces). The same goes for differential-graded algebras. Instead, we chose to give the statement for sheaves of usual associative algebras over an arbitrary site. This includes both the case of usual algebraic varieties, and the case of simplicial, cosimplicial and bisimplicial algebras (the latter suffices to deduce the A_∞-statement as well). Most likely, the proper generality for the theory is in any case that of a triangulated category equiped with some enhancement; however, this should be the topic of futher research.

Due to the peculiar power of the Deligne-Illusie method, our finiteness assumptions – and the reader will find them, together with the precise statement of the main Theorem, below in Section 5 – are rather weak. Essentially we need only three things: (1) the sheaf of algebras should be finitely generated in some sense, (2) everything should be of finite homological dimension, and (3) the Hochschild homology in question should be a finite-dimensional vector space over the base field. In particular, in the commutative case, one might get a statement by our method for at least some proper algebraic varieties that are not smooth (the Hochschild homology in this case should be understood as the Hochschild homology of the category of perfect complexes of coherent sheaves). We do not know whether this has any geometric significance. Also, it is known that the Deligne-Illusie method allows one to prove several strong vanishing theorems of the Kodaira type, see [EV]; we do not know whether our non-commutative version can be used in a similar way.

Another intriguing observation concernes the non-commutative Cartier isomorphism. In a sense, and this may be made quite precise, the Cartier isomorphism is only the visible part of an iceberg, which is the action of the Frobenius map on crystalline cohomology. In the non-commutative setting, there is certainly no Frobenius map. However, our results suggest that the Frobenius action on cohomology still exists. Among the many and varied applications of this Frobenius action, some might also make sense in the non-commutative world.

The paper is organized as follows. Section 1 contains the necessary preliminary material recast in the way convenient for our application – in particular, we systematically use the language of homology of small categories. Everything is standard and contained in some form in [L]. Then we introduce an analog of the so-called *conjugate filtration* in non-commutative cohomology, and define an analog of the Cartier map. We have tried to make the paper self-contained, and we have also tried to separate linear algebra from things that require tensor products. For this reason, we have split this material into Section 2, which is purely linear-algebraic, and Section 4, where associative algebras finally appear. Section 3 is taken up with the proof of Proposition 3.9. This is a technical fact needed for our degeneration theorem; the proof is not difficult at all, but annoyingly tedious, and the reader is strongly adviced to skip it at the first reading. Finally, Section 5 contains the degeneration theorems, both in characteristic 0 (Theorem 5.5) and in positive characteristic $p > 2$ (Theorem 5.1).

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1 Cyclic objects.

In this section, we recall standard basic facts about cyclic objects and cyclic categories needed in the rest of the paper; we mostly follow [FT2], see also [L, Chapter 6].

1.1 Homological preliminaries. We begin with homological generalities. Let $\mathcal{C}$ be a Grothendieck abelian category – or, in Grothendieck's language [G], an abelian category which additionally satisfies $AB3$, $AB4$, $AB3'$, $AB4'$ (in applications, $\mathcal{C}$ will be the category of B-modules for a commutative ring B, or, more generally, the category of sheaves of $\mathcal{B}$-modules for a sheaf of commutative rings $\mathcal{B}$ on some site). For any small category F, denote by $\operatorname{Fun}(\mathsf{F}, \mathcal{C})$ the abelian category of covariant functors from F to

$\mathcal{C}$, and denote by $\mathcal{D}^-(\mathsf{F},\mathcal{C})$ the derived category of complexes in $\mathrm{Fun}(\mathsf{F},\mathcal{C})$ bounded from above. The category $\mathrm{Fun}(\mathsf{F},\mathcal{C})$ is also a Grothendieck abelian category.

For any two small categories F, G and a functor $\sigma : \mathsf{F} \to \mathsf{G}$, denote by $\sigma^* : \mathrm{Fun}(\mathsf{G},\mathcal{C}) \to \mathrm{Fun}(\mathsf{F},\mathcal{C})$ the natural restriction functor given by

$$\sigma^*(E) = E \circ \sigma \in \mathrm{Fun}(\mathsf{F},\mathcal{C}), \qquad E \in \mathrm{Fun}(\mathsf{G},\mathcal{C}).$$

Since $\mathcal{C}$ is a Grothendieck category, σ^* has right and left adjoint functors called *Kahn extensions*. Denote the left and right Kahn extensions by $\sigma_! : \mathrm{Fun}(\mathsf{F},\mathcal{C}) \to \mathrm{Fun}(\mathsf{G},\mathcal{C})$ and $\sigma_* : \mathrm{Fun}(\mathsf{F},\mathcal{C}) \to \mathrm{Fun}(\mathsf{G},\mathcal{C})$, and denote their derived functors by $\mathsf{L}^\bullet\sigma_! : \mathcal{D}^-(\mathsf{F},\mathcal{C}) \to \mathcal{D}^-(\mathsf{G},\mathcal{C})$ and $\mathsf{R}^\bullet\sigma_! : \mathcal{D}^-(\mathsf{F},\mathcal{C}) \to \mathcal{D}^-(\mathsf{G},\mathcal{C})$.

In the particular case when $\mathsf{F} = \mathsf{pt}$ is the category with one object and one morphism, we have $\mathrm{Fun}(\mathsf{F},\mathcal{C}) \cong \mathcal{C}$, and specifying a functor $\sigma : \mathsf{F} \to \mathsf{G}$ is the same as specifying an object $[f] \in \mathsf{G}$. For any object $A \in C$, we denote

$$\begin{aligned} A_{[f]} &= \sigma_! A, \\ A^o_{[f]} &= \sigma_* A. \end{aligned}$$

If $\mathcal{C}$ is the category of abelian groups, and $A = \mathbb{Z}$ is the group $\mathbb{Z}$, for any $g \in \mathsf{G}$ the group $\mathbb{Z}_{[f]}([g])$ is naturally identified with the free abelian group spanned by the set $\mathrm{Hom}_{\mathsf{G}}([f],[g])$, and the group $\mathbb{Z}^o_{[f]}([g])$ is naturally identified with the abelian group of $\mathbb{Z}$-valued functions on the set $\mathrm{Hom}_{\mathsf{G}}([g],[f])$.

In the particular case when $\mathsf{G} = \mathsf{pt}$, there exists only one tautological functor $\sigma : \mathsf{F} \to \mathsf{G}$. For any $A \in \mathcal{C} \cong \mathrm{Fun}(\mathsf{G},\mathcal{C})$, we denote

$$A_{\mathsf{F}} = \sigma^* A,$$

or simply $A \in \mathrm{Fun}(\mathsf{F},\mathcal{C})$ when there is no danger of confusion. For any $E \in \mathrm{Fun}(\mathsf{F},\mathcal{C})$ we denote $\mathsf{L}^\bullet\sigma_!(E)$ by $H_\bullet(\mathsf{F},E)$ and we call it the *homology of the category* F *with coefficients in* $E \in \mathrm{Fun}(\mathsf{F},\mathcal{C})$. Analogously, we denote $\mathsf{R}^\bullet\sigma_*E = H^\bullet(\mathsf{F},E)$, and we call the *cohomology of the category* F *with coefficients in* E. We note that for any $\rho : \mathsf{F} \to \mathsf{G}$, and any $E \in \mathrm{Fun}(\mathsf{G},\mathcal{C})$, we have by adjunction a canonical map

$$\sigma_! : H_\bullet(\mathsf{F},\sigma^*\mathcal{C}) \to H_\bullet(\mathsf{G},C).$$

If $\mathcal{C} = B$-mod is the category of modules over a commutative ring B, we have by adjunction

$$H^\bullet(\mathsf{F},B_{\mathsf{F}}) = \mathrm{Ext}^\bullet(B_{\mathsf{F}},B_{\mathsf{F}}),$$

so that the cohomology $H^\bullet(\mathsf{F}, B_\mathsf{F}) = H^\bullet(\mathsf{F}, B)$ with coefficients in the constant functor $B \in \mathrm{Fun}(\mathsf{F}, B\text{-mod})$ carries a natural algebra structure. For any B-linear Grothendieck category $\mathcal{C}$ and any object $E \in \mathrm{Fun}(\mathsf{F}, \mathcal{C})$, the homology $H_\bullet(\mathsf{F}, E)$ is naturally a module over the algebra $H^\bullet(\mathsf{F}, B_\mathsf{F})$. For any F, G and a functor $\rho : \mathsf{F} \to \mathsf{G}$, the restriction functor ρ^* induces an algebra map $\rho^* : H^\bullet(\mathsf{G}, B) \to H^\bullet(\mathsf{F}, B)$. The maps ρ^* and $\rho_!$ are, as usual, related by the projection formula: we have

$$\sigma_!(\sigma^*(\alpha) \cdot e) = \alpha \cdot \sigma_!(e)$$

for any $\alpha \in H^\bullet(\mathsf{G}, B)$, $e \in H_\bullet(\mathsf{F}, \rho^* E)$.

We will also need to use the notion of a fibered category ([SGA]); let us briefly recall it. For any functor $\sigma : \mathsf{F} \to \mathsf{G}$ between small categories F and G and any object $[a] \in \mathsf{G}$, denote by $\mathsf{F}_{[a]}$ the subcategory in F of all objects $[a'] \in \mathsf{F}$ such that $\sigma([a']) = [a]$ and all morphisms f such that $\sigma(f) = \mathsf{id}_{[a]}$. This is called the *fiber of the functor* $\sigma : \mathsf{F} \to \mathsf{G}$ *over* $[a] \in \mathsf{G}$. For any $[a], [b] \in \mathsf{G}$, any $f : [a] \to [b]$ and any $[a'] \in \mathsf{F}_{[a]}$, $[b'] \in \mathsf{F}_{[b]}$, denote by $\mathsf{F}_f([a'], [b'])$ the set of all maps $f' : [a'] \to [b']$ such that $\sigma(f') = f$.

Definition 1.1. The functor $\sigma : \mathsf{F} \to \mathsf{G}$ is called a *fibration*, and F is called a *fibered category over* G, if for any $[a], [b] \in \mathsf{G}$, $f : [a] \to [b]$ and $[b'] \in \mathsf{F}_{[b]}$ the functor

$$[a'] \mapsto \mathsf{F}_f([a'] \to [b'])$$

from $\mathsf{F}_{[a]}$ to the category of sets is representable by an object $f^*([b]) \in \mathsf{F}_{[a]}$.

Passing to the opposite categories gives the dual notion of a cofibration and a cofibered category. A functor which is both a fibration and a cofibration is called a *bifibration.*

An example of a cofibered category is a so-called *discrete cofibration*: for any functor F from G to sets, pairs $\langle [a] \in \mathsf{G}, a \in F([a]) \rangle$ form a small category F naturally fibered over G (the fiber $\mathsf{F}_{[a]}$ is the set $F([a])$ cosidered as a category with no non-identical morphisms). The category F is called the *total space* of the functor F and denoted $\mathsf{Tot}(\mathsf{G}, F)$. It is easy to check that every cofibration whose fibers have no non-identical morphisms is obtained in this way. An opposite case is a cofibration $\mathsf{F} \to \mathsf{G}$ whose fibers are categories with one object. For example, every functor F from G to the category of groups defines a category F cofibered over G whose fiber $\mathsf{F}_{[a]}$ is a groupoid with one object with automorphism group $F([a])$. By abuse of notation, in this case we will also call F the total space of the functor F and denote it by $\mathsf{Tot}(\mathsf{G}, F)$. More generally, a cofibered category over G whose fibers are groupoids is usually called a *gerbe* over G. Gerbes of the form $\mathsf{Tot}(\mathsf{G}, F)$ are said to be *split*; there are gerbes which are not of this type (we will see one example below in Remark 2.4).

We will need the following standard properties of cofibrations (and the dual properties of fibrations, which are analogous and left to the reader).

Proposition 1.2. *Let $\sigma : \mathsf{F} \to \mathsf{G}$ is a cofibration.*

(i) *For any small category G' and a functor $\rho : \mathsf{G}' \to \mathsf{G}$, the pullback $\sigma' : \mathsf{G}' \times_{\mathsf{G}} \mathsf{F} \to \mathsf{G}'$ is also a cofibration, and we have a natural base change isomorphism*

$$\mathsf{L}^{\bullet}\sigma_! \circ \rho^* \cong \rho^* \circ \mathsf{L}^{\bullet}\sigma_!. \tag{1.1}$$

(ii) *For any Grothendieck category $\mathcal{C}$ and any object $E \in \mathrm{Fun}(\mathsf{G}, \mathcal{C})$, we have a natural isomorphism*

$$\mathsf{L}^{\bullet}\sigma_!\sigma^* E \cong E \otimes \mathsf{L}^{\bullet}\sigma_!\sigma^*\mathbb{Z}_{\mathsf{G}}, \tag{1.2}$$

and this isomorphism is functorial in E.

Sketch of a proof. To check that $\sigma' : \mathsf{G}' \times_{\mathsf{G}} \mathsf{F} \to \mathsf{G}'$ is indeed a cofibration is an elementary exercize left to the reader. Moreover, for any object $[a] \in \mathsf{G}'$ we have natural isomorphism $\pi : \sigma'_{[a']} \cong \sigma_{\pi([a'])}$. Now, by the definition of a cofibration, for any objects $M \in \mathcal{C}$, $[a] \in \mathsf{G}$, we have

$$\sigma^* M^o_{[a]} \cong \lim_{\leftarrow} M^o_{[b]} \cong \iota_*(M_{\sigma_{[a]}}),$$

where the limit is taken over $[b] \in \sigma_{[a]} \subset \mathsf{F}$, and $\iota : \sigma_{[a]} \to \mathsf{F}$ is the embedding of the fiber $\sigma_{[a]}$. Therefore for any $E \in \mathrm{Fun}(\mathsf{F}, \mathcal{C})$ and any $[a] \in \mathsf{G}$ we have

$$\mathsf{L}^{\bullet}\sigma_!(E)([a]) \cong H_{\bullet}(\sigma_{[a]}, E),$$

and this is functorial in E and in $[a]$. This immediately gives both the base change isomorphism (1.1) and the projection formula (1.2). □

Finally, in Section 2 we will need to use a bar resolution several times; we recall the setup. Let $F : \mathcal{C} \to \mathcal{C}$ be any exact functor from an abelian category $\mathcal{C}$ to itself. Assume that F is equipped with a surjective augmentation map $f : F \to \mathsf{Id}$, and denote by $P : \mathcal{C} \to \mathcal{C}$ the kernel of this map (P is obviously also an exact functor from $\mathcal{C}$ to iself). The *bar-resolution*, or the

Godement resolution associated to the pair $\langle F, f\rangle$ is constructed as follows. For any $l \geq 1$ we set

$$B_l(F, f) = F \circ P^{l-1} : \mathcal{C} \to \mathcal{C}, \tag{1.3}$$

and we define a map $d : B_{l+1}(F, f) \to B_l(F, f)$ as the composition of the projection $f : F \circ P^l \to \mathsf{Id} \circ P^l = P^l$ and the embedding $P^l = P \circ P^{l-1} \to F \circ P^{l-1}$. It will be convenient for us to extend this by setting $B_0 = \mathsf{Id}$, $d = f : B_1 = F \to \mathsf{Id}$, and we will omit f from notation when it is clear from the context what it is. Then $d^2 = 0$, and $\langle B_l(F), d\rangle$ is an complex of exact functors from $\mathcal{C}$ to itself. For any object $E \in \mathcal{C}$, $B_\bullet(F)(E)$ is a complex in $\mathcal{C}$ with $B_0(F)(E) = E$. Moreover, it is easy to check by induction that the complex $B_\bullet(F)(E)$ is actually acyclic, so that $B_{\geq 1}(F)(E)$ is a resolution of the object E – indeed, by construction we have a natural embedding of complexes

$$P(B_\bullet(F)(E))[1] \to B_\bullet(F)(E), \tag{1.4}$$

and the cokernel of this embedding is the acyclic complex $E \to E$.

The bar-construction works without any changes in a more general situation when F is functor from the category of complexes in $\mathcal{C}$ to itself. In this setting, for any $E \in \mathcal{C}$ the bar complex $B_\bullet(F)(E)$ is naturally a bicomplex; by abuse of notation, we will denote by $B_\bullet(F)(E)$ its total complex.

We note that formally, in order to define the complex $B_\bullet(F, f)$ we do not need P to be the kernel of the map $f : F \to \mathsf{Id}$ – all we need is a sequence

$$0 \longrightarrow P \longrightarrow F \longrightarrow \mathsf{Id} \longrightarrow 0 \tag{1.5}$$

of exact functors which is exact on the left and on the right. In this yet more general situation, we denote the bar complex by $B_\bullet(F, P)$. This construction is obviously functorial with respect to sequences of the form (1.5). If (1.5) is exact in the middle term, the complex $B_\bullet(F, P) = B_\bullet(F)$ is acyclic, if not, then not. However, if F and P in (1.5) are allowed to be exact functors from the category of complexes in $\mathcal{C}$ to itself, then the condition is weaker: the bar complex $B_\bullet(F, P)(E)$ is acyclic for any $E \in \mathcal{C}$ if the middle homology of the sequence (1.5) is acyclic – in other words, if (1.5) gives an exact triangle after passing to the derived category.

1.2 Cyclic categories – definition and combinatorics. The main small categories that we will need in the paper are the cyclic category Λ introduced by A. Connes, and its generalizations, the so-called paracyclic categories Λ_p introduced in [BG]. Let us recall the definitions. Consider the category Cycl of linearly ordered sets M equipped with an order-preserving automorphism $\sigma : M \to M$. Maps in Cycl are order-preserving maps which commute with σ. Consider the set $\mathbb{Z}$ of all integers as a linearly ordered set with the natural order, and for any positive $m \in \mathbb{Z}$, denote by $[m]$ the set $\mathbb{Z}$ equipped with an automorphism $\sigma : \mathbb{Z} \to \mathbb{Z}$ given by $a \mapsto a + m$. Let $\Lambda_\infty \subset$ Cycl be the full subcategory spanned by $[m] \in$ Cycl for all positive $m \in \mathbb{Z}$. We will treat Λ_∞ is a small category whose set of objects is the set of positive integers, and for any two positive integers $n, m \in \mathbb{Z}$, the set

$$\Lambda_\infty([m],[n])$$

of maps from $[m]$ to $[n]$ is the set of all order-preserving maps $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(a+m) = f(a)+n$. In particular, σ itself gives an automorphism $\sigma \in \Lambda_\infty([m],[m])$ for any object $[m]$, and for any $f \in \Lambda_\infty([m],[n])$, we have $\sigma \circ f = f \circ \sigma$. Therefore setting $f \mapsto f \circ \sigma$ gives an action of the map σ on $\Lambda_\infty([m],[n])$ which is compatible with compositions, and we can define a category Λ with the same objects as Λ_∞, and with morphism sets given by

$$\Lambda([m],[n]) = \Lambda_\infty([m],[n])/\sigma.$$

This is the Connes cyclic category. Moreover, for any positive $p \in \mathbb{Z}$ we can set

$$\Lambda_p([m],[n]) = \Lambda_\infty([m],[n])/\sigma^p$$

and obtain the paracyclic category Λ_p. For any $p \geq 1$, we obviously have a natural embedding $i_p : \Lambda_p \to \Lambda$, $[m] \mapsto [mp]$, and a natural projection $\pi_p : \Lambda_p \to \Lambda$, $[m] \mapsto [m]$. The projection $\pi_p : \Lambda_p \to \Lambda$ is a bifibration over Λ, with all fibers isomorphic to pt_p, the groupoid with one object with automorphism group $\mathbb{Z}/p\mathbb{Z}$. The projection $\rho_p : \Lambda_\infty \to \Lambda_p$ is also a bifibration, with all fibers isomorphic to pt_∞, the groupoid with one object with automorphism group $\mathbb{Z}$.

The category Λ_∞ is self-dual – the duality functor $D : \Lambda_\infty \to \Lambda_\infty^o$ is identical on objects, and for any $f \in \Lambda_\infty([m],[n])$ we set

$$D(f)(a) = \max\{b \in \mathbb{Z} \mid f(b) \leq a\}, \qquad a \in \mathbb{Z}.$$

This duality descends to the cyclic category Λ and to all the paracylic categories Λ_p.

Let Δ be the simplicial category, that is, the category of finite linearly ordered sets. Then the opposite category Δ^o is naturally embedded into Λ_∞: for any $[m]$, $[n]$, the set $\Delta([m],[n])$ is naturally identified with the set of those $f \in \Lambda_\infty([m],[n])$ that preserve $0 \in \mathbb{Z}$, $f(0) = 0$. Denote this embedding by $j_\infty : \Delta^o \to \Lambda_\infty$. For every $p \geq 1$, the embedding j_∞ induces an embedding from Δ^o to Λ_p, which we denote by $j_p : \Delta^o \to \Lambda_p$ (simply j if $p = 1$). By duality, we obtain the embeddings $j_p^o : \Delta \to \Lambda_p$. The bifibration $\pi_p : \Lambda_p \to \Lambda$ splits over $\Delta^o \subset \Lambda$ by means of the embedding j_p – in other words, the embedding $j_p : \Delta^o \to \Lambda_p$ extends to an embedding $\widetilde{j}_p : \Delta^o \times \mathsf{pt}_p \to \Lambda_p$. The composition $i_p \circ \widetilde{j}_p : \Delta^o \times \mathsf{pt}_p \to \Lambda$ factors through an embedding from $\Delta^o \times \mathsf{pt}_p$ to Δ^o which we also denote by i_p, by abuse of notation. We can collect all these data into a commutative diagram

$$\begin{array}{ccc}
\Delta^o & \xrightarrow{j} & \Lambda \\
{\scriptstyle i_p}\uparrow & & \uparrow{\scriptstyle i_p} \\
\Delta^o \times \mathsf{pt}_p & \xrightarrow{\widetilde{j}_p} & \Lambda_p \\
\downarrow & & \downarrow{\scriptstyle \pi_p} \\
\Delta^o & \xrightarrow{j} & \Lambda
\end{array}$$

of small categories and functors, with Cartesian squares. Explicitly, the embedding $i_p : \Delta^o \times \mathsf{pt}_p \to \Delta^o$ sends $[m] \in \Delta$ to $[p] \times [m]$ with lexicographical ordering; the automorphism group $\mathbb{Z}/p\mathbb{Z}$ acts by changing the ordering on the set $[p]$.

As far as the functor categories are concerned, one can also treat the embedding $j_p : \Delta^o \to \Lambda_p$ as a discrete cofibration, both for $p \in \mathbb{Z}$ and $p = \infty$. Indeed, let $\widetilde{\Delta^o}$ be the total space of the functor $\Lambda_p \to$ Sets given by $[n] \mapsto \Lambda_p([1],[n])$. Explicitly, $\widetilde{\Delta^o}$ is the category of pairs $\langle [n], a \rangle$ of the linearly ordered set $\mathbb{Z}$ with the map $\sigma : \mathbb{Z} \to \mathbb{Z}$ and a fixed element $a \in \mathbb{Z}$ defined modulo σ^p; maps in $\widetilde{\Delta^o}$ are maps in Λ_p which preserve the fixed element. Then the embedding $j_p : \Delta^o \to \Lambda_p$ naturally factors through an embedding $\Delta^o \to \widetilde{\Delta^o}$ which sends $[n]$ to $\langle [n], 0 \rangle$. It is easy to check that this embedding $\Delta^o \to \widetilde{\Delta^o}$ is an equivalence of categories, and in particular, it induces an equivalence $\operatorname{Fun}(\widetilde{\Delta^o}, \mathcal{C}) \cong \operatorname{Fun}(\Delta^o, \mathcal{C})$.

1.3 Periodicity. Fix a Grothendieck abelian category $\mathcal{C}$ and an integer $p \geq 1$, and consider the embedding $j_p : \Delta^o \to \Lambda_p$.

Lemma 1.3. *For any $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, there exists an isomorphism*

$$j_{p!} j_p^* E \cong E \otimes j_{p!} j_p^* \mathbb{Z},$$

this isomorphism is functorial in E, and $j_{p!}j_p^*\mathbb{Z}$ is isomorphic to $\mathbb{Z}_{[1]} \in$ $\operatorname{Fun}(\Lambda_p, \mathbb{Z}\text{-mod})$.

Proof. Replace $j_p : \Delta^o \to \Lambda_p$ with the discrete fibration $\xi : \widetilde{\Delta^o} \to \Lambda_p$; since Δ^o is equivalent to Δ^o, we have $j_{p!}j_p^* \cong \xi_!\xi^*$, and both claims follow from Lemma 1.2. $\square$

By duality, we also obtain a functorial isomorphism $j^o_{p*}j^{o*}_p E \cong E \otimes j^o_{p*}j^{o*}_p\mathbb{Z} \cong E \otimes \mathbb{Z}^o_{[1]}$, and since both $\mathbb{Z}_{[1]}$ and $\mathbb{Z}^o_{[1]}$ are functors into *flat* abelain groups, we conclude that both $j_{p!}j_p^*$ and $j^o_{p*}j^{o*}_p$ are exacts functors from $\operatorname{Fun}(\Lambda_p, \mathcal{C})$ to itself. We tautologically have $H_\bullet(\Lambda_p, j_{p!}j_p^*E) \cong H_\bullet(\Delta^o, j_p^*E)$.

Lemma 1.4. *For any $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, we have*

$$H_\bullet(\Lambda_p, j^o_{p*}j^{o*}_p E) = H_\bullet(\Lambda_p, E \otimes \mathbb{Z}^o_{[1]}) = 0.$$

Proof. Since $E \mapsto E \otimes \mathbb{Z}^o_{[1]}$ is an exact functor in E, the homology that we have to study is the derived functor of the right-exact functor

$$H_0(\Lambda_p, - \otimes \mathbb{Z}^o_{[1]}) : \operatorname{Fun}(\Lambda_p, \mathcal{C}) \to \mathcal{C}.$$

Therefore is suffices to prove that this right-exact functor vanishes. Indeed, the constant functor $\mathbb{Z} \in \operatorname{Fun}(\Lambda_p, \mathbb{Z}\text{-mod})$ embeds into $\mathbb{Z}^o_{[1]}$ by adjunction, and the cokernel of this embedding embeds into $\mathbb{Z}^o_{[2]}$; by adjunction, we have a functorial exact sequence

$$E([2])^{\oplus 2p} \xrightarrow{\ \rho\ } E([1])^{\oplus p} \longrightarrow H_0(\Lambda_p, E \otimes \mathbb{Z}^o_{[1]}) \longrightarrow 0,$$

and it remains to compute the map ρ and to notice that it is tautologically surjective (in fact, split). We leave it to the reader. $\square$

We can now deduce the main homological result on the embedding $j_p : \Delta^o \to \Lambda_p$.

Lemma 1.5. *There exists a map $B_p : \mathbb{Z}^o_{[1]} \to \mathbb{Z}_{[1]}$ such that the sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^o_{[1]} \xrightarrow{\ B_p\ } \mathbb{Z}_{[1]} \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{1.6}$$

is an exact sequence in $\operatorname{Fun}(\Lambda_p, \mathbb{Z}\text{-mod})$. *The cohomology algebra $H^\bullet(\Lambda_p, \mathbb{Z})$ is the free algebra $\mathbb{Z}[u]$ in one generator $u = u(p)$, and this generator is represented by Yoneda by the exact sequence* (1.6). *The embedding $i_p : \Lambda_p \to \Lambda$ sends the generator $u(1)$ to $u(p)$.*

Proof. The construction of the map B_p for $p = 1$ is standard, see e.g. [FT2]. In the case $p > 1$, we notice that $i_p^*(\mathbb{Z}_{[1]})$ is canonically isomorphic to $\mathbb{Z}_{[1]}$, and $i_p^*(\mathbb{Z}^o_{[1]})$ is canonically isomorphic to $\mathbb{Z}^o_{[1]}$. This gives the map B_p and, by Yoneda, the universal generator $u = u(p) \in H^2(\Lambda_p, \mathbb{Z})$. Finally, to show that $H^{\bullet}(\Lambda_p, \mathbb{Z})$ is freely generated by u, it suffices to extend (1.6) to a resolution of the constant object $\mathbb{Z} \in \operatorname{Fun}(\Lambda_p, \mathbb{Z}\text{-mod})$ and to notice that by adjunction,

$$H^{\bullet}(\Lambda_p, \mathbb{Z}^o_{[1]}) = H^{\bullet}(\mathsf{pt}, \mathbb{Z}) = \mathbb{Z},$$

while $H^{\bullet}(\Lambda_p, \mathbb{Z}_{[1]}) = 0$ by the statement dual to Lemma 1.4. □

Corollary 1.6. *For any object $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, we have a functorial exact sequence*

$$0 \longrightarrow E \longrightarrow j^o_{p*} j^{o*}_p E \longrightarrow j_{p!} j^*_p E \longrightarrow E \longrightarrow 0 \tag{1.7}$$

and a functorial exact triangle

$$H_{\bullet}(\Delta^o, j^*_p E) \longrightarrow H_{\bullet}(\Lambda_p, E) \xrightarrow{\ u\ } H_{\bullet}(\Lambda_p, E)[2] \longrightarrow, \tag{1.8}$$

where $u : H_{\bullet}(\Lambda_p, E) \to H_{\bullet}(\Lambda_p, E)[2]$ is given by the action of the universal generator $u \in H^2(\Lambda_p, \mathbb{Z})$. □

It will be convenient for us to rewrite (1.7) as a functorial short exact sequence

$$0 \longrightarrow E[1] \longrightarrow j^{\dagger}_p E \longrightarrow E \longrightarrow 0 \tag{1.9}$$

in the category of complexes of objects in $\operatorname{Fun}(\Lambda_p, \mathcal{C})$, where $E[1]$ is, as usually, the complex consisting of E placed in degree -1, and $j^{\dagger}_p E$ is the complex $j^o_{p*} j^{o*}_p E \to j_{p!} j^*_p E$ placed in degrees 0 and -1. Then (1.8) amounts to a canonical isomorphism $H_{\bullet}(\Delta^o, j^*_p E) \cong H_{\bullet}(\Lambda_p, j^{\dagger}_p E)$ and the long exact sequence associated to (1.9). We note that $j^{\dagger}_p$ is an exact functor from $\operatorname{Fun}(\Lambda_p, \mathcal{C})$ to the category of complexes in $\operatorname{Fun}(\Lambda_p, \mathcal{C})$; in particular, it trivially extends to a functor $j^{\dagger}_p : \mathcal{D}^-(\Lambda_p, \mathcal{C}) \to \mathcal{D}^-(\Lambda_p, \mathcal{C})$ on the derived categories. If $p = 1$, one usually denotes

$$\begin{aligned} H_{\bullet}(\Delta^o, j^* E) &= HH_{\bullet}(E), \\ H_{\bullet}(\Lambda, E) &= HC_{\bullet}(E), \end{aligned} \tag{1.10}$$

and calls them the *Hochschild* and *cyclic* homology of the cyclic object E. The map u is the Connes' periodicity map, and the exact triangle (1.8) is known as the Connes' exact sequence.

1.4 Compatibility for paracyclic embeddings. We finish this section with a somewhat surprising result which shows that for any cyclic object E and any integer p, one can find a p-cyclic object with the same homology (this is the key point in our construction of the non-commutative Cartier operator).

Proposition 1.7. *For any Grothendieck abelian category $\mathcal{C}$ and any object $E \in \operatorname{Fun}(\Lambda, \mathcal{C})$, the natural map*

$$i_{p!} : H_\bullet(\Lambda_p, i_p^* E) \to H_\bullet(\Lambda, E)$$

is a quasiisomorphism.

Proof. By Lemma 1.5 the map $i_{p!}$ is compatible with the periodicity map u; therefore by (1.8) it suffices to prove that the natural map

$$i_{p!} : H_\bullet(\Delta^o, i_p^* E') \to H_\bullet(\Delta^o, E')$$

is a quasiisomorphism, where $E' = j^* E$, and i_p is now treated as the embedding from Δ^o to itself. We will in fact prove this claim for any $E' \in \operatorname{Fun}(\Delta^o, E')$. To do this, it suffices to consider the case when E' goes over a family of generators of the category $\operatorname{Fun}(\Delta^o, \mathcal{C})$; to construct such a family, it suffices to take a generator S of the category $\operatorname{Fun}(\Delta^o, \mathbb{Z}\text{-mod})$ and tensor it with all objects in $\mathcal{C}$ in turn. Thus we are reduced to the case $\mathcal{C} = \mathbb{Z}\text{-mod}$, $E' = S$ is a generator of $\operatorname{Fun}(\Delta^o, \mathbb{Z}\text{-mod})$.

Consider the category of bimodules over the polynomial algebra $B = \mathbb{Z}[t]$ in one variable t over $\mathbb{Z}$, and let S be the standard bar-resolution of the diagonal bimodule B (we have $S([n]) = B^{\otimes(n+1)}$). This is a simplicial abelian group which in addition inherits the standard grading from the polynomial algebra B. It is easy to check that the degree-l component S^l of the simplicial $\mathbb{Z}$-module S is isomorphic to the functor $\mathbb{Z}_{[l]}$ (the standard l-simplex); therefore $S = \bigoplus S^l$ is indeed a generator of the category $\operatorname{Fun}(\Delta^o, \mathbb{Z}\text{-mod})$. On the other hand, it is well-known that the homology of Δ^o with coefficients in a simplicial abelian group can be computed by the standard simplicial complex, and since S is a resolution, we have

$$H_0(\Delta^o, S) = B$$

and $H_i(\Delta^o, S) = 0$ for $i \geq 1$. It remains to notice that S is a simplicial flat B-bimodule, and

$$i_p^* S \cong S \otimes_B S \otimes_B \cdots \otimes_B S,$$

where the product contains p terms. Since $B \otimes_B B \cong B$, we conclude that $i_p^* S$ is another resolution for the same bimodule B, and the natural map $H_\bullet(\Delta^o, i_p^* S) \to H_\bullet(\Delta^o, S)$ is indeed a quasiisomorphism, as required. $\square$

2 Filtrations on cyclic homology.

2.1 The Hodge filtration. Fix a Grothendieck abelian category $\mathcal{C}$, and consider an arbitrary object $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$. Iterating (1.7), one obtains a canonical resolution $E_\bullet$ of the object E; using this resolution, one can refine (1.9) in the following way. Consider the stupid filtration on the complex $E_\bullet$, and let $F^l E_\bullet$, $l \geq 0$, denote the $2l$-th term of this stupid filtration. Then $\langle E, F^\bullet \rangle$ is a filtered complex, and it defines an object in the filtered derived category $\mathcal{D}\mathsf{F}^-(\Lambda_p, \mathcal{C})$. The following is a reformulation of Corollary 1.6.

Lemma 2.1. *The correspondence $E \mapsto \langle E_\bullet, F^\bullet \rangle$ extends to a functor*

$$\mathcal{D}^-(\Lambda_p, \mathcal{C}) \to \mathcal{D}\mathsf{F}^-(\Lambda_p, \mathcal{C}),$$

and we have a functorial exact triangle

$$j_p^\dagger E \longrightarrow E \xrightarrow{u} E[2] \longrightarrow ,$$

where the filtration in the right-hand side is shifted by 1, *and u is the canonical periodicity map.* $\square$

Definition 2.2. The filtration $F^\bullet$ on the object $E \in \mathcal{D}^-(\Lambda_p, \mathcal{C})$ is called the *Hodge filtration*.

By the standard formalism of filtered complexes, the Hodge filtration induces a spectral sequence in homology which starts with $HH_\bullet(E)[v]$, the space of polynomials in one variable $v = u^{-1}$ of degree -2, and converges to $HC_\bullet(E)$. We call it the *Hodge-to-de Rham*, or *Hochschild-to-cyclic* spectral sequence. It is this spectral sequence whose degeneration we are going to study in the rest of the paper.

Assume now that the integer $p \geq 1$ is actually an odd prime, and assume that the category $\mathcal{C}$ is k-linear over a field k of characteristic p. In this case, there is another way of looking at the Hodge filtration. Namely, consider the projection $\pi = \pi_p : \Lambda_p \to \Lambda$ (from now on, we will fix p and drop it from the notation whenever there is no danger of confusion). By definition, for any $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$ we have

$$H_\bullet(\Lambda_p, E) \cong H_\bullet(\Lambda, \mathsf{L}^\bullet \pi_!(E)),$$

where $\mathsf{L}^\bullet\pi_! : \mathcal{D}^-(\Lambda_p, \mathcal{C}) \to \mathcal{D}^-(\Lambda, \mathcal{C})$ is the derived functor of the direct image functor $\pi_!$. We also have $H^\bullet(\Lambda, \mathsf{R}^\bullet\pi_* k) \cong H^\bullet(\Lambda_p, k)$; consider the associated spectral sequence (the Hochschild-Serre spectral sequence for the fibration $\pi : \Lambda_p \to \Lambda$). Recall that the fibers of the fibration $\pi : \Lambda_p \to \Lambda$ are all isomorphic to the groupoid pt_p, so that taking $\mathsf{L}^\bullet\pi_!$ amounts to taking homology of the finite group $\mathbb{Z}/p\mathbb{Z}$. Recall also that the cohomology algebra $H^\bullet(\mathbb{Z}/p\mathbb{Z}, k)$ of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ with coefficients in the trivial module k is the free graded-commutative algebra generated by a generator ε of degree 1 and a generator u of degree 2.

Lemma 2.3. *The graded cyclic k-vector space $\mathsf{R}^\bullet\pi_* k$ is a free module over the cohomology algebra $H^\bullet(\mathbb{Z}/p\mathbb{Z}, k) = k[u]\langle\varepsilon\rangle$; in particular, for any $l \geq 0$ we have $\mathsf{R}^l\pi_* k \cong k \in \operatorname{Fun}(\Lambda, k\text{-mod})$. The Hochschild-Serre spectral sequence for the fibration $\pi : \Lambda_p \to \Lambda$ degenerates at E_3, we have*

$$E_2^{r,s} = H^s(\Lambda, \mathsf{R}^r k) = \begin{cases} k, & s \textit{ is even}, \\ 0, & s \textit{ is odd}, \end{cases}$$

and the differential $d_2 : E_2^{r,s} \to E_2^{r-1,s+2}$ vanishes for even r and induces an isomorphism for odd r.

Proof. To prove the first claim, it suffices to evaluate $\mathsf{R}^\bullet\pi_* k([n])$ for all $[n] \in \Lambda$ and to notice that by base change, we have

$$\mathsf{R}^\bullet\pi_* k([n]) = H^\bullet(\mathbb{Z}/p\mathbb{Z}, k([n])) = H^\bullet(\mathbb{Z}/pZ, k).$$

To compute $E_2^{r,s}$, combine this with Lemma 1.5. Again by Lemma 1.5, the class $\varepsilon \in E_2^{1,0}$ cannot survive in the E_∞-term of the spectral sequence; therefore $d_2(\varepsilon)$ generates $E_2^{0,2}$. Once again by Lemma 1.5, the class $u \in E_2^{2,0}$ must survive in E_∞, so that in particular $d_2(u) = 0$. The claim about d_2 follows by multiplicativity. This implies that $E_3^{r,s} = 0$ unless $s = 0, 1$ and $r = 2l$ is even, and the spectral sequence degenerates at E_3 by dimension reasons. $\square$

Remark 2.4. This Lemma shows, in particular, that Λ_p considered as a gerbe over Λ is *not* split – conversely, its class in $H^2(\Lambda, \mathbb{Z}/p\mathbb{Z})$ is the generator class u.

Lemma 2.5. *For any $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, the complex $j_p^\dagger(E) \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$ consists of objects acyclic for the functors $\pi_!, \pi_* : \operatorname{Fun}(\Lambda_p, \mathcal{C}) \to \operatorname{Fun}(\Lambda, \mathcal{C})$.*

Proof. By the projection formula, it suffices to prove that $j_{p!}(k), j_{p*}(k) \in \mathrm{Fun}(\Lambda_p, k\text{-mod})$ are acyclic for $\pi_!$ and π_*. By Proposition 1.2, this has to be checked for every fiber $\pi_{[n]}$ of the bifibration $\pi : \Lambda_p \to \Lambda$. This fiber is equivalent to pt_p, so that we have to check that $j_{p!}(k)([n])$ and $j_{p*}(k)([n])$ are regular representations of the group $\mathbb{Z}/p\mathbb{Z}$. This is immediate. □

By virtue of Lemma 2.5, one can compute $\mathsf{L}^\bullet\pi_!E$ by using the standard periodic resolution $E_\bullet$ assembled out of the complexes $j_p^\dagger(E)$. This defines a filtration $F^\bullet$ on $\mathsf{L}^\bullet\pi_!E$ and equips it with the structure of a filtered complex in $\mathrm{Fun}(\Lambda, \mathcal{C})$; by abuse of terminology, we will also call the filtration $F^\bullet\mathsf{L}^\bullet\pi_!E$ the *Hodge filtration.*

2.2 The conjugate filtration. It turns out, however, that the complex $\mathsf{L}^\bullet\pi_!E$ carries another canonical filtration, which goes in the opposite direction; we call it the conjugate filtration. We start with the following.

Lemma 2.6. *An object $F \in \mathrm{Fun}(\Lambda_p, k\text{-mod})$ is acyclic for the functor $\pi_!$ if and only if it is acyclic for the functor π_*; moreover, there exists a functorial map $T : \pi_!F \to \pi_*F$ which is an isomorphism for acyclic F. For any k-linear Grothendieck category $\mathcal{C}$, any $E\,\mathrm{Fun}(\Lambda_p, \mathcal{C})$ and any acyclic $F \in \mathrm{Fun}(\Lambda_p, k\text{-mod})$, $F \otimes E \in \mathrm{Fun}(\Lambda_p, \mathcal{C})$ is acyclic for both $\pi_!$ and π_*, and $T : \pi_!(E \otimes F) \to \pi_*(E \otimes F)$ is an isomorphism.*

Proof. As in the proof of Lemma 2.5, it suffices by Proposition 1.2 to check the statement for every fiber $\pi_{[n]}$ of the bifibration $\pi : \Lambda_p \to \Lambda$; since $\pi_{[n]} \cong \mathsf{pt}_p$, this reduces to standard facts about the homology of finite groups – in this case, the group is $\mathbb{Z}/p\mathbb{Z}$. The map T is the Tate trace map: for every $[n] \in \Lambda$, $T([n]) = 1 + \sigma + \cdots + \sigma^{p-1}$ is the averaging over $\mathbb{Z}/p\mathbb{Z}$, where $\sigma \in \mathbb{Z}/p\mathbb{Z}$ is the generator. □

Therefore we can take a left acyclic resolution $I_\bullet$ and a right acyclic resolution $I^\bullet$ of the constant functor $k \in \mathrm{Fun}(\Lambda_p, k\text{-mod})$ and combine them into an unbouded complex $I_\#$ by taking the cone of the natural map $I_\bullet \to kI^\bullet$ (we normalize the grading so that $I_\#$ is an extension of $I_\bullet[1]$ by $I^\bullet$).

Definition 2.7. For k-linear Grothendieck category $\mathcal{C}$ and for any $E \in \mathrm{Fun}(\Lambda_p, \mathcal{C})$, the unbounded complex

$$\pi_\#(E) = \pi_*(E \otimes I_\#) \in D(\Lambda, \mathcal{C})$$

is called the *relative Tate homology complex* of E with respect to $\pi : \Lambda_p \to \Lambda$.

The Tate homology complex $\pi_\#(E) \in D(\Lambda, \mathcal{C})$ is obviously independent of the choice of resolutions $I_\bullet$, $I^\bullet$ and functorial in E. We have a functorial exact triangle

$$\mathsf{L}^\bullet\pi_!(E)[1] \longrightarrow \pi_\#(E) \longrightarrow \mathsf{R}^\bullet\pi_*(E) \longrightarrow$$

in $\mathcal{D}(\Lambda, k\text{-mod})$ and a periodicity map $u : \pi_\#(E) \to \pi_\#(E)[2]$ which is compatible with the periodicity maps on $\mathsf{L}^\bullet\pi_!(E)$ and $\mathsf{R}^\bullet\pi_*(E)$.

Definition 2.8. The *conjugate filtration* $W_\bullet$ on the complex $\pi_\#(E)$ is defined as its canonical truncation

$$W_l\pi_\#(E) = \tau_{-2l-1}\pi_\#(E).$$

This is also functorial and independent of choices. Moreover, by construction we see that W_l for $l \geq 1$ is actually a filtration on $\mathsf{L}^\bullet\pi_!(E)[1] \cong \pi_!(E \otimes I_\bullet)[1]$. We will adopt this point of view to avoid dealing with unbounded complexes. However, for various reasons it will be more convenient to consider also the canonical truncation $\tau_{\leq 0}\pi_\#(E)$ equipped with the filtration induced by $W_\bullet$, and we will denote this filtered complex by

$$\pi_\flat(E) = \tau_{\leq 0}\pi_\#(E).$$

The complex $\pi_\flat(E)$ is an extension of $\mathsf{L}^\bullet\pi_!(E)[1]$ by $\pi_*(E)$.

We note that if we compute the complex $\pi_\#(E)$ by using the standard periodic resolution $E_\#$, then the periodicity map u is induced by an invertible map of complexes $u : E_\# \to E_\#[2]$. We denote the inverse map by

$$v = u^{-1} : E_\#[2] \to E_\#.$$

This map shifts the conjugate filtration by 1 – we actually have a series of maps

$$v : W_lE_\#[2] \to W_{l+1}E_\#. \tag{2.1}$$

2.3 Periodicity in the bar resolutions. To analyse the conjugate filtration, we will need to use several acyclic resolution in Lemma 2.6. We use the machinery of bar resolutions described in Subsection 1.1. Consider the category $\mathrm{Fun}(\Lambda_p, \mathcal{C})$ of p-cyclic objects in a Grothendieck abelian category $\mathcal{C}$. There are two natural exact functors from $\mathrm{Fun}(\Lambda_p, \mathcal{C})$ to itself that can play the role of F in (1.3): firstly, we can take the functor $j_{p!}j_p^*$, secondly, we can take the functor $j_p^\dagger$ (the second one takes values in complexes in $\mathrm{Fun}(\Lambda_p, \mathcal{C})$).

Moreover, in the second case we can take $P = \mathsf{Id}[1]$, with the natural embedding $\mathsf{Id}[1] \to j_{p*}j_p^*[1] \to j_p^\dagger$. This gives three possible functorial resolution related by the following natural maps

$$B_\bullet(j_{p!}j_p^*)(E) \longrightarrow B_\bullet(j_p^\dagger)(E) \longleftarrow B_\bullet(j_p^\dagger, \mathsf{Id}[1])(E). \tag{2.2}$$

All these maps are quasiisomorphisms, and all the complexes are acyclic. To simplify notation, denote $B_\bullet(j_{p!}j_p^*) = B_\bullet^!$, $B_\bullet(j_p^\dagger) = B_\bullet^\dagger$ and $B_\bullet(j_p^\dagger, \mathsf{Id}[1]) = \overline{B}_\bullet^\dagger$. By Lemma 2.5, for any $l \geq 1$, all three objects $B_l^!(E)$, $B_l^\dagger(E)$ and $\overline{B}_l^\dagger(E)$ are acyclic for the functor π_*, so that all three resolutions can be used in Lemma 2.6 to compute the relative Tate homology complex $\pi_\flat(E)$. For any $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$,

$$\pi_*(B_\bullet^!(E)) \cong \pi_*(B_\bullet^\dagger(E)) \cong \pi_*(\overline{B}_\bullet^\dagger(E)) \cong \pi_\flat(E). \tag{2.3}$$

The complex $\overline{B}_\bullet^\dagger(E)$ is nothing but the standard periodic resolution of E composed of the complexes $j_p^\dagger(E)[2l]$, $l \geq 0$, and the filtration on $\overline{B}_\bullet^\dagger(E)$ is the conjugate filtration. The canonical map (1.4) for this complex is a map

$$v : \overline{B}_\bullet^\dagger(E)[2] \to \overline{B}_\bullet^\dagger(E), \tag{2.4}$$

which coincides with the periodicity map (2.1). For the complex $B_\bullet^\dagger(E)$, the map (1.4) is the map

$$\overline{j}_p^\dagger(B_\bullet^\dagger(E))[1] \to B_\bullet^\dagger(E), \tag{2.5}$$

where we denote by $\overline{j}_p^\dagger$ the kernel of the natural map $j_p^\dagger \to \mathsf{Id}$. Since the natural map $\mathsf{Id}[1] \to \overline{j}_p^\dagger$ is a quasiisomorphism, the natural map

$$B_\bullet^\dagger(E) \to \overline{B}_\bullet^\dagger(E)$$

is a quasiisomorphism for every $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$. To see the periodicity map v, we have to compose the map (2.5) with the natural quasiisomorphism $B_\bullet^\dagger[1](E) \to \overline{j}_p^\dagger(B_\bullet^\dagger(E))$ induced by the quasiisomorphism $\mathsf{Id}[1] \to \overline{j}_p^\dagger$.

In Section 3, we will need to use the complex $B_\bullet^!(E)$, and in particular, to interpret the periodicity in terms of this complex. To do this, we note that $\pi_* \circ j_{p*} \cong j_*$, and by Lemma 2.5 and Lemma 2.6 we have $\pi_* \circ j_{p!} \cong \pi_! \circ j_{p!} \cong j_!$. Therefore $\pi_*(j_p^\dagger k) \cong j^\dagger k$, and by adjunction, we have a map

$$\varphi : \pi^*(j^\dagger k) \to j_p^\dagger k. \tag{2.6}$$

Both sides are complexes of length 2 with homology in degree 0 and 1 equal to the constant functor k; the right-hand side is a Yoneda representation of the periodicity class $u \in H^(\Lambda_p, k)$, and the left-hand side represents $\pi^*(u) \in H^2(\Lambda_p, k)$, which is equal to 0 by Lemma 2.3. The map φ is an isomorphism on homology in degree 1, and since $\pi^*(u) = 0$, it is trivial on homology in degree 0. Thus the map φ actually maps $\pi^*(j^\dagger k)$ into $\overline{j}_p^\dagger k \subset j_p^\dagger k$ and thus extends the canonical embedding $k[1] \to \overline{j}_p^\dagger k$. Composing φ with the canonical map (2.5), we extend the periodicity map (2.4) to a map

$$v : \pi^*(j^\dagger(k)) \otimes B_\bullet^\dagger(E)[1] \to B_\bullet^\dagger(E). \tag{2.7}$$

We will now denote by $k(1) \subset j_! k \in \operatorname{Fun}(\Lambda_p, k\text{-mod})$ the kernel of the canonical map $j_! k \to k$, and for every $E \in \operatorname{Fun}(\Lambda, \mathcal{C})$, or $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, we will denote $E(1) = E \otimes k(1)$, resp. $E(1) = E \otimes \pi^* k(1)$. The map φ in (2.6) induces in particular a map $\varphi : E(1) \to j_{p!} j_p^* E$, which actually goes into the kernel of the canonical map $j_{p!} j_p^* E \to E$. Composing this map φ with the canonical map (1.4) for the complex $B_\bullet^!(E)$, we obtain a functorial map

$$\overline{v} : B_\bullet^!(E)(1)[1] \to B_\bullet^!(E). \tag{2.8}$$

Now, the object $k(1) \in \operatorname{Fun}(\Lambda, k\text{-mod})$ is also the cokernel of the canonical map $k \to j_* k$. Therefore by Lemma 1.4 we have

$$H_\bullet(\Lambda, E(1)) \cong H_\bullet(\Lambda, E)[1]$$

for any $E \in \operatorname{Fun}(\Lambda, \mathcal{C})$. More generally, by the projection formula, we have

$$H_\bullet(\Lambda, \pi_*(E(1))) \cong H_\bullet(\Lambda, \pi_*(E)(1)) \cong H_\bullet(\Lambda, \pi_* E)[1] \tag{2.9}$$

for any $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$;

Lemma 2.9. *For any $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, the map*

$$H_\bullet(\Lambda, \pi_* B_\bullet^!(E)(1)[1]) \to H_\bullet(\Lambda, \pi_* B_\bullet^!(E))$$

induced by the map $\overline{v}$ from (2.8) *and the map*

$$H_\bullet(\Lambda, \pi_* \overline{B}_\bullet^\dagger(E)[2]) \to H_\bullet(\Lambda, \pi_* \overline{B}_\bullet^\dagger(E))$$

induced by the periodicity map v from (2.4) *become equal under the identification* (2.9).

Proof. By construction, the diagram (2.2) extends to a diagram

$$\begin{tikzcd} B^!_\bullet(E)[1](1) \arrow[r] \arrow[d, "\overline{v}"'] & B^\dagger_\bullet[1](E) \otimes \pi^*\overline{j}^\dagger k \arrow[d, "v"] & \overline{B}^\dagger_\bullet(E)[2] \arrow[l] \arrow[d, "v"] \\ B^!_\bullet(E) \arrow[r] & B^\dagger_\bullet(E) & \overline{B}^\dagger_\bullet(E), \arrow[l] \end{tikzcd} \tag{2.10}$$

and by the projection formula, it suffices to prove that for any complex E' in $\operatorname{Fun}(\Lambda, \mathcal{C})$ – in particular, for $E' = \pi_* B^!_\bullet(E)$ and such – the natural maps

$$H_\bullet(\Lambda, E'(1)) \longrightarrow H_\bullet(\Lambda, E' \otimes \overline{j}^\dagger(k)) \longleftarrow H_\bullet(\Lambda, E'[1])$$

are quasiisomorphisms. Indeed, the map on the right-side is a quasiisomorphism already in $\operatorname{Fun}(\Lambda, \mathcal{C})$, while the cone of the map on the left-hand side is $H_\bullet(\Lambda, j_* j^* E')$, which is trivial by Lemma 1.4. □

As an application, Lemma 2.9 allows to see the conjugate filtration in the bar resolutions $B^!_\bullet(E)$ and $B^\dagger_\bullet(E)$. Namely, by definition $\overline{B}^\dagger_\bullet(E)$ carries the conjugate filtration $W_\bullet$, and in particular, we have $W_1\overline{B}^\dagger_\bullet(E) = \tau_{\leq -1}\overline{B}^\dagger_\bullet(E)$. Set $W_1 B^!_\bullet(E) = \tau_{\leq -1}\mathsf{B}^!_\bullet(E)$, $W_1 B^\dagger_\bullet(E) = \tau_{\leq -1}\mathsf{B}^\dagger_\bullet(E)$, and note that all the vertical maps in (2.10) are injective. Therefore we can define inductively

$$\begin{aligned} W_l B^!_\bullet(E) &= v\left(W_{l-1} B^!_\bullet(E)[1](1)\right), \\ W_l B^\dagger_\bullet(E) &= v\left(W_{l-1} B^\dagger_\bullet(E)[1](1) \otimes \pi^*\overline{j}^\dagger k\right). \end{aligned}$$

This turns $B^!_\bullet(E)$ and $B^\dagger_\bullet(E)$ into filtered complexes, and all the maps in (2.2) are filtered maps.

Corollary 2.10. *The natural map* $\pi_*(\overline{B}^\dagger_\bullet(E)) \to \pi_*(B^\dagger_\bullet(E))$ *is a filtered quasiisomorphism, while the natural map* $\pi_*(B^!_\bullet(E)) \to \pi_*(B^\dagger_\bullet(E))$ *becomes a filtered quasiisomorphism aftr applying the cyclic homology functor* $HC_\bullet$.

Proof. By Lemma 2.9, it suffices to notice that the map $k[1] \to j^\dagger k$ is a quasiisomorphism in $\operatorname{Fun}(\Lambda, k\text{-mod})$, while the map $k(1) \to j^\dagger k$ becomes a filtered quasiisomorphism after applying $HC_\bullet(-) = H_\bullet(\Lambda, -)$. □

2.4 Tight $\mathbb{Z}/p\mathbb{Z}$-modules. We will now use the bar resolutions to compute, under some assumptions on the p-cyclic object $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, the associated graded quotients of the conjugate filtration on $\pi_\flat(E)$.

Let V be a vector space over the field k – or, more generally, an object in a k-linear Grothendieck category $\mathcal{C}$ – equipped with an action of the group $\mathbb{Z}/p\mathbb{Z}$. Denote by $\mathsf{K}_\bullet\langle V\rangle$ the complex

$$V_{\mathbb{Z}/p\mathbb{Z}} \xrightarrow{T} V^{\mathbb{Z}/p\mathbb{Z}} \tag{2.11}$$

placed in degrees 0 and -1 (here T is the Tate trace map from coinvariants to invariants with respect to the group $\mathbb{Z}/p\mathbb{Z}$). The natural map $V^{\mathbb{Z}/p\mathbb{Z}} \to V \to V_{\mathbb{Z}/p\mathbb{Z}}$ factors through a map

$$\varphi : H_0(\mathsf{K}_\bullet\langle V\rangle) \to H_1(\mathsf{K}_\bullet\langle V\rangle).$$

The complex $\mathsf{K}_\bullet\langle V\rangle$ is actually a piece of the standard periodic complex which computes the homology $H_\bullet(\mathbb{Z}/p\mathbb{Z}, V)$, so that we have

$$\begin{aligned} H_0(\mathsf{K}_\bullet\langle V\rangle) &\cong H_{odd}(\mathbb{Z}/p\mathbb{Z}, V),\\ H_1(\mathsf{K}_\bullet\langle V\rangle) &\cong H_{even}(\mathbb{Z}/p\mathbb{Z}, V),\end{aligned}$$

and the map φ is given by the action of the standard generator ε of the first cohomology group $H^1(\mathbb{Z}/p\mathbb{Z}, k)$.

Definition 2.11. The representation V of the group $\mathbb{Z}/p\mathbb{Z}$ is called *tight* if the map φ is an isomorphism.

For example, a trivial representation k is tight, and so is the regular representation $k[\mathbb{Z}/p\mathbb{Z}]$. In other words, a representation is tight if $\varepsilon : H_{odd}(\mathbb{Z}/p\mathbb{Z}, V) \to H_{even}(\mathbb{Z}/p\mathbb{Z}, V)$ is an isomorphism. We note that since $\varepsilon^2 = 0$, this automatically implies that $\varepsilon : H_{even}(\mathbb{Z}/p\mathbb{Z}, V) \to H_{odd}(\mathbb{Z}/p\mathbb{Z}, V)$ is a trivial map. For any tight $\mathbb{Z}/p\mathbb{Z}$-module V in a category $\mathcal{C}$, we will denote by $\mathsf{I}\langle V\rangle$ the object $H_0(\mathsf{K}_\bullet\langle V\rangle) \cong H_1(\mathsf{K}_\bullet\langle V\rangle) \in \mathcal{C}$.

Assume given an object $E \in \mathrm{Fun}(\Lambda_p, \mathcal{C})$; then $E([m])$ is a $\mathbb{Z}/p\mathbb{Z}$-module in $\mathcal{C}$ for every $[m] \in \Lambda_p$, and all the complexes $\mathsf{K}_\bullet\langle E([m])\rangle$ fit together into a single complex

$$\pi_! E \to \pi_* E,$$

which we denote by $\mathsf{K}_\bullet\langle E\rangle \in \mathrm{Fun}(\Lambda, \mathcal{C})$. The maps φ for various $E([m])$ fit into a single map

$$\varphi = \varphi_E : H_0(\mathsf{K}_\bullet\langle E\rangle) \to H_1(\mathsf{K}_\bullet\langle E\rangle). \tag{2.12}$$

Definition 2.12. An object $E \in \mathrm{Fun}(\Lambda_p, \mathcal{C})$ is called *tight* if the map φ_E is an isomorphism – or, equivalently, if for any $[m] \in \Lambda$ the object $E([m])$ is tight with respect to $\mathsf{pt}_p \times [m] = \pi_{[m]} \subset \Lambda_p$. For a tight object $E \in \mathrm{Fun}(\Lambda_p, \mathcal{C})$, we denote by $\mathsf{I}\langle E\rangle \in \mathrm{Fun}(\Lambda, \mathcal{C})$ the object $H_0(\mathsf{K}_\bullet\langle E\rangle) \cong H_1(\mathsf{K}_\bullet\langle E\rangle)$.

Lemma 2.13. *For any tight $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, the periodicity map v induces a filtered exact triangle*

$$\pi_\flat(E)[1] \xrightarrow{\ v\ } \pi_\flat(E) \longrightarrow \mathsf{K}_\bullet\langle E\rangle \longrightarrow \tag{2.13}$$

in $\mathcal{D}\mathsf{F}^-(\Lambda, \mathcal{C})$, where the conjugate filtration on the left in shifted by 1, and the complex $\mathsf{K}_\bullet\langle E\rangle$ is equipped with the canonical filtration. Moreover, for any $l \geq 1$ the map v induces a filtered exact triangle

$$W_{l+1}\pi_\flat E \longrightarrow W_l\pi_\flat E \longrightarrow j^\dagger\mathsf{I}\langle E\rangle[2l] \longrightarrow\ , \tag{2.14}$$

where again the conjugate filtration on the left is shifted by 1, and the complex $j^\dagger\mathsf{I}\langle E\rangle$ is equipped with a trivial one-step filtration of degree l.

Proof. Recall that we have filtered quasiisomorphisms

$$\pi_\flat(E) \cong \pi_*(\overline{B}^\dagger_\bullet(E)) \cong \pi_*(B^\dagger_\bullet(E)).$$

By definition, $\pi_*(\overline{B}^\dagger_0(E)) \cong \pi_*(E)$, and $H_1(\pi_*(\overline{B}^\dagger_{\geq 1}(E)) \cong \pi_!(E)$, so that we have a natural projection $\pi_*(\overline{B}^\dagger_\bullet(E)) \to \mathsf{K}_\bullet\langle E\rangle$, and it obviously fits into a sequence

$$0 \longrightarrow \pi_*(\overline{B}^\dagger_\bullet(E))[2] \xrightarrow{\ v\ } \pi_*(\overline{B}^\dagger_\bullet(E)) \longrightarrow \mathsf{K}_\bullet\langle E\rangle \longrightarrow 0$$

of filtered complexes. This sequence is not exact in the middle term, but its homology is an acyclic complex with trivial one-step filtration, so that after passing to the filtered derived category we obtain an exact triangle. This is the triangle (2.13). To prove (2.14), we may assume by induction that $l = 1$. The periodicity map v is filtered by construction and identical in degrees $\neq 1$; we have to prove that $\mathsf{gr}^W_1 \pi_\flat(E) \cong j^\dagger\mathsf{I}\langle E\rangle$. Indeed, by the projection formula, the canonical filtered map (2.7) induces a map

$$j^\dagger k \otimes \pi_*(B^\dagger_\bullet(E)) \cong j^\dagger\pi_*(B^\dagger_\bullet(E)) \to \pi_*(B^\dagger_\bullet(E)),$$

which in particular gives a map

$$\varphi : j^\dagger(\mathsf{gr}^W_0 \pi_*(B^\dagger_\bullet(E))) \to \mathsf{gr}^W_1 \pi_*(B^\dagger_\bullet(E)).$$

Since $\mathsf{gr}^W_0 \pi_*(B^\dagger_\bullet(E)) \cong H_0(\mathsf{K}_\bullet\langle E\rangle)$ by (2.13), it suffices to prove that this map φ is a quasiisomorphism. Both sides are complexes with non-trivial homology in degrees 0 and 1 only, and, again by (2.13), the homology in degree 1 on both sides is equal to $H_0(\mathsf{K}_\bullet\langle E\rangle)$ and identified by φ. In degree

0, the homology on the right-hand side is $H_1(\mathsf{K}_\bullet\langle E\rangle)$, and the homology on the left-hand side is again $H_0(\mathsf{K}_\bullet\langle E\rangle)$. It is elementary to check that the induced map $\varphi : H_0(\mathsf{K}_\bullet\langle E\rangle) \to H_1(\mathsf{K}_\bullet\langle E\rangle)$ is the same map as in (2.12); since E is assumed tight, φ is an isomorphism. $\square$

As a corollary, we see that for any tight $E \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, the associated graded quotients $\operatorname{gr}^W_l HC_\bullet(E)$ are isomorphic to the cyclic homology $HC_\bullet(j^\dagger \mathsf{I}\langle E\rangle)$, which is by definition equal to the *Hochschild* homology $HH_\bullet(\mathsf{I}\langle E\rangle)$. This is the linear-algebraic origin of our non-commutative Cartier isomorphism.

3 Conjugate filtration in detail.

We will now investigate the conjugate filtration on cyclic homology in some detail. Unfortunately, it seems that apart from Lemma 2.13, nothing can be said for general p-cyclic objects, even for those which are tight. Thus in general, we can identify the associated graded quotients of the conjugate filtration but cannot study the extension data between these quotients. In order to have any control over the extension data, we need to impose an additional symmetry on our objects, which we will call *polycyclic*.

3.1 Polycyclic groups. For any finite set S, denote by G_S the abelian group $\mathbb{Z}/p\mathbb{Z}[S]$ – the free $\mathbb{Z}/p\mathbb{Z}$-module generated by S. Fix a finite set S and assume given a G_S-module in a k-linear abelian category $\mathcal{C}$ over a field k of characteristic p. For every subset $I \subset S$, denote

$$V_I = H_0(G_I, H^0(G_{\overline{I}}, V)),$$

where $\overline{I} = S \setminus I$ is the complement to $I \subset S$. For any two subsets $I \subset I' \subset S$, we have natural transition maps

$$\tau_{I,I'} : V_{I'} \to V_I \qquad \sigma_{I,I'} : V_I \to V_{I'}$$

induced by the natural embedding $H^0(G_{\overline{I}}, V) \to H^0(G_{\overline{I}'}, V)$ and the trace map $H^0(G_{\overline{I'}}, V) \to H^0(G_{\overline{I}}, V)$. Fix a linear order on S, so that $S = \{i, 1 \leq i \leq n\}$, where $n = |S|$ is the number of elements in S. For any $I \subset S$, $i \in \overline{I}$, let $l(i, I)$ be the number of elements in I which are less than i. Then we define a map

$$d_I = \sum_{i \in \overline{I}} (-1)^{l(i,I)} \tau_{I, I \cup \{i\}} : V_I \to \bigoplus_{i \in \overline{I}} V_{I \cup \{i\}},$$

and if we set

$$\mathsf{K}_l\langle V\rangle = \bigoplus_{|I|=l} V_I, \qquad d_l = \bigoplus_{|I|=l} d_I,$$

then $d_{l+1} \circ d_l = 0$, so that $\langle \mathsf{K}_\bullet\langle V\rangle, d_\bullet\rangle$ becomes a complex of length $|S|+1$. For any $i \in S$, define a map $v_i : \mathsf{K}_\bullet\langle V\rangle \to \mathsf{K}_{\bullet+1}\langle V\rangle$ by

$$\varphi_i = \sum_{i \in I} (-1)^{l(i,\overline{I})} \sigma_{I \setminus \{i\}, I}. \tag{3.1}$$

Then φ_i, $i \in S$ anticommute with each other and with the differential d, so that the homology $H_\bullet(\mathsf{K}_\bullet\langle V\rangle)$ becomes a module over the exterior algebra $\Lambda^\bullet\langle\varphi_1,\dots,\varphi_n\rangle$ generated by $\varphi_1,\dots,\varphi_n$.

Definition 3.1. A G_S-module V is called *tight* if the homology $H_\bullet(\mathsf{K}_\bullet\langle V\rangle)$ is the free $\Lambda\langle\varphi_1,\dots,\varphi_n\rangle$-module generated by $H_0(\mathsf{K}_\bullet\langle V\rangle)$.

It is easy to see that when $n = |S| = 1$, this reduces to Definition 2.11, so that the terminology and notation is consistent with Subsection 2.4.

We will now give a different construction of the complex $\mathsf{K}_\bullet\langle V\rangle$ which is more canonical (and in particular, manifestly independent of the choice of an order on S). Namely, let $\mathcal{C}'$ be the category of G_S-modules in $\mathcal{C}$. For any $i \in S$, let $R_i = k[\mathbb{Z}/p\mathbb{Z}]$ be the regular representation of $\mathbb{Z}/p\mathbb{Z}$ over the field k considered as a G_S-module by means of the natural projection $G_S \to G_{\{i\}} = \mathbb{Z}/p\mathbb{Z}$. Define a functor $F : \mathcal{C}' \to \mathcal{C}'$ by

$$F(V) = \bigoplus_{i \in S} V \otimes R_i.$$

This is obviously an exact functor; the trace projections $R_i \to k$ induce a map $F \to \mathsf{Id}$. Therefore we can set up a bar resolution as described in Subsection 1.1 and obtain a functorial resolution $B_\bullet(F)(V)$. We form a canonical complex $\mathsf{C}_\bullet\langle V\rangle$ by setting

$$\mathsf{C}_\bullet\langle V\rangle_\bullet = H^0(G_S, B_\bullet(F)(V)). \tag{3.2}$$

Explicitly, we have

$$B_l(F)(V) = V \otimes \left(\bigoplus R_i\right)^{\otimes l} = V \otimes \bigoplus_{i_1,\dots,i_l \in S} R_{i_1} \otimes \dots \otimes R_{i_l},$$

and this space is naturally graded by the number of different multiples on the right-hand side: we set

$$B_{l-m,m}(V) = V \otimes \bigoplus_{|\{i_1,\dots,i_l\}|=m} R_{i_1} \otimes \dots \otimes R_{i_l}. \tag{3.3}$$

It is easy to see that this is a bicomplex – the differential d splits into a sum of a component $d^{1,0}$ of bidegree $(1,0)$ and a component $d^{0,1}$ of bidegree $(0,1)$. This grading and the bicomplex structure descend to the complex $\mathsf{C}_\bullet\langle V\rangle$. The grading can be further refined by specifying precisely the subset $\{i_1,\dots,i_l\} \subset S$ – for any $I \subset S$, we set

$$B_{l-|I|,I}(V) = V \otimes \bigoplus_{\{i_1,\dots,i_l\}=I} R_{i_1} \otimes \dots \otimes R_{i_l}.$$

Lemma 3.2. *For every $I \subset S$, the complex $B_{\bullet,I}(V)$ is a free resolution of V considered as a G_I-module. Moreover, we have*

$$H_0(\mathsf{C}_\bullet\langle V\rangle, d^{0,1}) \cong \mathsf{K}_\bullet\langle V\rangle$$

– in other words, $\mathsf{K}_\bullet\langle V\rangle$ is the 0-th homology complex of the bicomplex $\mathsf{C}_\bullet\langle V\rangle$ with respect to the differential $d^{0,1}$.

Proof. Every product

$$R = R_{i_1} \otimes \dots \otimes R_{i_l}$$

is obviously a free G_I-module, where $I = \{i_1,\dots,i_l\}$, while $G_{\overline{I}}$ acts trivially on R. Therefore the complex $B_{\bullet,I}(V)$ indeed consists of free G_I-modules, and moreover,

$$H^0(G_S, B_{\bullet,I}(V)) \cong H^0(G_I, B_{\bullet,I}(H^0(G_{\overline{I}}, V))) \cong H_0(G_I, B_{\bullet,I}(H^0(G_{\overline{I}}, V))).$$

Thus the second claim follows from the first, and to prove the first, it suffices to prove that the complex $B_{\bullet,I}(V)$ is indeed a resolution of V. It obviously suffices to consider $\mathcal{C} = k$-mod, $V = k$ with the trivial G_S-action. By induction on $n = |S|$, it suffices to consider $I = S$, and we may assume the claim proved for all proper subsets $I \subset S$. Then instead of proving that $B_{\bullet,n} \to k$ is a quasiisomorphism, we may prove that $B_\bullet \to k_\bullet$ is a quasiisomorphism, where $k_\bullet$ is the complex given by

$$k_m = \bigoplus_{|I|=m} k, \qquad d = \sum_{i \in \overline{I}} (-1)^{l(i,I)} \operatorname{\mathsf{id}}.$$

This latter complex is easily seen to be acyclic, while the bar-complex $B_\bullet$ is acyclic by construction. This finishes the proof. □

We see that the order on S that we used to construct the complex $\mathsf{K}_\bullet\langle V\rangle$ is an artefact of the particular explicit construction: the bar resolution does not depend on the order and gives the same thing. To see the maps φ_i in this language, we note that the subgroup of G_S-invariants in R_i is the trivial G_S-module $\mathbb{Z}$; this gives a canonical embedding $k \to R_i$ and a functorial map $\varphi_i : \mathsf{Id} \to F$. As in (1.4), this gives rise to functorial maps

$$\varphi_i : B_\bullet(V)[1] \to B_\bullet, \qquad \varphi_i : \mathsf{C}_\bullet\langle V\rangle \to \mathsf{C}_\bullet\langle V\rangle$$

and on the level of homology $H_0(\mathsf{C}_\bullet\langle V\rangle, d^{0,1})$, this is the same map φ_i as in (3.1). We can also collect all the maps φ_i together into a single map

$$\varphi : k[S] \otimes B_\bullet(V)[1] \to B_\bullet(V). \tag{3.4}$$

One final result we will need is the following. Consider the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$, and let it act on $G_S = \mathbb{Z}/p\mathbb{Z}[S]$ by dilations. Assume that a G_S-module V is tight, and that the G_S-action on V is extended to an action of the semidirect product $G_S \rtimes (\mathbb{Z}/p\mathbb{Z})^*$. Moreover, assume that the induced $(\mathbb{Z}/p\mathbb{Z})^*$-action on the homology $H_0(\mathsf{K}_\bullet\langle V\rangle)$ is trivial.

Lemma 3.3. *In the assumptions above, the canonical map*

$$\tau_{[0,l]}\mathsf{C}_\bullet\langle V\rangle^{(\mathbb{Z}/p\mathbb{Z})^*} \to \tau_{[0,l]}\mathsf{K}_\bullet\langle V\rangle$$

is a quasiisomorphism whenever $l \leq 2(p-2)$.

Proof. The canonical maps $\varphi_i : H_\bullet(\mathsf{K}_\bullet\langle V\rangle) \to H_{\bullet+1}(\mathsf{K}_\bullet\langle V\rangle)$ are obviously $(\mathbb{Z}/p\mathbb{Z})^*$-equivariant; therefore in the assumptions above, $(\mathbb{Z}/p\mathbb{Z})^*$ acts trivially on the homology $H_l(\mathsf{K}_\bullet\langle V\rangle)$ for any l. By Lemma 3.2, it suffices to prove that for every $I \subset S$, the natural map

$$H_\bullet(G_I, H^0(G_{\overline{I}}, V))^{(\mathbb{Z}/p\mathbb{Z})^*} \to V_I$$

is a quasiisomorphism in degrees $\leq 2(p-2)$ – in other words, we have to check that $H_l(G_I, H^0(G_{\overline{I}}, V))$ has no $(\mathbb{Z}/p\mathbb{Z})^*$-invariant elements when $0 < l \leq 2(p-2)$. But indeed, by the same assumption of tightness, we have

$$H_l(G_I, H^0(G_{\overline{I}}, V)) \cong V_I \otimes H_l(G_I, k),$$

and $(\mathbb{Z}/p\mathbb{Z})^*$ acts through the second factor on the right-hand side, so that it suffices to check that $H^0((\mathbb{Z}/p\mathbb{Z})^*, H_l(G_I, k)) = 0$ for $0 < l \leq 2(p-1)$. This is obvious: the homology space $H_\bullet(G_I, k)$ is the free module of rank 1 over the free graded-commutative algebra – or rather, coalgebra – $k[v_1, \dots, v_m, \varphi_1, \dots, \varphi_m]$ generated by $m = |I|$ elements $v_1, \dots, v_m$ of degree -2 and m elements $\varphi_1, \dots, \varphi_m$ of degree -1, and all the generators are $(\mathbb{Z}/p\mathbb{Z})^*$-eigenvectors with weight 1. $\square$

Remark 3.4. It is rather unfortunate that we have to define the complex $\mathsf{C}_\bullet\langle V\rangle$ as an explicit complex, by fixing an explicit resolution $B_\bullet(V)$. It would be much nicer to be able to define it as a derived functor of some type. Then $|S| = 1$, it is possible, and we indeed do this in Lemma 2.6: the complex $\mathsf{C}_\bullet\langle V\rangle$ can be equivalently defined as the negative part of the canonical filtration in the Tate homology complex $H_\#(G_S, V)$; if V is tight, the homology of the complex $\mathsf{C}_\bullet\langle V\rangle$ is isomorphic to

$$H_\#(G_S, V)/H^\bullet(G_S, V) = H^\bullet(G_S, V) \otimes_{k[[u]]} k[u^{-1}],$$

where $u \in H^2(G_S, \mathbb{Z})$ is the periodicity generator, and none of this depends on any choices of a free resolution. When $n = |S| > 1$, we still have

$$H_\bullet(\mathsf{C}_\bullet\langle V\rangle) \cong H^\bullet(G_S, V) \otimes_{k[[u_1,\dots,u_n]]} k[u_1^{-1}, \dots, u_n^{-1}],$$

but this is not a good description from the general homological point of view (in particular, it makes no sense in the derived category and it is not sufficiently functorial in S). A derived category interpretation is unknown, and none of the obvious candidates give the correct answer. A similar situation occurs, for instance, in the representation theory of affine Kac-Moody algebras (see e.g. [FF]): one can construct a certain type of homology by an explicit resolution, and while it is intuitively clear that the choice of a resolution should be irrelevant, there is no general categorical framework which makes it precise.

3.2 Polycyclic categories. For any set X, define a natural contravariant functor $X^\# : \Lambda^{opp} \to \text{Sets}$ by

$$X^\#([m]) = \text{Maps}(\Lambda([1],[m]), X) = X^m. \tag{3.5}$$

If $X = G$ is a group, then $G^\#$ can be treated as a functor from Λ^{opp} to the category of groups.

Definition 3.5. The *wreath product* $G \wr \Lambda$ is the fibered category over Λ which is the total space of the functor $G^\#$. The *polycyclic categories* B_p, $p \geq 1$, and B_∞ are the wreath products $\mathsf{B}_p = (\mathbb{Z}/p\mathbb{Z}) \wr \Lambda$, $\mathsf{B}_\infty = \mathbb{Z} \wr \Lambda$.

Explicitly, objects of $G \wr \Lambda$ are $[n]$, $n \geq 1$, and for any $[n], [m]$ the set of morphisms from $[m]$ to $[n]$ is given by

$$\begin{aligned}(G \wr \Lambda)([m],[n]) &= G^m \times \Lambda([m],[n]) \\ &= \left\{\langle f', f\rangle \mid f' \in \text{Hom}^\sigma([m], G), f \in \text{Hom}_{\text{Cycl}}([m],[n])/\sigma\right\},\end{aligned}$$

with composition defined by $\langle g', g\rangle \circ \langle f', f\rangle = \langle (g' \circ f)f', g \circ f\rangle$. Here we recall that Cycl is the category of linearly ordered sets equipped with an order-preserving automorphism σ, for any $l \geq 1$, $[l] \in$ Cycl is the linearly ordered set $\mathbb{Z}$ with $\sigma : x \mapsto x + l$, and we interpret G^m as the set of σ-invariant maps from $[m] = \mathbb{Z}$ to G.

In the particular case $G = \mathbb{Z}$, $G \wr \Lambda = \mathsf{B}_\infty$, we define for any $[m]$, $[n]$ a subset $\overline{\mathsf{B}}_\infty([m], [n]) \subset \mathsf{B}_\infty([m], [n])$ by

- $\langle f', f\rangle \in \overline{\mathsf{B}}_\infty([m], [n])$ if and only if for any l, $0 \leq l < m$, we have

$$0 \leq f(l) + nf'(l) < n. \tag{3.6}$$

It is easy to see that (3.6) is preserved by the composition law, so that it defines a subcategory $\overline{B}_\infty \subset \mathsf{B}_\infty$. Moreover, for any $f \in \mathrm{Hom}_{\mathrm{Cycl}}([m], [n])$ there is exactly one f' such that $\langle f', f\rangle$ satisfy (3.6). We conclude that $\overline{\mathsf{B}}_\infty \cong \Lambda_\infty$, so that we have a canonical embedding $\lambda_\infty : \Lambda_\infty \to \mathsf{B}_\infty$. Reducing this modulo p, we obtain a canonical embedding $\lambda_p : \Lambda_p \to \mathsf{B}_p$.

For any small category Σ equipped with a functor $\Sigma \to \Lambda$, we denote $G \wr \Sigma = \Sigma \times_\Lambda G \wr \Lambda$. In particular, we have the simplicial category Δ^o equipped with the functor $j : \Delta^o \to \Lambda$, and we can form the wreath product $G \wr \Delta^o$. Denote $\Delta^o_p = (\mathbb{Z}/p\mathbb{Z}) \wr \Delta^o$. We have a natural embedding $\Delta^o_p \to \mathsf{B}_p$, and a Cartesian diagram

$$\begin{array}{ccc} \Delta^o \times \mathsf{pt}_p & \xrightarrow{j_p} & \Lambda_p \\ \downarrow & & \downarrow{\scriptstyle \lambda_p} \\ \Delta^o_p & \longrightarrow & \mathsf{B}_p. \end{array} \tag{3.7}$$

But by definition, for any $[m] \in \Delta^o$, the set $\Lambda([1], j([m]))$ has a distinguished element; therefore for any set X the pullback $j^*X^\#$ admits a canonical projection $j^*X^\# \to X_{\Delta^o}$ onto the constant functor $X_{\Delta^o} : \Delta^o \to$ Sets. If $X = \mathbb{Z}/p\mathbb{Z}$, then this is compatible with the group structure and induces a projection $\Delta^o_p \to \Delta^o \times \mathsf{pt}_p$. Denote

$$\overline{\Delta^o}_p = \Delta^o_p \times_{\Delta^o \times \mathsf{pt}_p} \Delta^o.$$

Then the Cartesian diagram (3.7) extends to a diagram

$$\begin{array}{ccccc} \Delta^o & \longrightarrow & \Delta^o \times \mathsf{pt}_p & \xrightarrow{j_p} & \Lambda_p \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \lambda_p} \\ \overline{\Delta^o}_p & \longrightarrow & \Delta^o_p & \longrightarrow & \mathsf{B}_p \end{array} \tag{3.8}$$

with Cartesian squares. By abuse of notation, we will denote the composition $\overline{\Delta^o}_p \to \Delta^o_p \to \mathsf{B}_p$ of the embeddings in the bottom row by the same letter j_p.

Finally, note that the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ acts on $\mathbb{Z}/p\mathbb{Z}$ and consequently on B_p, so that we can form the semidirect product $\widetilde{\mathsf{B}}_p = \mathsf{B}_p \rtimes (\mathbb{Z}/p\mathbb{Z})^*$: it has the same objects $[n]$, $n \geq 1$, we set

$$\widetilde{\mathsf{B}}_p([m],[n]) = \left\{ \langle f, f' \rangle | f \in \mathsf{B}_p([m],[n]), f' \in (\mathbb{Z}/p\mathbb{Z})^* \right\},$$

and the composition law is $\langle g, g' \rangle \circ \langle f, f' \rangle = \langle f'(g)f, g'f' \rangle$. We have a natural embedding $\mathsf{B}_p \subset \widetilde{\mathsf{B}}_p$, and the embedding $\lambda_p : \Lambda_p \to \mathsf{B}_p$ extends to an embedding $\widetilde{\lambda}_p : \Lambda_p \to \widetilde{\mathsf{B}}_p$. We will call $\operatorname{Fun}(\widetilde{\mathsf{B}}_p, \mathcal{C})$ *extended polycyclic category*, and we will call its objects *extended polycyclic*.

3.3 Tight polycyclic objects. For any $E \in \operatorname{Fun}(\mathsf{B}_p, \mathcal{C})$, we can form the bar resolution $B^!_\bullet(\lambda_p^* E) = B_\bullet(j_{p!} j_p^*)(\lambda_p^* E) \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$ as in Subsection 2.3. Applying the base change to the diagram (3.8), we see that the functor $j_{p!} j_p^*$ extends to a functor $j_{p!} j_p^* : \operatorname{Fun}(\mathsf{B}_p, \mathcal{C}) \to \operatorname{Fun}(\mathsf{B}_p, \mathcal{C})$, so that the resolution $B^!_\bullet(\Lambda_p^* E)$ comes from a polycyclic resolution $B^!_\bullet(E) \in \operatorname{Fun}(\mathsf{B}_p, \mathcal{C})$. For any $[n] \in \mathsf{B}_p$, the complex $B^!_\bullet(E)([n])$ is the functorial bar resolution $B_\bullet(E)(E([n]))$ of the polycyclic module $E([n])$ that we have considered in Subsection 3.1. If we apply the direct image functor $\chi_* : \operatorname{Fun}(\mathsf{B}_p, \mathcal{C}) \to \operatorname{Fun}(\Lambda, \mathcal{C})$, then we have

$$\chi_*(B^!_\bullet(E))([n]) \cong \mathsf{C}_\bullet \langle E([n]) \rangle_\bullet,$$

where $\mathsf{C}_\bullet \langle E([n]) \rangle_\bullet$ is as in (3.2). The additional grading (3.3) on $B^!_\bullet(E)([n])$ is not compatible with maps $[n] \to [n']$ in the category Λ. However, one checks easily that the associated filtration is preserved by the maps: if we set

$$F_q B^!_l(E)([n]) = \bigoplus_{m \geq q} B_{m,l-m}(E([n]))$$

for every $[n]$, then these fit together into a subobject $F_q B^!_l(E) \subset B^!_l(E)$. Denote

$$\mathsf{K}_\bullet \langle E \rangle = \chi_*(B^!_\bullet(E)) / (\chi_*(F_1 B^!_\bullet(E)) + d\chi_*(F_1 B^!_\bullet(E))),$$

where d is the differential in the complex $B^!_\bullet(E)$. Then $\mathsf{K}_\bullet \langle E \rangle$ is a well-defined complex in $\operatorname{Fun}(\Lambda, \mathcal{C})$, functorial in E, and Lemma 3.2 immediately shows that we have

$$\mathsf{K}_\bullet \langle E \rangle([n]) = \mathsf{K}_\bullet \langle E([n]) \rangle$$

for any $[n] \in \Lambda$.

Recall that we have defined in Subsection 2.3 a canonical map $\pi^*(j_!k) \to j_{p!}k$ of p-cyclic objects; the reader will check easily that this map is compatible with the polycyclic stucture on $j_{p!}k$ and gives a map $\varphi : \chi^*(j_!k) \to j_{p!}k$ of objects in $\mathrm{Fun}(\mathsf{B}_p, k\text{-mod})$. Composing this with (1.4), we obtain a functorial map

$$\varphi : \chi^* j_! k \otimes B^!_\bullet(E)[1] \to B^!_\bullet(E)$$

for any $E \in \mathrm{Fun}(\mathsf{B}_p, \mathcal{C})$, and as in Subsection 2.3, this allows to introduce the conjugate filtration $W_\bullet$ on $B^!_\bullet(E)$ and on $\chi_*(B^!_\bullet(E)$. We note that if we equip $\mathsf{K}_\bullet\langle E\rangle$ with the canonical filtration, then the natural map $\chi_*(B^!_\bullet(E)) \to \mathsf{K}_\bullet\langle E\rangle$ is a filtered map. By the projection formula, the map φ induces a map

$$\varphi : j_! k \otimes \chi_* B^!_\bullet(E)[1] \to \chi_* B^!_\bullet(E).$$

We note that for any $[n] \in \Lambda$, we have $j_!k([n]) = k[S]$, where the finite set S is the set of maps $\mathrm{Maps}_\Lambda([1],[n])$, and φ evaluated at $[n]$ is just the canonical map (3.4). In particular, φ induces a map

$$\varphi : j_! k \otimes \mathsf{K}_\bullet\langle E\rangle[1] \to \mathsf{K}_\bullet\langle E\rangle.$$

Restricting this to $k(1) \subset j_!k$, we obtain a map

$$v : \mathsf{K}_\bullet\langle E\rangle(1)[1] \to \mathsf{K}_\bullet\langle E\rangle.$$

It turns out that a statement completely analogous to Lemma 2.13, (2.13) is true for the complex $\mathsf{K}_\bullet\langle E\rangle$. Namely, for any $E \in \mathrm{Fun}(\mathsf{B}_p, \mathcal{C})$ and any $[n] \in \mathsf{B}_p$, the object $E([n]) \in \mathcal{C}$ by definition carries a representation of the polycyclic group $(\mathbb{Z}/p\mathbb{Z})^n$ – or, more precisely, of the group $\mathbb{Z}/p\mathbb{Z}[S]$, where $S = \mathrm{Maps}([1],[n])$.

Definition 3.6. The object $E \in \mathrm{Fun}(\mathsf{B}_p, \mathcal{C})$ is called *tight* if

(i) for any object $[n] \in \mathsf{B}_p$, the $(\mathbb{Z}/p\mathbb{Z})^n$-module $E([n])$ is tight in the sense of Definition 3.1, and,

(ii) the object $\lambda_p^* E \in \mathrm{Fun}(\Lambda_p, \mathcal{C})$ is tight in the sense of Definition 2.12, and the natural map
$$\lambda_p^* : \mathsf{I}\langle E\rangle \to \mathsf{I}\langle \lambda_p^* E\rangle$$
is an isomorphism (where we denote $\mathsf{I}\langle E\rangle = H_0(\mathsf{K}_\bullet\langle E\rangle)$).

Remark 3.7. The condition (ii) of Definition 3.6 is probably redundant, but were not able to prove it, and in practical applications, it is very easy to check it by hand.

Lemma 3.8. *Let $E \in \operatorname{Fun}(\mathsf{B}_p, \mathcal{C})$ be tight in the sense of Definition* 3.6. *Then there exists a triangle*

$$\mathsf{K}_\bullet\langle E\rangle(1)[1] \xrightarrow{\ v\ } \mathsf{K}_\bullet\langle E\rangle \longrightarrow \mathsf{K}_\bullet\langle\lambda_p^* E\rangle \longrightarrow$$

which, if we equip all the terms with the canonical filtration, becomes a filtered exact triangle after applying the cyclic homology functor $HC_\bullet$.

Proof. By definition, we have to prove that for any $l \geq 0$, the cyclic homology functor $HC_\bullet$ turns the sequence

$$0 \longrightarrow \mathsf{K}_{l-1}\langle E\rangle(1) \xrightarrow{\ v\ } \mathsf{K}_l\langle E\rangle \longrightarrow \mathsf{K}_l\langle\lambda_p^* E\rangle \longrightarrow 0 \tag{3.9}$$

into an exact triangle. By Corollary 1.6, we may equally well use the Hochschild homology functor $HH_\bullet$. For $l = 0$, the sequence is exact already in $\operatorname{Fun}(\Lambda, \mathcal{C})$ by Definition 3.6 (ii). By Definition 3.1, we have

$$\mathsf{K}_l\langle E\rangle \cong \mathsf{I}\langle E\rangle \otimes \Lambda^l j_! k,$$

and the map v is induced by the exterior multiplication map $m : j_!k \otimes \Lambda^{l-1} j_! k \to \Lambda^l j_1 k$ restricted to $k(1) \subset j_! k$ (here, sadly, Λ with an upper index has to mean exterior power in the sense of the pointwise tensor product in $\operatorname{Fun}(\Lambda, k\text{-mod})$, and we hope that this does not cause any confusion). By Definition 2.11, $\mathsf{K}_l\langle\lambda_p^* E\rangle$ is isomorphic to $\mathsf{I}\langle E\rangle$ for $l = 1$ and trivial for $l \geq 2$. Consider the multiplication map

$$m : \Lambda^{l-1} j_1 k \otimes k(1) \to \Lambda^l j_! k. \tag{3.10}$$

If $l = 1$, then this is just the embedding map $k(1) \to j_! k$; it is injective, its cokernel is by definition the constant functor k, and we conclude that (3.9) is exact for $l = 1$, also already in $\operatorname{Fun}(\Lambda, \mathcal{C})$. For $l \geq 2$, we restrict terms to $\operatorname{Fun}(\Delta^o, k\text{-mod})$ and use the Dold-Thom equivalence DT [DT] between simplicial objects in an abelian category and non-positively graded complexes in the same category. Under this equivalence, $\mathsf{DT}(j^* k(1)) \cong k[1]$, $\mathsf{DT}(j_! k) \cong k[1] \oplus k$, and moreover, $\mathsf{DT}(\Lambda^l j_! k) \cong k[l] \oplus k[l-1]$. Therefore (3.10) would be an isomorphism, were the functor DT compatible with the tensor product. Since it is only compatible with the tensor product up to a quasiisomorphism, we conclude that $\mathsf{DT}(m)$ is a quasiisomorphism. Therefore for any $E' \in \operatorname{Fun}(\Delta^o, \mathcal{C})$, in particular for $E' = j^* \mathsf{I}\langle E\rangle$, the map

$$\mathsf{DT}(m) : \mathsf{DT}(E' \otimes \Lambda^{l-1} j_1 k \otimes k(1)) \to \mathsf{DT}(E' \otimes \Lambda^l j_! k)$$

is a quasiisomorphism. Since for any $E' \in \operatorname{Fun}(\Delta^o, \mathcal{C})$, $H_\bullet(\Delta^o, E')$ is the homology of the complex $\mathsf{DT}(E')$, we conclude that v in (3.9) becomes a quasiisomorphism after applying $H_\bullet(\Delta^o, -) = HH_\bullet(-)$, so that (3.9) indeed becomes exact. $\square$

3.4 Comparison maps. Below in Section 4, in our applications to cyclic homology for associative algebras, the computation at some point passes through a tight polycyclic object E, and it turns out that the associated complex $\mathsf{K}_\bullet\langle E\rangle$ is rather easy to control. However, it is the complex $\pi_\flat\lambda_p^*E$ that is related to the cyclic homology. Both $HC_\bullet(\mathsf{K}_\bullet\langle E\rangle)$ and $HC_\bullet(\pi_\flat\lambda_p^*)$ have very similar structure – in particular, both are equipped with filtrations whose associated graded pieces are the same (and isomorphic to $HH_\bullet(\mathsf{I}\langle E\rangle)$). It would be very nice to know that the two complexes themselves are quasiisomorphic, possibly under additional natural assumptions on E. Unfortunately, we were not able to prove it, and there are reasons to believe that it is not true: the extension data between associated graded pieces of the corresponding filtrations *are* different. In particular, there is no natural map from one complex to the other. We had to settle for a weaker comparison result: there is a third complex which maps both into $\mathsf{K}_\bullet\langle E\rangle$ and into $\pi_\flat\lambda_p^*E$, and after we take cyclic homology, both maps become quasiisomorphisms *in degrees* $\geq -2(p-2)$. Consequently, $\mathsf{K}_\bullet\langle E\rangle$ and $\pi_\flat\lambda_p^*$ do have equal cyclic homology in low degrees, which turns out to be enough in application to the Hodge-to-de Rham degeneration. An instance of this phenomenon occurs already in the commutative situation considered in [DL] (where the characteristic of the base field had to be greater than the dimension of the algebraic variety in question).

The comparison result that we prove is as follows. Assume given a polycyclic object $E \in \operatorname{Fun}(\mathsf{B}_p, \mathcal{C})$ which is tight in the sense of Definition 3.6. Moreover, assume that E comes from an *extended* polycyclic object $E \in \operatorname{Fun}(\widetilde{\mathsf{B}}_p, \mathcal{C})$. It is easy to check that the bar resolution $B^!_\bullet(E)$ together with its conjugate filtration $W_\bullet$ is compatible with the extended polycyclic structure, so that we have a natural comparison map

$$\widetilde{\chi}_*(B^!_\bullet(E)) \to \chi_*(B^!_\bullet(E)) \to \mathsf{K}_\bullet\langle E\rangle, \tag{3.11}$$

which becomes a map of filtered complexes in $\operatorname{Fun}(\Lambda, \mathcal{C})$ if we equip $\mathsf{K}_\bullet\langle E\rangle$ with the canonical filtration. On the other hand, we have a natural filtered map

$$\widetilde{\chi}_*(B^!_\bullet(E)) \to \pi_*(B^!_\bullet(\lambda_p^*E)). \tag{3.12}$$

Let us say that a map $f : E_\bullet \to E'_\bullet$ of (filtered) complexes in $\operatorname{Fun}(\Lambda, \mathcal{C})$ is a (filtered) quasiisomorphism *up to homology* in degrees $\geq m$ if the correponding map $f : HC_\bullet(E_\bullet) \to HC_\bullet(E'_\bullet)$ is a (filtered) quasiisomorphism in degrees $\geq m$.

Proposition 3.9. *In the assumptions above, the comparison maps* (3.11) *and* (3.12) *are filtered quasiisomorphisms up to homology in degrees* $\geq -2(p-2)$.

Proof. The map (3.11) is a quasiisomorphism in degrees $\geq -2(p-2)$ by Lemma 3.3, and since it is compatible with the periodicity map, it is a filtered quasiisomorphism up to homology in the said degrees by Lemma 3.8. The map (3.12) is also compatible with the periodicity map, and by the definition of the conjugate filtration, it suffices to prove that the induced map

$$\mathsf{gr}^W_1 \widetilde{\chi}_*(B^!_\bullet(E)) \to \mathsf{gr}^W_1 \pi_*(B^!_\bullet(\lambda_p^* E))$$

is a quasiisomorphism up to homology in degrees $\geq -2(p-2)$. By definition, both sides are complexes concentrated in degrees ≤ -1. In degree -1, the homology of the left-hand side isomorphic to $j_!j^*\mathsf{I}\langle E\rangle$, while the homology of the right-hand side is $j_!j^*\mathsf{I}\langle\lambda_p^* E\rangle$, and the map is an isomorphism by Definition 3.6 (ii); thus it suffices to prove that $\tau_{\leq -2}\,\mathsf{gr}^W_1 \widetilde{\chi}_*(B^!_\bullet(E))$ and $\tau_{\leq -2}\,\mathsf{gr}^W_1 \pi_*(B^!_\bullet(\lambda_p^* E))$ are trivial up to homology in degrees $\geq -2(p-2)$. For the first complex, we can apply the map (3.11) and deduce the statement. For the second complex, we note that for any tight p-cyclic $E' \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$, by Lemma 2.9 and Corollary 2.10 $\tau_{\leq -2}\,\mathsf{gr}^W_1 \pi_* B^!_\bullet(E')$ is trivial up to homology in all degrees. □

4 Associative algebras.

We now fix a perfect field k of odd positive characteristic p, and we assume given a k-linear Grothendieck abelian category $\mathcal{C}$. Moreover, we assume that $\mathcal{C}$ is equipped with a symmetric tensor product with unit object $k \in \mathcal{C}$.

4.1 Definitions. Let A be an associative unital algebra object in the tensor category $\mathcal{C}$. We define a canonical cyclic object $A_\# \in \operatorname{Fun}(\Lambda, \mathcal{C})$ in the following way. For any $m \geq 1$, we set

$$A_\#([m]) = A^{\otimes m} = \bigotimes_{i \in \overline{[m]}} A,$$

where we number the factors in the tensor product by elements of the set $\overline{[m]} = \Lambda([1], [m])$. For any map $f \in \Lambda([m], [n])$, we set

$$f_\# = \bigotimes_{i \in \overline{[n]}} f_i,$$

where

$$f_i : \bigotimes_{j \in \overline{[f]}^{-1}(i)} A \cong A^{\otimes|\overline{f}^{-1}(i)|} \to A$$

is the multiplication map given by the algebra structure on A, and $\overline{f} : \overline{[m]} \to \overline{[m]}$ is the natural map induced by $f \in \Lambda([m],[n])$ (if $\overline{f}^{-1}(i)$ is empty, we let $A^{\otimes 0} = k$ be the unit object in $\mathcal{C}$, and we take $f^i : k \to A$ to be the unit embedding).

This is a well-defined cyclic object in $\mathcal{C}$, so that we have the Hochschild homology complex $HH_\bullet(A_\#) \in D^-(\mathcal{C})$ and the cyclic homology complex $HC_\bullet(A_\#) \in D^-(\mathcal{C})$.

To proceed further, we specialize to the following situation. Assume fixed a small category Z equipped with a Grothendieck topology J. From now on, and until Section 5, let $\mathcal{C}$ be the category of sheaves of k-vector spaces on $\langle Z, J\rangle$. Moreover, assume given and fixed a cohomological functor $H^\bullet(-)$ from $\mathcal{C}$ to the category of k-vector spaces (for example, this may be the cohomology of the site $\langle Z, J\rangle$, or the cohomology with some fixed supports).

Definition 4.1. The *Hochschild and cyclic homology* of the algebra $A \in \mathcal{C}$ is given by

$$HH_\bullet(A) = H^\bullet(\langle Z, J\rangle, HH_\bullet(A_\#)) \qquad HC_\bullet(A) = H^\bullet(\langle Z, J\rangle, HC_\bullet(A_\#)),$$

where $HH_\bullet(A_\#)$ and $HC_\bullet(A_\#)$ are as in (1.10).

The Hodge and the conjugate filtrations on $HC_\bullet(A_\#)$ induce filtrations on $HC_\bullet(A)$. We will be interested in the conjugate filtration $W_\bullet$. By abuse of notation, for any $l \geq 0$ we will denote

$$\begin{aligned} W_l HC_\bullet(A) &= H^\bullet(\langle Z, J\rangle, W_l \pi_! i_p^* A_\#), \\ \mathsf{gr}_l^W HC_\bullet(A) &= H^\bullet(\langle Z, J\rangle, \mathsf{gr}_l^W \pi_! i_p^* A_\#), \end{aligned}$$

although the map $W_l HC_\bullet(A) \to HC_\bullet(A)$ does not have to be injective, so that $W_\bullet HC_\bullet(A)$ is only a filtration in the generalized sense.

For any $V \in \mathcal{C}$, denote by $V^{(1)} = \mathsf{Fr}^*(V)$ the sheaf V with k-vector space structure twisted by the Frobenius map $\mathsf{Fr} : k \to k$.

Lemma 4.2. *For any $V \in \mathcal{C}$, the tensor power $V^{\otimes p}$ equipped with the natural action of $\mathbb{Z}/p\mathbb{Z}$ by transpositions is tight in the sense of Definition* 2.11*, and there is a canonical isomorphism $\mathsf{I}\langle V^{\otimes p}\rangle \cong V^{(1)}$.*

Proof. Since taking the associated sheaf is an exact functor, it suffices to prove the statement for the trivial topology J on Z, so that $\mathcal{C}$ is the category of presheaves of k-vector spaces on Z. This in turn reduces to proving the statement for the vector spaces $V(X)$ for all objects $X \in Z$. Thus we may assume that Z is trivial and $V = k[S]$ is just a k-vector space with some basis S. Then $V^{\otimes p} = k[S^p] = k[S] \oplus V'$, where $S \subset S^p$ is the diagonal, and $V' = k[S^p \setminus S]$ is the natural complement to $V = k[S] \subset V^{\otimes p}$. The group $\mathbb{Z}/p\mathbb{Z}$ acts freely on $S^p \setminus S$; therefore V' is a free $\mathbb{Z}/p\mathbb{Z}$-module, it is trivially tight, and we have $\mathsf{I}\langle V' \rangle = 0$. The action on $V \subset V^{\otimes p}$ is trivial, so that V also tight, and $\mathsf{I}\langle V \rangle \cong V$. Therefore $V^{\otimes p} = V \oplus V'$ is tight, and $\mathsf{I}\langle V^{\otimes p} \rangle \cong V$. Finally, to construct a functorial isomorphism $V^{(1)} \cong \mathsf{I}\langle V^{\otimes p} \rangle$, we note that the embedding $V \to V^{\otimes p}$ defined by the basis S sends V into the subspace of $\mathbb{Z}/p\mathbb{Z}$-invariant vectors, and moreover, coincides modulo $T(V^{\otimes p})$ with

$$v \mapsto (v \otimes v \otimes \dots \otimes v) \mod T(V^{\otimes p}) \in V^{\otimes p}/T(V^{\otimes p}).$$

This is a well-defined map, it is explicitly indepedent of any bases, it is *a posteriori* additive, and the k-vector space structure on both sides obviously differs by the Frobenius map $\mathsf{Fr} : k \to k$, $\lambda \mapsto \lambda^p$. $\square$

Corollary 4.3. *For any associative unital algebra A in the tensor category $\mathcal{C}$, the object $i_p^* A_\# \in \operatorname{Fun}(\Lambda_p, \mathcal{C})$ is tight, and we have $\mathsf{I}\langle i_p^* A_\# \rangle \cong A_\#^{(1)}$, so that*

$$\mathsf{gr}_l^W \, \mathsf{L}^\bullet \pi_! i_p^* A_\# \cong j^\dagger A_\#^{(1)}[2l] \tag{4.1}$$

for any $l \geq 1$, and $\mathsf{gr}_l^W \, HC_\bullet(A) \cong HH_\bullet(A^{(1)})[2l]$.

Proof. By definition, for any $m \geq 1$ we have

$$i_p^* A_\#([m]) \cong A_\#([pm]) \cong (A_\#([m]))^{\otimes p},$$

this is tight by Lemma 4.2, and $\mathsf{I}\langle i_p^* A_\#([m]) \rangle \cong A_\#([m])^{(1)}$. We leave it to the reader to check that the isomorphism is compatible with the action of maps $f \in \Lambda([m],[n])$, $n, m \geq 1$. The last claim is (2.14) of Lemma 2.13. $\square$

Remark 4.4. The isomorphism (4.1) and the induced isomorphism

$$\mathsf{gr}_l^W \, HC_\bullet(A) \cong HH_\bullet(A^{(1)})$$

is probabely the closest analog of the usual Cartier isomorphism in our non-commutative theory.

4.2 Polycyclic structure. We now note that if the algebra A is equipped with an action of a group G, then the object $A_\# \in \mathrm{Fun}(\Lambda, \mathcal{C})$ has a natural structure of a functor from $G \int \Lambda$ to $\mathcal{C}$. In particular, if $G = \mathbb{Z}/p\mathbb{Z}$, we have a polycyclic object $A_\# \in \mathrm{Fun}(\mathsf{B}_p, \mathcal{C})$. On the other hand, the length-2 complex $\mathsf{K}_\bullet\langle A\rangle$ has a natural structure of a DG algebra in $\mathcal{C}$.

Lemma 4.5. *In the assumptions above, we have*

$$\mathsf{K}_\bullet\langle A_\#\rangle \cong \mathsf{K}_\bullet\langle A\rangle_\#.$$

Proof. Evaluating on $[n] \in \mathsf{B}_p$, we have $A_\#([n]) = A^{\otimes n}$, and the action of the group $(\mathbb{Z}/p\mathbb{Z})^n$ on this object is induced by the $(\mathbb{Z}/p\mathbb{Z})$-action on each factor in $A^{\otimes n} = A \otimes \cdots \otimes A$. Therefore for any $I \subset S$, $S = \Lambda([1],[n])$, we have

$$A_\#([n])_I \cong \left(\bigotimes_{i \in I} H_0(\mathbb{Z}/p\mathbb{Z}, A)\right) \otimes \left(\bigotimes_{i \notin I} H^0(\mathbb{Z}/p\mathbb{Z}, A)\right).$$

This identification is by definition compatible with the differentials and gives an isomorphism $\mathsf{K}_\bullet\langle A_\#([n])\rangle \cong \mathsf{K}_\bullet\langle A\rangle^{\otimes n}$. It remains to check that these isomorphisms are compatible with the maps $[m] \to [n]$; this is easy and left to the reader. □

In particular, for any associative unital algebra A in $\mathcal{C}$, the p-th power $A^{\otimes p}$ has a natural algebra structure, and this algebra $A^{\otimes p}$ is acted upon by the cyclic group $\mathbb{Z}/p\mathbb{Z}$ – and in fact, by the whole symmetric group S_p on p letters. Therefore we can form a polycyclic object $A^{\otimes p}_\# \in \mathrm{Fun}(\mathsf{B}_p, \mathcal{C})$, and this object is in fact extended polycyclic.

Lemma 4.6. *The polycyclic object $A^{\otimes p}_\#$ is tight in the sense of Definition* 3.6*; we have $\lambda_p^* A^{\otimes p}_\# \cong i_p^* A_\#$ and $\mathsf{I}\langle A^{\otimes p}_\#\rangle \cong \mathsf{I}\langle i_p^* A_\#\rangle \cong A^{(1)}_\#$.*

Proof. The isomorphism $\lambda_p^* A^{\otimes p}_\# \cong i_p^* A_\#$ immediately follows from the definitions, and the condition (i) of Definition 3.6 is an immediate corollary of Lemma 4.5. Moreover, by the same lemma we have $\mathsf{I}\langle A^{\otimes p}_\#([n])\rangle \cong \mathsf{I}\langle A^{\otimes p}\rangle^{\otimes n} \cong \mathsf{I}\langle A^{\otimes p}\rangle_\#([n])$; together with Corollary 4.3, this gives an isomorphism $\mathsf{I}\langle A^{\otimes p}_\#\rangle \cong \mathsf{I}\langle i_p^* A_\#\rangle \cong A^{(1)}_\#$, which yields Definition 3.6 (ii). □

Definition 4.7. For any $V \in \mathcal{C}$, denote $V^\dagger = \mathsf{K}_\bullet\langle V^{\otimes p}\rangle$.

Lemma 4.8. *Assume that for an associative algebra A in $\mathcal{C}$, the DG algebra $A^\dagger$ is formal – in other words, $A^\dagger$ is quasiisomorphic to its homology, the trivial square-zero extension of $A^{(1)}$ by $A^{(1)}[1]$. Then the spectral sequence for the conjugate filtration computing $W_1 HC_\bullet(A)$ degenerates up to the term E^{p-2}.*

Proof. Since the conjugate filtration is periodic, it suffices to analyse the top terms of the spectral sequence – that is, we have to show that

$$W_{[1,(p-2)]}HC_\bullet(A) \cong \bigoplus_{1 \leq l \leq p-2} \mathsf{gr}^W_l HC_\bullet(A).$$

In other words, we have to analyse the cyclic homology of $W_{[1,(p-2)]}\pi_\flat i_p^* A_\#$. By Proposition 3.9, we may replace it with $\tau_{[-1,-(p-2)]}\mathsf{K}_\bullet\langle A_\#^{\otimes p}\rangle$, and by Lemma 4.6, this is isomorphic to $\tau_{[-1,-(p-2)]}A^\dagger_\#$. In other words, we have to show that the canonical filtration on the complex $A^\dagger_\#$ splits in a certain range of degrees. But by assumption, we have $A^\dagger \cong A^{(1)} \oplus A^{(1)}[1]$ – which means that the complex $A^\dagger_\#$ is quasiisomorphic to the sum of its homology in all degrees. $\square$

4.3 Splittings. We will now study the formality of the DG algebra $A^\dagger$. To do this, we need to refine (and explain) Lemma 4.2.

Let V be a k-vector space. Denote by

$$\rho^0 : V \to H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \tag{4.2}$$

the map which sends v to $v \otimes \dots \otimes v$. This map, although not additive, is functorial with respect to V. Therefore for any small category Z and a presheaf $V \in \mathrm{Fun}(Z^o, k\text{-mod})$ of k-vector spaces on Z, we have a natural map $\rho : V \to H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$. In particular, we may take $Z = \Delta$, the category of linearly ordered finite sets. By the Dold-Thom Theorem [DT], the category $\mathrm{Fun}(\Delta^o, k\text{-mod})$ is equivalent to the category of non-positively graded complexes of k-vector spaces. Take a k-vector space V, consider the complex $V[1]$, denote the associated simplicial vector space by $V(1)$, and consider the map

$$\rho^1 : V(1) \to H^0(\mathbb{Z}/p\mathbb{Z}, V(1)^{\otimes p}).$$

This map extends to an *additive* map

$$\rho^1 : k[V(1)] \to H^0(\mathbb{Z}/p\mathbb{Z}, V(1)^{\otimes p}). \tag{4.3}$$

Apply now the Dold-Thom equivalence DT. The left-hand side becomes the standard bar complex $C_\bullet(V,k)$ which computes the homology of the vector space V – considered as an abelian group – with coefficients in the field k. One the other hand, one checks easily that we have an isomorphism of $(\mathbb{Z}/p\mathbb{Z})$-modules

$$\mathsf{DT}_\bullet(V(1)^{\otimes p}) \cong V^{\otimes p} \otimes \mathsf{DT}_\bullet(k(1)^{\otimes p}),$$

and the complex $\mathsf{DT}(k(1)^{\otimes p})$ is quasiisomorphic to $k[p]$ and concentrated in degrees ≤ -1, $\mathsf{DT}_1(k(1)^{\otimes p})$ is the trivial $(\mathbb{Z}/p\mathbb{Z})$-module k, and all the $\mathsf{DT}_l(k(1)^{\otimes p})$, $l \geq 2$ are regular representations of $(\mathbb{Z}/p\mathbb{Z})$. Therefore we can take the standard periodic resolution $I_\#(k)$ of the trivial $(\mathbb{Z}/p\mathbb{Z})$-module k, set $I_\flat(k) = \tau_{\leq 1} I_\#(k)$ as in Subsection 2.2, and choose a $(\mathbb{Z}/p\mathbb{Z})$-module map $\psi : \mathsf{DT}(k(1)^{\otimes p}) \to I_\flat(k)[2]$ which is a quasiisomorphism in degrees $\geq -p$. Composing this with $\mathsf{DT}(\rho)$, we obtain a canonical map

$$\begin{aligned} C_\bullet(V,k) \to H^0(\mathbb{Z}/p\mathbb{Z}, DT_\bullet(V(1)^{\otimes p})) \to \\ \to H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p} \otimes I_\flat(k))[2] \cong H^0(\mathbb{Z}/p\mathbb{Z}, I_\flat(V))[2]. \end{aligned} \tag{4.4}$$

Note that that the only choice of the whole construction was the choice of the map ψ, which does not depend on V at all. Therefore (4.4) is completely functorial in V. In particular, we may just as well let V be a presheaf on some category Z, and moreover, by applying the associated sheaf functor we may even consider sheaves with respect to some topology J.

Truncating the right-hand side of (4.4), we obtain a canonical map

$$C_\bullet(V,k) \to V^\dagger[1] \tag{4.5}$$

which is again completely functorial in V. In degree 1, this is the canonical map $V \cong H_1(V,k) \to H_0(V^\dagger)$ constructed in Lemma 4.2. In degree 2, we obtain a canonical cocycle $\rho_V \in C^2(V, H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}))$.

Definition 4.9. Denote by $\widetilde{V}$ the group obtained as an extension of V by $H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$ given by the cocycle ρ.

One checks easily that the 2-cocycle ρ_V is in fact symmetric, so that the group $\widetilde{V}$ is commutative. The group $\widetilde{V}$ is *not* a k-vector space and not a group of characteristic p: multiplication by p is given by the map

$$\widetilde{V} \to V \to H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \subset \widetilde{V}.$$

Assume from now on that the field k is perfect; then one checks easily that $\widetilde{V}$ is in fact a module over the second Witt vectors ring $W_2(k)$. Since the cocycle ρ_V comes from a map of complexes (4.5), it reduction modulo $V \subset H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$ is the cocycle for the extension

$$H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \to V$$

split as a map of sets by (4.2)), so that we have $\widetilde{V}/p \cong H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$.

The construction of the cocycle φ_V is also compatible with tensor products in the following sense.

Lemma 4.10. *Assume given two k-vector spaces V, W, and let m be the natural map*

$$H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \otimes H^0(\mathbb{Z}/p\mathbb{Z}, W^{\otimes p}) \to H_0(\mathbb{Z}/p\mathbb{Z}, (V \otimes W)^{\otimes p}).$$

Then $m(\rho_V \otimes \rho^0_W) = \rho_{V\otimes W}$, where ρ^0_W is the map ρ^0 in (4.2) *for the vector space W.*

Proof. To obtain $\rho_V \otimes \rho^0_W$, one uses the same procedure as for the cocycle ρ_V, but for the simplicial abelian group $V(1)\otimes W$ instead of $V(1)$. The map m is induced by the natural map $V(1) \otimes W \to (V \otimes W)(1)$. Since (4.2) is functorial with respect to any maps of simplicial groups, we get the desired compatibility. $\square$

As before, all of the above obviously works not only for k-vector spaces, but also for presheaves and sheaves of k-vector spaces on a site $\langle Z, J\rangle$. Take a k-vector space or a sheaf V. Denote by $\widetilde{V^\dagger}$ the complex

$$H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \xrightarrow{T} \widetilde{V}$$

placed in degrees 0 and -1. This is a complex quasiisomorphic to $V^{(1)}$, and we have a natural map $\widetilde{V^\dagger} \to V^\dagger$ which induces an isomorphism on H_0. If $V = A$ is an associative unital algebra, then by Lemma 4.10 $\widetilde{A^\dagger}$ is a DG algebra, and the canonical map $\widetilde{A^\dagger} \to A^\dagger$ is a DG algebra map. Adding the tautological embedding $A^{(1)}[1] \cong H_1(A^\dagger)[1] \to A^\dagger$, we obtain a DG algebra quasiisomorphism

$$\widetilde{A^\dagger} \oplus A^{(1)}[1] \to A^\dagger.$$

Therefore the DG algebra $A^\dagger$ is *always* quasiisomorphic to the sum of tis homology, but in the wrong category: $\widetilde{A^\dagger}$ is a DG algebra over $W_2(k)$, not over k.

Lemma 4.11. *Assume that there exists an associative flat $W_2(k)$-algebra $\widetilde{A}$ in* $\mathrm{Shv}(Z, J)$ *such that $\widetilde{A}/p \cong A^{(1)}$. Then the DG k-algebra $A^\dagger$ is formal.*

Proof. We have two extensions of the algebra $A^{(1)}$: $\widetilde{A}$ and $\widetilde{A^\dagger}$. Let $\overline{A^\dagger}$ be their Baer difference – that is, the middle cohomology of the complex

$$A^{(1)} \longrightarrow \widetilde{A} \oplus \widetilde{A^\dagger} \longrightarrow A^{(1)},$$

where the left-hand side map is the sum, and the right-hand side map is the difference of the natural maps. Then $\overline{A^\dagger}$ is a DG algebra over $W_2(k)$, and moreover, p acts trivially on it, so that $\overline{A^\dagger}$ is in fact a DG algebra over k. On the other hand, $\overline{A^\dagger}$ is quasiisomorphic to $A^{(1)}$, and we have a natural map $\overline{A^\dagger} \to A^\dagger$. Adding the embedding $A^{(1)}[1] \to A^\dagger$, we obtain the desired quasiisomorphism $A^{(1)} \oplus A^{(1)}[1] \cong \overline{A^\dagger} \oplus A^{(1)}[1] \to A^\dagger$. □

Remark 4.12. One can show without much difficulty that the converse to this statement is also true: liftings of V to a flat $W_2(k)$-module ar in functorial one-to-one correspondence with splittings of the complex $V^\dagger$ (understood in appropriate way), and this is compatible with algebra structures, if they are present. We do not go into this to save space.

Remark 4.13. It is very interesting to repeat our construction of the map ρ^1 in (4.3) for different shifts – one takes $V[l]$, $l \geq 0$, instead of $V[1]$, and obtains a map ρ^l. If one goes to the limit $l \mapsto \infty$, then the left-hand side of (4.3) becomes the *stable* homology of the group V – this is a complex which depends funtorially on V, and whose homology is isomorphic to V tensored with the dual to the Steenrod algebra. The right-hand side stabilizes in a straightforward manner and becomes quasiisomorphic to the truncated Tate homology $H_{\geq -1}(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$. The limit map ρ^∞ is essentially just the Steenrod p-th power map. The cocycle ρ_V survives in the stable situation and becomes the Bokstein homomorphism (this is the topological underlying reason for Lemma 4.11 and Remark 4.12). We note that the structure of Steenrod algebra is well-known; in particular, after the Bokstein homomorphism class in degree 1, there is a non-trivial class in degree p (and in other higher degrees). We suspect that ρ^∞ sends these classes to some non-trivial classes in $H_\bullet(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$, so that even if V is lifted to a $W_2(k)$-module, this Tate homology complex does not split as a whole in a functorial way – it only splits in degrees $\geq -(p-1)$. Thus the restriction on degree in Lemma 4.8 and Proposition 3.9 is unavoidable: this is how things really are. We note that our trick of using the additional $(\mathbb{Z}/p\mathbb{Z})^*$-symmetry is lifted out the standard computation of Steenrod powers found in any topology textbook.

5 Hodge to de Rham degeneration.

We are now ready to study the Hodge-to-de Rham spectral sequence and its degeneration. Assume given a field k and a small category Z equipped with a Grothendieck topology J. Denote by $\mathcal{C}$ the category of sheaves of k-vector spaces on $\langle Z, J\rangle$. Assume also given a cohomological functor $H^{\bullet}(-)$ from $\mathcal{C}$ to the category of k-vector spaces.

We start with the positive characteristic case.

Theorem 5.1. *Assume given an associative unital algebra $A \in \mathcal{C}$ over a field k in the category $\mathcal{C}$ of sheaves of k-vector spaces on a site $\langle Z, J\rangle$. Assume that k is a perfect field of positive odd characteristic $\operatorname{\mathsf{char}} k = p > 2$. Moreover, assume that*

(i) *the diagonal A-bimodule A admits a finite flat resolution of length n,*

(ii) *for some integer $m > 0$ and every sheaf $E \in \mathcal{C}$, the cohomology groups $H^l(\langle Z, J\rangle, E)$ are trivial whenever $l > m$,*

(iii) *we have $p > n + m$, and*

(iv) *for any integer l, the Hochschild homology group $HH_l(A)$ is a finite-dimensional vector space over k.*

Finally, assume that there exists a flat algebra $\widetilde{A} \in \operatorname{Shv}(Z, J)$ over the ring $W_2(k)$ of second Witt vectors of the field k such that $\widetilde{A}/p \cong A$.

Then the Hochschild-to-cyclic spectral sequence $HH_{\bullet}(A)[v] \Rightarrow HC_{\bullet}(A)$ degenerates.

Lemma 5.2. *In the assumptions of Theorem 5.1, the spectral sequence associated to the conjugate filtration on $W_1HC_{\bullet}(A)$ degenerates, so that we have an isomorphism $W_1HC_{\bullet}(A) \cong HH_{\bullet}(A^{(1)})[v]$.*

Proof. By Lemma 4.8, the conjugate spectral sequence for $W_1HC_{\bullet}(A)$ degenerates up to term E^{p-2}. By the assumption (iii) of Theorem 5.1, it degenerates in all the following terms for dimension reasons. $\square$

We now recall that by definition, we have an exact triangle

$$(5.1) \quad W_1HC_{\bullet}(A) \longrightarrow HC_{\bullet}(A) \longrightarrow H^{\bullet}(\langle Z, J\rangle, HC_{\bullet}(\overline{A}_{\#})) \longrightarrow$$

of complexes of k-vector spaces, where the cyclic object $\overline{A}_{\#} \in \operatorname{Fun}(\Lambda, \mathcal{C})$ is the quotient

$$\overline{A}_{\#} = \pi_! i_p^* A_{\#} / A_{\#}^{(1)}.$$

Lemma 5.3. *In the assumptions of Theorem* 5.1*, the Hochschild homology sheaf* $HH_l(\overline{A}_\#) \in \mathcal{C}$ *is trivial whenever* $l > mp$.

Proof. As in the proof of Lemma 4.2, we may assume that $\langle Z, J\rangle$ is trivial, so that $\mathcal{C}$ is the category of k-vector spaces. Moreover, the homology of $A^{(1)}_\#$ is bounded by Theorem 5.1 (i), so that we might just as well prove our statement for $\pi_! i_p^* A_\#$ instead of its quotient $\overline{A}_\#$.

For any A-bimodule M, denote by $M_\natural$ the cokernel of the commutator map $A \otimes M \to M$ – equivalently, we have $M_\natural = M \otimes_{A \otimes A^{opp}} A$. Moreover, denote

$$M_{p\natural} = (M \otimes_A M \otimes_A \cdots \otimes_A M)_\natural,$$

where we have p multiples on the right-hand side. The reader will check easily that this is construction is actually cyclically symmetric in all p multiples (but since paper is two-dimensional, we cannot represent this symmetry in convenient notation). Moreover, $M \mapsto M_{p\natural}$ is obviously functorial in M.

Recall that by the Dold-Thom Theorem [DT], for any abelian category $\mathcal{C}$ the category $\operatorname{Fun}(\Delta^o, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ is equivalent to the category of complexes in $\mathcal{C}$ concentrated in non-positive degree. For any $M \in \operatorname{Fun}(\Delta^o, \mathcal{C})$, the corresponding complex represents the homology $H_\bullet(\Delta^o, M)$. If the category $\mathcal{C}$ is equipped with a symmetric tensor product, then the category $\operatorname{Fun}(\Delta^o, \mathcal{C})$ also has a natural symmetric tensor structure, and this structure is homotopy-invariant in the following strong sense. For any group G, denote by $\mathcal{C}[G]$ the category of objects in $\mathcal{C}$ equipped with an action of G. For any $M \in \operatorname{Fun}(\Delta^o, \mathcal{C})$, the p-fold tensor product $M^{\otimes p}$ equipped with the transposition action of the symmetric group S_n is naturally an object in $\operatorname{Fun}(\Delta^o, \mathcal{C}[S_n])$ – or, by Dold-Thom equivalence, a complex in $\mathcal{C}[S_n]$. Then if $M_0, M_1 \in \operatorname{Fun}(\Delta^o, \mathcal{C})$ are homotopy-equivalent – that is, the corresponding complexes are equivalent up to chain homotopy of complexes – then the complexes $M_o^{\otimes p}$, $M_1^{\otimes p}$ are homotopy-equivalent in $\mathcal{C}[S_n]$.

If the category $\mathcal{C}$ is tensor but not symmetric, there is no natural group action on tensor powers $M^{\otimes p}$. However, if $\mathcal{C} = A$-bimod is the category of A-bimodules, then for any simplicial A-bimodule $M \in \operatorname{Fun}(\Delta^o, \mathcal{C})$ we can form $M_{p\natural} \in \operatorname{Fun}(\Delta^o, \mathcal{C}[\mathbb{Z}/p\mathbb{Z}])$, and this is also homotopy-invariant in the above sense.

Take now a flat resolution $P_\bullet$ of the diagonal A-bimodule A of length m whose existence is assumed by Theorem 5.1 (i), and treat it as a simplicial A-bimodule $P_\bullet \in \operatorname{Fun}(\Delta^o, A\text{-bimod})$. Moreover, let $P'_\bullet \in \operatorname{Fun}(\Delta^o, A\text{-bimod})$ be the standard simplicial bar-resolution of A. Then by definition of the cyclic object $A_\# \in \operatorname{Fun}(\Lambda, k\text{-mod})$, we have

$$j^* i_p^* A_\# \cong \left(P'_\bullet\right)_{p\natural} \in \operatorname{Fun}(\Delta^o, k[\mathbb{Z}/p\mathbb{Z}]\text{-mod}) = \operatorname{Fun}(\Delta^o \times \mathsf{pt}_p, k\text{-mod}),$$

and since $P_\bullet$ must be homotopy-equivalent to $P'_\bullet$, this is homotopy-equivalent to $(P_\bullet)_{p\natural}$ as a complex of $k[\mathbb{Z}/p\mathbb{Z}]$-modules. We conclude that

$$j^* \pi_! i_p^* A_\# \cong H_0\left(\mathbb{Z}/p\mathbb{Z}, \left(P'_\bullet\right)_{p\natural}\right)$$

is homotopy equivalent to $H_0(\mathbb{Z}/p\mathbb{Z}, (P_\bullet)_{p\natural}) \in \operatorname{Fun}(\Delta^o, k\text{-mod})$. Applying the Dold-Thom equivalence, we see that the latter complex is manifestly trivial in degrees $> mp$. $\square$

Proof of Theorem 5.1. Consider the exact triangle (5.1). We have

$$HC_\bullet(A) \cong H^\bullet(\langle Z, J\rangle, H_\bullet(\Lambda_p, i_p^* A_\#)) \cong H^\bullet(\langle Z, J\rangle, H_\bullet(\Lambda, \mathsf{L}^\bullet \pi_! i_p^* A_\#)),$$

and by Lemma 2.3, the generator $u \in H^2(\Lambda, k)$ of the cohomology algebra $H^\bullet(\Lambda, k)$ acts trivially on the right-hand side. Therefore the cokernel of the natural map $W_1 HC_\bullet(A) \to HC_\bullet(A)$ lies inside the subspace in $H^\bullet(\langle Z, J\rangle, HC_\bullet(\overline{A}_\#))$ annihilated by u. This subspace is a quotient of $H^\bullet(\langle Z, J\rangle, HH_\bullet(\overline{A}_\#))$, and by Lemma 5.3, the latter space is trivial in degrees $\ll 0$. We conclude that the map $W_1 HC_\bullet(A) \to HC_\bullet(A)$ is trivial in degrees $\ll 0$. By Lemma 5.2, this implies that

$$\dim HC_l(A) \leq \bigoplus_m \dim HH_{l+2m}(A^{(1)}) = \bigoplus_m \dim HH_{l+2m}(A)$$

for $l \ll 0$ (the sum on the right-hand side is over all integers m, and it is bounded and finite by our assumptions). By the standard criterion [D], this implies that the Hochschild-to-cyclic spectral sequence $HH_\bullet(A)[v] \Rightarrow HC_\bullet(A)$ degenerates in degrees $\ll 0$. Since the sequence is periodic, it degenerates everywhere. $\square$

This finishes the positive characteristic case. The characteristic 0 statement would follow from this by a standard and well-known procedure, but there is a problem: interesting non-commutative algebras are often not Noetherian. Therefore it is unclear how to "spread out" an algebra given over a field of characteristic 0 so that it acquires fibers over fields of positive characteristic. In this paper, we have decided to settle for the following comporimise: we give a statement "in mixed characteristic" – that is, we assume that the algebra is already defined and flat over a $\mathbb{Z}$-algebra of finite type

– and we give a second statement which shows that such a such a $\mathbb{Z}$-model exists in the Noetherian situation. This is enough for some applications but certainly not for all of them. This is a technical limitation, since the finiteness assumptions we impose in order to have the degeneration should also be sufficient to construct a $\mathbb{Z}$-model. In a separate paper, we will handle this problem by using the A_∞-methods.

Theorem 5.4. *Assume given an associative unital algebra $A \in \mathrm{Shv}(Z, J)$ flat over a intergal domain O of finite type over $\mathbb{Z}$. Moreover, assume that*

(i) *the diagonal A-bimodule A admits a finite flat resolution,*

(ii) *for some integer $m > 0$ and every sheaf $E \in \mathrm{Shv}(Z, J)$ of K-vector spaces on $\langle Z, J\rangle$, the cohomology groups $H^l(\langle Z, J\rangle, E)$ are trivial whenever $l > m$, and*

(iii) *for any integer l, the Hochschild homology group $HH_l(A)$ is a finitely generated O-module.*

Then the Hochschild-to-cyclic spectral sequence $HH_\bullet(A)[v] \Rightarrow HC_\bullet(A)$ degenerates.

Proof. Denote by n the length of the resolution in (i). It suffices to show that the differentials in the spectral sequence vanish after reduction modulo all maximal ideals $\mathfrak{m} \subset O$, and moreover, it suffices to consider all the maximal ideals outside of a closed subset in $\operatorname{Spec} O$. Therefore we may assume that $\mathsf{char}\, O/\mathfrak{m} > n + m$. Since O is of finite type over $\mathbb{Z}$, the field $O/\mathfrak{m}$ is perfect, and we are done by Theorem 5.1. $\square$

Theorem 5.5. *Assume given a field K of characteristic 0, and assume given an associative unital Noetherian K-algebra $A \in \mathrm{Shv}(Z, J)$. Moreover, assume that*

(i) *the diagonal A-bimodule A admits a finite flat resolution,*

(ii) *for some integer $m > 0$ and every sheaf $E \in \mathrm{Shv}(Z, J)$ of K-vector spaces on $\langle Z, J\rangle$, the cohomology groups $H^l(\langle Z, J\rangle, E)$ are trivial whenever $l > m$, and*

(iii) *for any integer l, the Hochschild homology group $HH_l(A)$ is a finite-dimensional K-vecotr space.*

Then the Hochschild-to-cyclic spectral sequence $HH_\bullet(A)[v] \Rightarrow HC_\bullet(A)$ degenerates.

Proof. Since A is Noetherian, it is in particular generated by a finite number of local sections with a finite number of relations. Therefore there exists a subalgebra $O \subset K$ of finite type over $\mathbb{Z}$ such that there exists an O-algebra $A_O \in \operatorname{Shv}(Z, J)$ with $A \cong A_O \otimes_O K$. Again since A is Noetherian, we may localize O so that it still is of finite type over $\mathbb{Z}$, but A_O satisfies all the assumptions of Theorem 5.4. □

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