Derived equivalences by quantization

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Introduction

The goal of this paper is to find generalization of the so-called *McKay equivalence* of derived categories, as described in M. Reid's well-known preprint [R]. We briefly recall the setup. One considers a vector space V over a field of characteristic 0 and a finite subgroup $G \subset SL(V)$. Moreover, one assumes given a smooth crepant resolution X of the quotient variety

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$Y = V/G$ (crepant in this context means that the canonical bundle K_X is trivial). In these assumptions, the *McKay correspondence* predicts certain numerical data of X, such as its Betti numbers, starting from the combinatorics of G-action on V. This was described in [R], with precise conjectures (which were later proved, [Ba], [DL]). However, M. Reid went further: in trying to explain the geometry underlying his numerical predictions, he proposed a series of statements, each one stronger than the preceding one. The strongest of them was the following: one conjectures that the derived category $\mathcal{D}^b_{coh}(X)$ of coherent sheaves on X is equivalent to the derived category of G-equivariant coherent sheaves on V.

Five years ago, a spectacular proof of this conjecture was given in [BKR], under assumption $\dim V = 3$, and for some specific crepant resolution X (whose existence the authors also prove). Since then, there has been a lot of progress in $\dim 3$, and some partial results on adapting the methods of [BKR] to some cases of higher dimension ([Go]). However, to the best of our knowledge, the only relatively general result in higher dimension was obtained very recently by the author jointly with R. Bezrukavnikov, [BK2]. We establish the McKay equivalence in arbitrary dimension, but under one additional assumption: we require V to be a *symplectic* vector space, and we want the finite group G to preserve the symplectic form.

But there was a different development still in dimension 3. It was realized by T. Bridgeland [Br] that the methods of [BKR] can work in a more general situation. One still considers a crepant resolution X of a singular affine algebraic variety Y, but one no longer requires Y to be a quotient singularity. In [Br], Bridgeland considers instead a so-called *small contraction* $X \to Y$ of a smooth 3-dimensional manifold X with trivial K_X – that is, he assumes given a proper birational map $X \to Y$ whose only exceptional fibers are curves. Bridgeland's results were extended and re-cast in a different language by M. Van den Bergh [V1], and in this form, they are very similar to the McKay equivalence: the derived category $D^b_{coh}(X)$ of coherent sheaves on X is shown to be equivalent to the derived category of finite-generated left modules over a certain non-commutative algebra R. The algebra R has a structure formalized by Van den Bergh [V2] under the name of a *non-commutative resolution* of the affine variety Y; in particular, R has a large center, which is identified with the algebra $A = H^0(Y, \mathcal{O}_Y)$ of functions on Y, and generically on Y – that is, after tensoring with the fraction field of A – the algebra R is a matrix algebra. It also enjoys some other nice properties, such as finite global homological dimension.

We note that the McKay equivalence can be stated in exactly the same way, with R being the so-called *smash product algebra* $S^\bullet(V^*)\#G$ (see e.g.

[BK2]); the category of finitely generated left R-modules is then immediately seen to be equivalent with the category of G-equivariant coherent sheaves on V. All in all, it seems that Van den Bergh's non-commutative resolution picture is the proper framework for generalizing McKay equivalence, and it is this picture that one should try to find in higher dimensions.

In this paper, we do this under the same additional assumption as in [BK2]: we assume given a *symplectic* resolution X of a normal irreducible affine variety Y, and we construct a non-commutative resolution R of the variety Y and an equivalence between $D^b_{coh}(X)$ and the bounded derived category of finitely generated left R-modules. Unfortunately, we can only do this locally on Y – that is, we fix a point $y \in Y$, and in the course of our construction we may have replace Y with an étale neighborhood of the point y. However, we impose no additional restrictions on X. We also prove that if X, X' are two different symplectic resolutions of the same variety Y, then, locally on Y, the derived categories $D^b_{coh}(X)$ and $D^b_{coh}(X')$ are equivalent (this generalizes the particular case of [BO1, Section 3, Conjecture] proved by Y. Kawamata in [Ka]: in Kawamata's language, "K-equivalence implies D-equivalence"). As in [BK2], our main technical tool is reduction to positive characteristic and applying the Fedosov quantization procedure, which has recently been worked out in positive characteristic in [BK3].

The paper is organized as follows. In Section 1 we give the precise statements of our results, and also do some reformulation; in particular, we prove that the main Theorem is equivalent to X having a so-called *tilting generator* $\mathcal{E}$. In Section 2, we recall additional material needed for the construction: in Subsection 2.1, we introduce a certain one-parameter deformation of the symplectic manifold X called the *twistor deformation*, and in Subsection 2.2, we recall the necessary facts about reduction to positive characteristic and quantization over positive characteristic fields. At this point, we are able to construct a tilting vector bundle $\mathcal{E}$ on X. To prove the main Theorem, it remains to show that $\mathcal{E}$ is a generator; this is done in Section 4, after preliminary technical estimates are established in Section 3. Finally, in Section 5 we give some applications and generalizations of our main result.

One final remark is perhaps in order. In [BK2], when dealing with symplectic resolutions of quotient singularities, not only did we establish an equivalence $D^b_{coh}(X) \cong D^b(R\text{-mod}^{\mathrm{fg}})$, but we were actually able to compute the algebra R. In this paper, we do not try to do this. However, we expect that this is a meaningful thing to do, and that the resulting algebras should be related to the theory of quantum groups. For more details, see Remark 2.8.

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1 Statements.

Let X be a smooth manifold – that is, a regular finite-type scheme – over some field k. We start with the following general definition.

Definition 1.1. A coherent sheaf $\mathcal{E}$ on X is called a *tilting generator* of the bounded derived category $D^b_{coh}(X)$ of coherent sheaves on X if the following holds:

(i) The sheaf $\mathcal{E}$ is a tilting object in $D^b_{coh}(X)$ – that is, for any $i \geq 1$ we have $\operatorname{Ext}^i(\mathcal{E},\mathcal{E}) = 0$

(ii) The sheaf $\mathcal{E}$ generates the derived category $D^-_{coh}(X)$ of complexes bounded from above – that is, if for some object $\mathcal{F} \in D^-_{coh}(X)$ we have $\operatorname{RHom}^\bullet(\mathcal{E},\mathcal{F}) = 0$, then $\mathcal{F} = 0$.

Lemma 1.2. *Let $\mathcal{E}$ be a tilting generator of the derived category $D^b_{coh}(X)$, and denote $R = \operatorname{End}(\mathcal{E})$. Then the correspondence $\mathcal{F} \mapsto \operatorname{RHom}^\bullet(\mathcal{E},\mathcal{F})$ extends to an equivalence*

$$D^b_{coh}(X) \to D^b(R\text{-mod}^{\mathrm{fg}}) \tag{1.1}$$

between the bounded derived category $D^b_{coh}(X)$ of coherent sheaves on X and the bounded derived category $D^b(R\text{-mod}^{\mathrm{fg}})$ of finitely generated left R-modules.

Proof. The functor $a : \mathcal{F} \mapsto \operatorname{Hom}(\mathcal{E}, \mathcal{F})$ is a left-exact functor from the abelian category of coherent sheaves on X to the abelian category of finitely generated R-modules. Its derived functor is therefore a well-defined functor $A : D^b_{coh}(X) \to D^b(R\text{-mod}^{\mathrm{fg}})$. It is easy to see that a has a right-exact left-adjoint functor $b : M \mapsto M \otimes_R \mathcal{E}$; its derived functor $B : D^-(R\text{-mod}^{\mathrm{fg}}) \to D^-_{coh}(X)$ is adjoint to A. The composition

$$A \circ B : D^-(R\text{-mod}^{\mathrm{fg}}) \to D^-(R\text{-mod}^{\mathrm{fg}})$$

sends R to $\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \mathcal{E})$; since $\mathcal{E}$ is tilting, we have $A \circ B(R) = R$. But R generates $D^-(R\text{-mod}^{\mathrm{fg}})$. Therefore $B : D^-(R\text{-mod}^{\mathrm{fg}}) \to D^-_{coh}(X)$ is fully faithful. Moreover, for any $\mathcal{F} \in D^-_{coh}(X)$, the cone of the adjunction map $B(A(\mathcal{F})) \to \mathcal{F}$ is annihilated by A, which by Definition 1.1 (ii) means that $\mathcal{F} \cong B(A(\mathcal{F}))$. Therefore A and B are mutually inverse equivalences between $D^-_{coh}(X)$ and $D^-(R\text{-mod}^{\mathrm{fg}})$. This in particular means that B is adjoint to A *both on the right and on the left*; since A sends $D^b_{coh}(X)$ into $D^b(R\text{-mod}^{\mathrm{fg}})$ and has bounded cohomological dimension, this implies that B sends $D^b(R\text{-mod}^{\mathrm{fg}})$ into $D^b_{coh}(X)$, so that A and B also induce mutually inverse equivalences between $D^b_{coh}(X)$ and $D^b(R\text{-mod}^{\mathrm{fg}})$. $\square$

Remark 1.3. In Definition 1.1, we require $\mathcal{E}$ to be a coherent sheaf on X, not just an arbitrary object in $D^b_{coh}(X)$ or $D^-_{coh}(S)$. This might not be strictly necessary for Lemma 1.2, but it simplifies the proof, and this level of generality is sufficient for our purposes. In our applications, $\mathcal{E}$ will in fact be not just a sheaf, but a vector bundle.

Assume now that the base field k has characteristic 0, and that X is symplectic – that is, we are given a non-degenerate closed 2-form $\Omega \in H^0(X, \Omega^2_X)$. Moreover, assume that X is equipped with a projective birational map $\pi : X \to Y$ onto a normal irreducible affine algebraic variety Y (we note that under these assumptions we must have $Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$, so that the existence of Y and π is a condition on X, not some new data). The main result of the paper is the following.

Theorem 1.4. *Under the assumptions above, for any point $y \in Y$ there exists an étale neighborhood $U_y \to Y$ such that the pullback $X_y = X \times_Y U_y$ admits a tilting generator $\mathcal{E}$. Moreover, this tilting generator is in fact a vector bundle on X_y.*

Remark 1.5. The standard t-structure on $D^b(R\text{-mod}^{\mathrm{fg}})$ gives by (1.1) a non-standard t-structure on $D^b(X)$ and defines a notion of a perverse coherent sheaf on X (it is this non-standard t-structure, rather than the equivalence (1.1), that was discovered in [Br]). Our claim that the tilting generator $\mathcal{E}$ is a vector bundle is then obviously equivalent to the fact that every skyscraper coherent sheaf on X is perverse.

We also prove the following.

Theorem 1.6. *Assume given two smooth projective resolutions X, X' of a normal affine irreducible variety Y, assume that the canonical bundles K_X, $K_{X'}$ are trivial, and assume moreover that X admits a closed non-degenerate 2-form $\Omega \in H^0(X, \Omega^2_X)$. Then every point $y \in Y$ admits an étale neighborhood $U_y \to Y$ such that the derived categories $D^b_{coh}(X \times_Y U_y)$ and $D^b_{coh}(X' \times_Y U_y)$ are equivalent.*

This is a very particular case of [Ka, Conjecture 1.2], which in turn essentially goes back to [BO1, Section 3] (see also [BO2]). In the language of Y. Kawamata, "K-equivalence implies D-equivalence" (for symplectic resolutions, and locally over the base). In the original, slightly less precise language of A. Bondal and D. Orlov, "if two smooth symplectic varieties are related by a flop, their derived categories are equivalent".

The need to pass to an étale neighborhood of a point is unfortunate, but, seemingly, unavoidable in the general situation. In practice, the problem can sometimes be alleviated by presence of an additional structure. We prove one result of this sort.

Definition 1.7. An action of the multiplicative group $\mathbb{G}_m$ on an affine scheme Y with fixed closed point $y \in Y$ is said to have *positive weights* if the weights of the $\mathbb{G}_m$-action are non-negative on the function algebra $H^0(Y, \mathcal{O}_Y)$, and strictly positive on the maximal ideal $\mathfrak{m} \subset Y$ which defines the point y.

Theorem 1.8. (i) *Assume that a smooth symplectic scheme X is projective over an affine variety Y, and assume that Y admits an action of the group $\mathbb{G}_m$ with fixed closed point $y \in Y$ and positive weights. Then the $\mathbb{G}_m$-action extends to a $\mathbb{G}_m$-action on X.*

(ii) *Assume that a scheme X is projective over an affine variety Y, and assume that Y admits an action of the group $\mathbb{G}_m$ with fixed closed point $y \in Y$ and positive weights which lifts to a $\mathbb{G}_m$-action on X. Assume also that for some étale neighborhood U_y of the point $y \in Y$, the pullback $X \times_Y U_y$ admits a tilting generator $\mathcal{E}_y$. Then $\mathcal{E}_y$ is obtained by pullback from a $\mathbb{G}_m$-equivariant tilting generator $\mathcal{E}$ on X.*

Finally, we prove one cohomological consequence of the existence of tilting generators.

Theorem 1.9. *Assume that a smooth manifold X is projective over an affine local Henselian scheme Y/k and admits a tilting generator $\mathcal{E}$. Then the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$ admits a finite resolution by vector bundles of the form $\mathcal{E}_i \boxtimes \mathcal{F}_i$, where $\mathcal{E}_i$, $\mathcal{F}_i$ are some vector bundles on X.*

Corollary 1.10. *Assume that a smooth manifold X is projective over an affine scheme Y, and let $F \subset X$ be the fiber over a closed point $y \in Y$. Assume that Y admits a positive-weight $\mathbb{G}_m$-action that fixes $y \in Y$, and assume that X admits a tilting generator $\mathcal{E}$. Then the cohomology groups $H^\bullet(F)$ of the scheme F are generated by classes of algebraic cycles.*

In this Corollary we are deliberately vague as to what particular cohomology groups $H^\bullet(F)$ one may take. In fact, every cohomology theory with the standard weight formalism will suffice; in particular, the statement is true for l-adic cohomology and for analytic cohomology when the base field k is $\mathbb{C}$.

2 Preliminaries.

2.1 Twistor deformations. Assume given a smooth manifold X over a field k equipped with a non-degenerate closed 2-form $\Omega \in H^0(X, \Omega^2_X)$ (from now on, we will call such a form a symplectic form). Assume also given a projective birational map $\pi : X \to Y$ onto a normal irreducible affine scheme Y of finite type over k. Choose a line bundle L on X. Denote $S = \operatorname{Spec} k[[t]]$, the formal disc over k, and let $o \in S$ be the special point (given by the maximal ideal $tk[[t]] \subset k[[t]]$).

Definition 2.1. By a *twistor deformation* Z associated to the pair $\langle X, L\rangle$ we will understand a triple of a smooth scheme $\mathfrak{X}/S$, a line bundle $\mathcal{L}$ on $\mathfrak{X}$ and a symplectic form Ω_Z on the total space Z of the $\mathbb{G}_m$-torsor associated to $\mathcal{L}$ (in other words, on the total space of $\mathcal{L}$ without the zero section) such that

(i) The form Ω_Z is $\mathbb{G}_m$-invariant, and the map $\rho : Z \to S$ is the moment map for the $\mathbb{G}_m$-action on Z – that is, we have $\Omega_Z \lrcorner \xi_0 = \rho^* dt$, where $\xi_0 \in H^0(Z, T_Z)$ is the infinitesimal generator of the $\mathbb{G}_m$-action.

(ii) The restriction $\langle \mathfrak{X}_o, \mathcal{L}_o \rangle$ of the pair $\langle \mathfrak{X}, \mathcal{L}$ to the special point $o \in S$ is identified with the pair $\langle X, L \rangle$, and the restriction of the form Ω_Z to the special fiber $Z_o \subset Z$ coincides under this identification with the pullback of the given form $\Omega \in H^0(X, \Omega^2_X)$.

Assume that the base field k has characteristic 0. Then for any $i \geq 1$ we have $H^i(X, K_X) = 0$ by the Grauert-Riemenschneider Vanishing Theorem, and since the top power of the symplectic form Ω trivializes the canonical bundle K_X, this implies that $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. Therefore the variety X falls within the assumptions of the paper [K1].

Lemma 2.2. *For any line bundle L on X, there exists a twistor deformation $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z \rangle$ associated to the pair $\langle X, L \rangle$. Moreover, $\mathfrak{X}$ is projective over $\mathcal{Y} = \operatorname{Spec} H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, while $\mathcal{Y}$ is normal and flat over S.*

Proof. This is a particular case of [K1, Theorem 1.4]; the second statement is [K1, Theorem 1.5]. □

There is also a certain uniqueness statement in [K1, Theorem 1.4], but we will not need it. What is important is that the construction is sufficiently functorial. This allows to prove the following.

Definition 2.3. A twistor deformation $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z \rangle$ is called *exact* if the symplectic form Ω_Z is exact, $\Omega_Z = d\alpha_Z$, and moreover the 1-form α is $\mathbb{G}_m$-equivariant and satisfies $\alpha \lrcorner \xi_0 = \rho^* t$, where ξ_0 is the infinitesimal generator of the $\mathbb{G}_m$-action on Z, t is the coordinate on $S = \operatorname{Spec} k[[t]]$, and $\rho : Z \to S$ is the natural projection.

Lemma 2.4. *Assume that the symplectic form Ω on X is exact. Then any twistor deformation $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z \rangle$ associated to the pair $\langle X, L \rangle$ by Lemma 2.2 is also exact.*

Proof. Since Ω is non-degenerate, $\alpha = \Omega \lrcorner \xi$ for some vector field $\xi \in H^0(X, T_X)$ on X. By the Cartan Homotopy formula, $d\alpha = \Omega$ is equivalent to $\mathcal{L}_\xi(\Omega) = \Omega$, where $\mathcal{L}_\xi$ is the Lie derivative along ξ. Since $H^1(X, \mathcal{O}_X) = 0$, the line bundle L can be made equivariant with respect to the vector field ξ. Thus ξ is an infinitesimal automorphism of the pair $\langle X, L \rangle$, and it dilates the symplectic form. In terms of the Poisson structure on X, this is equivalent to [K1, (1.2)]. By [K1, Proposition 1.5] the vector field ξ extends to a $\mathbb{G}_m$-invariant vector field ξ_Z on Z such that $\xi_Z(\rho^* t) = -t$. By the Cartan

Homotopy Formula, $\Omega_Z = d(\Omega_Z \lrcorner \xi_z)$, so that we can take $\alpha_Z = \Omega_Z \lrcorner \xi_Z$. Since both ξ_z and Ω_Z are $\mathbb{G}_m$-invariant, so is α_Z. Moreover, we have

$$\alpha_Z \lrcorner \xi_0 = \Omega_Z \lrcorner (\xi_z \wedge \xi_0) = -\Omega_Z \lrcorner (\xi_0 \wedge \xi_Z) = -\rho^* dt \lrcorner \xi_Z = -\xi_Z(\rho^* t) = \rho^* t,$$

which proves that Z is indeed exact in the sense of Definition 2.3. $\square$

We note that by [K2, Corollary 2.8], the symplectic form Ω is always exact over a formal neighborhood of any closed point $y \subset Y$. Finally, we will need the following.

Lemma 2.5. *Assume that the line bundle L on X is ample, and consider the twistor deformation $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z \rangle$ associated to the pair $\langle X, L \rangle$ by Lemma 2.2. Let $\widehat{A} = H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, $\mathcal{Y} = \operatorname{Spec} \widehat{A}$, and let $\pi : \mathfrak{X} \to \mathcal{Y}$ be the natural map. Then the map π is projective, and one-to-one over the complement $S \setminus \{o\}$. Moreover, if Y is the spectrum of a Henselian local k-algebra, so that $\widehat{A}$ is a local k-algebra with maximal ideal $\mathfrak{m} \subset \widehat{A}$, then there exists a finitely generated k-subalgebra $\widetilde{A} \subset \widehat{A}$ such that*

(i) *the t-adic completion of the Henselization of the algebra $\widetilde{A}$ in $\mathfrak{m} \cap \widetilde{A} \subset \widetilde{A}$ coincides with $\widehat{A}$, and*

(ii) *all the data $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z \rangle$ are defined over $\widetilde{A}$.*

Proof. (The argument uses a standard trick which probably goes back to [F]. Compare [H, Propositon 4.1] and [N, Section 2, Claim 3 on p. 18].)

The generic fiber $\mathcal{Y}_\eta = \mathcal{Y} \times_S \eta$ is open in the normal scheme $\mathcal{Y}$ (indeed, it is the complement to the closed special fiber $Y \subset \mathcal{Y}$). Since the scheme $\mathcal{Y}$ is normal, the open subscheme $\mathcal{Y}_\eta \subset \mathcal{Y}$ is also normal. Since the map $\pi : \mathfrak{X} \to \mathcal{Y}$ is birational, it suffices to prove that it is finite over the generic point $\eta \in S$. Moreover, by construction the map $\pi : \mathfrak{X}_\eta \to \mathcal{Y}_\eta$ is projective. Therefore by [EGA, IV, Théorème 8.11.1] it suffices to prove that it is quasifinite – in other words, that its fibers do not contain any compact curves.

Let $\iota : C \to \mathfrak{X}_\eta$ be an arbitrary map from a compact curve C/η to $\mathfrak{X}_\eta$. Replacing C with its normalization, we can assume that the curve C/η is connected and smooth. The $\mathbb{G}_m$-equivariant symplectic form Ω_Z induces by descent a relative symplectic form $\Omega_{\mathfrak{X}} \in H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}})$, and by [K1, Lemma 1.5], its de Rham cohomology class $[\Omega_{\mathfrak{X}}]$ is equal to $[\Omega] + c_1(L)t$, where $c_1(L)$ is the first Chern class of the line bundle L. Since C is a curve, $\iota^*\Omega_{\mathfrak{X}}$ is trivial; therefore

$$0 = \iota^*[\Omega_{\mathfrak{X}}] = \iota^*[\Omega] + t\iota^* c_1(L),$$

where we have identified $H^2_{DR}(\mathfrak{X}/S)$ with $H^2_{DR}(X)[[t]]$ by means of the Gauss-Manin connection. Differentiating this equality with respect to t, we obtain $\iota^* c_1(L) = 0$. Since L is an ample line bundle, its extension $\mathcal{L}$ is also ample. The first Chern class $c_1(\mathcal{L})$ is constant with respect to the Gauss-Manin connection; therefore $\iota^* c_1(\mathcal{L}) = \iota^* c_1(L) = 0$, and since $\mathcal{L}$ is ample, this is possible only if $\iota : C \to \mathfrak{X}_\eta$ maps the curve C to a point.

Finally, to show that the formal scheme $\mathcal{Y}$ comes from an algebraic variety $\operatorname{Spec} \widetilde{A}$, we note that since $\mathfrak{X}$ is smooth and $\pi : \mathfrak{X} \to \mathcal{Y}$ is bijective outside of the special fiber $Y \subset \mathcal{Y}$, the scheme $\mathcal{Y}$ is smooth outside of $Y \subset \mathcal{Y}$. Then the existence of a finitely generated subalgebra $\widetilde{A} \subset \widehat{A}$ which satisfies (i) is insured by [A, Theorem 3.9]. Localizing $\widetilde{A}$ if necessary, we can also achieve (ii). □

2.2 Quantization. We will now assume that the base field k has odd positive characteristic p. In this section, we prove the main quantization result which we need in the paper, Proposition 2.6; the proof depends heavily on notions introduced in [BK3, Section 1] (although the statement ought to be comprehensible to a reader who is not familiar with that paper, and it can be used as a black box).

For any scheme Z over k, we denote by $Z^{(1)}$ the twist of the scheme Z with respect to the Frobenius map $k \to k$. As a topological space, $Z^{(1)}$ coincides with Z, and if the scheme Z is reduced, the structure sheaf $\mathcal{O}_{Z^{(1)}}$ is canonically identified with the subalgebra $\mathcal{O}_Z^p \subset \mathcal{O}_Z$ generated by p-th powers of functions on Z. On level of schemes, the embedding $\mathcal{O}_{Z^{(1)}} \cong \mathcal{O}_Z^p \subset \mathcal{O}_Z$ is the Frobenius map $\mathsf{Fr}_Z : Z \to Z^{(1)}$. In particular, if $S = \operatorname{Spec} k[[t]]$, then $S^{(1)}$ is $\operatorname{Spec} k[[t^p]]$, and we have the Frobenius map $\mathsf{Fr}_S : S \to S^{(1)}$.

Consider the power series algebra $k[[t,h]]$ in two variables t, h, and denote $S_h = \operatorname{Spec} k[[t,h]]$. Define a map $s : k[[t^p]] \to k[[t,h]]$ by setting $s(t^p) = t^p + th^{p-1}$. By definition, $S = \operatorname{Spec} k[[t]]$ is canonically embedded into S_h; the map $s : S_h \to S^{(1)}$ extends the Frobenius map $\mathsf{Fr}_S : S \to S^{(1)}$ to $S_h \supset S$. This extension is *not* the obvious one, and it behaves differently. In particular, the fiber $s^{-1}(o) \subset S_h$ over the special point $o \in S^{(1)}$ is the union of p formal lines $t = ah$, $a \in \mathbb{Z}/p\mathbb{Z}$ in the formal affine plane S_h. The natural projection $\sigma : s^{-1}(o) \to \operatorname{Spec} k[[h]]$ is the Artin-Schreier covering.

Assume now given a symplectic manifold X/k equipped with a projective birational map $\pi : X \to Y$ onto a normal irreducible affine variety Y/k; moreover, assume given a line bundle L on X and a twistor deformation $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z \rangle$ of the pair $\langle X, L \rangle$. Denote by $\mathfrak{X}_h^{(1)}$ the scheme

$$\mathfrak{X}_h^{(1)} = \mathcal{S}pec(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^p \widehat{\otimes}_{k[[t^p]]} k[[t,h]]), \tag{2.1}$$

where $k[[t^p]]$ is embedded into $k[[t,h]]$ by means of the map s, and $\widehat{\otimes}$ stands for h-adic completion of the tensor product. The scheme $\mathfrak{X}_h^{(1)}$ is regular and projective over

$$\mathcal{Y}_h^{(1)} = \operatorname{Spec} H^0\left(\mathfrak{X}_h^{(1)}, \mathcal{O}^p_{\mathfrak{X}_h^{(1)}}\right) = \operatorname{Spec}\left(H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \widehat{\otimes}_{k[[t^p]]} k[[t,h]]\right), \tag{2.2}$$

where $\mathcal{Y}$ is the scheme $\operatorname{Spec} H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Denote by $\mathfrak{X}_o^{(1)} \subset \mathfrak{X}_h^{(1)}$ the special fiber of the natural flat projection $\mathfrak{X}_h^{(1)} \to \operatorname{Spec} k[[h]]$. As a topological space, $\mathfrak{X}_o^{(1)}$ coincides with $\mathfrak{X}$, while the structure sheaf is given by $\mathcal{O}_{\mathfrak{X}}^p \otimes_{k[[t^p]]} k[[t]] \subset \mathcal{O}_{\mathfrak{X}}$. In particular, $\mathcal{O}_{\mathfrak{X}}$ is a coherent sheaf on $\mathfrak{X}_o^{(1)}$, and its restriction $\mathcal{O}_{\mathfrak{X}}/t$ to $X^{(1)} \subset \mathfrak{X}_o^{(1)}$ coincides with the direct image $\mathsf{Fr}_*\mathcal{O}_X$ under the Frobenius map (thus $\mathcal{O}_{\mathfrak{X}}$ is in fact a vector bundle of rank $p^{\dim X}$ on $\mathfrak{X}_o^{(1)}$).

Proposition 2.6. *Assume that the twistor deformation $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z \rangle$ is exact in the sense of Definition 2.3, and assume that we have $H^i(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = 0$ for $i \geq 1$. Then there exists a coherent sheaf $\mathcal{O}_h$ of algebras on $\mathfrak{X}_h^{(1)}$, flat over $k[[h]]$, such that the restriction $\mathcal{O}_h|_{\mathfrak{X}_o^{(1)}}$ to $\mathfrak{X}_o^{(1)} \subset \mathfrak{X}_h^{(1)}$ is identified with $\mathcal{O}_{\mathfrak{X}}$, while over the complement $\mathfrak{X}_h^{(1)} \setminus \mathfrak{X}$, the algebra $\mathcal{O}_h$ is isomorphic to the endomorphism algebra of a vector bundle $\mathcal{E}$.*

Proof. We freely use the notions from [BK3, Section 1] – in particular, that of a restricted Poisson algebra, [BK3, Definition 1.8]. Recall that a restricted structure on a Poisson algebra A is defined by a non-additive "restricted power" operation $x \mapsto x^{[p]}$ on A compatible with the multiplication and the Poisson bracket in a certain specified way; if the Poisson bracket on A is trivial, then the restricted power must be additive, and it must be a "Frobenius-derivation" – that is, we have $(ab)^{[p]} = a^{[p]}b^p + a^p b^{[p]}$. Let $K_0 : k[[t]] \to k[[t]]$ be the (unique) Frobenius-derivation such that $K_0(t) = t$. Let α_Z be the 1-form on $\mathfrak{X}$ whose existence is required by Definition 2.3. By [BK3, Proposition 2.6], setting

$$f^{[p]} = H_f^{[p]} \lrcorner \alpha_Z - H_f^{p-1}(H_f \lrcorner \alpha_Z),$$

defines a restricted Poisson structure on Z – here H_f is the Hamiltonian vector field associated to the function f, and $H_f^{[p]}$ is its restricted p-th power with respect to usual restricted Lie algebra structure on the Lie algebra of vector fields. Since α_Z is $\mathbb{G}_m$-invariant, this restricted Poisson structure is $\mathbb{G}_m$-equivariant; in particular, it descends to a restricted Poisson structure on the quotient $\mathfrak{X} = Z/\mathbb{G}_m$. Moreover, since $H_{\rho^*t} = \xi_0$, the differential of the $\mathbb{G}_m$-action, we have

$$(\rho^*t)^{[p]} = \xi_0^{[p]} \lrcorner \alpha_Z - \xi_0^{p-1}(\xi_0 \lrcorner \alpha_Z).$$

Since ξ_0 is the differential of an action of the multiplicative group, we have $\xi_0^{[p]} = \xi_0$. By assumption $\xi_0 \lrcorner \alpha_z = \rho^*t$; in particular, it is $\mathbb{G}_m$-invariant. We conclude that $(\rho^*t)^{[p]} = \xi_0 \lrcorner \alpha_Z = \rho^*t$. This means that the restricted Poisson structure on $\mathfrak{X}/S$ is compatible with the Frobenius-derivation K_0 in the sense of [BK3, Corollary 1.13].

We now notice that $B = k[[t,h]]$ has a natural structure of a quantization base in the sense of [BK3, Definition 1.15]. To define it, one considers the map $s_0 : k[[t]] \to k[[h,t]]$, $s_0(t) = t$, and notices that for any $f \in k[[t]]$, the difference $s(f^p) - s_0(f)^p \in k[[h,t]]$ is divisible by h^{p-1}; thus there is a unique map $K : k[[t] \to k[[t,h]]$ such that $K(f)h^{p-1} + s_0(f)^p = s(f^p)$, and we extends it to $k[[t,h]]$ by setting $K(f) = K(f \mod h)$. We note that $K(t) = t$; therefore $K(f) = K_0(f) \mod h$ for any $f \in k[[t]]$, and we can apply [BK3, Theorem 1.23] to the restricted Poisson structure on $\mathfrak{X}/S$. By [BK3, Proposition 1.24], the result is a sheaf of algebras $\mathcal{O}_h$ on $\mathfrak{X}_h^{(1)}$, flat over $k[[h]]$, whose restriction to $\mathfrak{X}_o^{(1)}$ is identified with $\mathcal{O}_{\mathfrak{X}}$ – or actually, a whole set $Q(B,\mathfrak{X})$ of isomorphism classes of such sheaves. Moreover, on the complement $\mathfrak{X}_h^{(1)} \setminus \mathfrak{X}_o^{(1)}$, every sheaf $\mathcal{O}_h$ is an Azumaya algebra. Finally, the class of this Azumaya algebra in the Brauer group is also described in [BK3, Proposition 1.24] and, as noted there, one can choose an element in $Q(B,\mathfrak{X})$ so that the corresponding Azumaya algebra is split, $\mathcal{O}_h \cong \mathcal{E}nd(\mathcal{E})$ on $\mathfrak{X}_h^{(1)} \setminus \mathfrak{X}_o^{(1)}$ for some vector bundle $\mathcal{E}$. □

Lemma 2.7. *In the assumptions of Proposition* 2.6*, we have* $\operatorname{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ *for* $i \geq 1$.

Proof. Since $\mathcal{E}nd(\mathcal{E}) \cong \mathcal{O}_h$ on $\mathfrak{X}_h^{(1)} \setminus \mathfrak{X}_o^{(1)}$, it suffices to prove that $\mathcal{O}_h$ has no cohomology on this complement. By base change, it suffices to prove that it has no cohomology on the whole $\mathfrak{X}_h^{(1)}$. To compute $H^{\bullet}(\mathfrak{X}_h^{(1)}, \mathcal{O}_h)$ one can use the spectral sequence associated to the h-adic filtration on the $k[[h]]$-flat sheaf $\mathcal{O}_h$, and since the quotient $\mathcal{O}_h/h$ is supported on $\mathfrak{X}_o^{(1)}$, the term E^1 of this sequence is

$$H^i(\mathfrak{X}_o^{(1)}, \mathcal{O}_h/h)[[h]].$$

Since $\mathfrak{X}_o^{(1)} = \mathfrak{X}$ as a topological space and $\mathcal{O}_h/h \cong \mathcal{O}_{\mathfrak{X}}$, this vanishes for $i \geq 1$ by assumption. □

Remark 2.8. As we have noted in the Introduction, we make no attempt to describe the algebra $R = \operatorname{End}(\mathcal{E})$ obtained by this construction, nor its lifting to characteristic 0 which we will construct later in Section 4. However, if one replaces X with a semisimple algebraic group G considered as a Poisson-Lie group, then the resulting algebra R is very similar to the dual to G. Lusztig's quantum envelopping algebra specialized at a p-th root of unity. The similarity becomes even more pronounced if one restricts the quantum group to a symplectic leaf in G, as done, for instance, by C. De Concini and C. Procesi in [CP]. It must be noted, however, that Lusztig – and consequently, De Concini-Procesi – work by explicit computation. Our approach only works for symplectic X, but it makes no use of any group structure, and it somewhat clarifies the geometric picture. In a nutshell, what happens is this: after one reduces a Poisson variety X to positive characteristic, it acquires the Frobenius map $F : X \to X$. This map of course cannot in general be lifted back to characteristic 0. What we prove for symplectic X is that it does admit a lifting *if we first deform the function algebra of X to a quantized function algebra*. The result is a quantum analog of Frobenius map in characteristic 0 – when $X = G$ is a semisimple group, this is Lusztig's quantum Frobenius map. Motivated by this analogy, we expect that with appropriate modifications, out construction should work for at least some Poisson manifolds which are not symplectic. A natural question is to find the proper conditions on the Poisson manifold X which allow to construct a quantum Frobenius map.

3 Estimates.

3.1 Generalities on algebra sheaves. We start with some generalities on algebra sheaves. Consider a scheme X flat over some affine scheme $S = \operatorname{Spec} B$, and assume that X is equipped with a coherent flat algebra sheaf $\mathcal{A}$. Denote by $A = H^0(X, \mathcal{A})$ the algebra of global sections of the sheaf $\mathcal{A}$. Taking global sections is then a left-exact functor from the category $\operatorname{Shv}(X, \mathcal{A})$ of quasicoherent sheaves of left $\mathcal{A}$-modules on X to the category A-mod of left A-modules. We denote this functor – for reasons of convenience – by $\pi^X_* : \operatorname{Shv}(X, \mathcal{A}) \to A\text{-mod}$. It has a left-adjoint right-exact functor $\pi^*_X : A\text{-mod} \to \operatorname{Shv}(X, \mathcal{A})$ given by $\pi^*(M) = M \otimes_A \mathcal{A}$.

Denote by $R^{\bullet}\pi^X_*$ and $L^{\bullet}\pi^*_X$ the derived functors of π^X_* and π^*_X. Assume that the category $\operatorname{Shv}(X, \mathcal{A})$ has finite homological dimension; then $R^{\bullet}\pi^X_*$ and $L^{\bullet}\pi^*_X$ are a pair of adjoint functors between the derived categories

$\mathcal{D}^-(X,\mathcal{A}) = \mathcal{D}^- \operatorname{Shv}(X,\mathcal{A})$ and $\mathcal{D}^-(A) = \mathcal{D}^- A$-mod. The composition

$$R^\bullet\pi^X_* L^\bullet\pi^*_X : D^-(A) \to D^-(A)$$

comes equipped with the adjunction map $\mathsf{Id} \to R^\bullet\pi^X_* L^\bullet\pi^*_X$. It is easy to see that this map is an isomorphism if and only if $H^i(X,\mathcal{A}) = 0$ for $i \geq 1$: indeed, since every module has a free resolution, it suffices to check that $A \to R^\bullet\pi^X_* L^\bullet\pi^*_X A$ is an isomorphism, and the right-hand side by definition coincides with $H^\bullet(X,\mathcal{A})$.

Assume from now on that this holds: $H^i(X,\mathcal{A}) = 0$ for $i \geq 0$. The composition $L^\bullet\pi^*_X R^\bullet\pi^X_*$ also comes equipped with a canonical adjunction map $L^\bullet\pi^*_X R^\bullet\pi^X_* \to \mathsf{Id}$; our goal in this Subsection is to find a way to measure whether it is an isomorphism or not.

To do this, consider the product $X \times_S X$ and equip it with the coherent algebra $\mathcal{A}^{opp} \boxtimes \mathcal{A}$ ($\mathcal{A}^{opp}$ denotes the opposite algebra). Since $H^i(X,\mathcal{A}) = 0$ for $i \geq 1$, the algebra $A = H^0(X,\mathcal{A})$ is flat over B, and by the Künneth formula we have $H^0(X \times X, \mathcal{A}^{opp} \boxtimes \mathcal{A}) \cong A^{opp} \otimes_B A$. Denote by $\pi_2 : X \times_S X \to X$ the projection onto the second factor, and let π_{2*} be the associated direct image functor. For any object $\mathcal{K} \in D^-(X \times_S X, \mathcal{A}^{opp} \boxtimes \mathcal{A})$, we define the kernel functor $F(\mathcal{K}) : D^-(X,\mathcal{A}) \to D^-(X,\mathcal{A})$ by setting

$$F(\mathcal{K})(\mathcal{F}) = R^\bullet\pi_{2*}\left((\mathcal{F} \boxtimes \mathcal{A}) \overset{\mathsf{L}}{\otimes}_{\mathcal{A}\boxtimes\mathcal{A}^{opp}} \mathcal{K}\right)$$

for any $\mathcal{F} \in \mathcal{D}^-(X,\mathcal{A})$. An obvious example is the identity functor; we denote the corresponding kernel by $\mathcal{A}_\Delta \in \operatorname{Shv}(X \times X, \mathcal{A}^{opp} \boxtimes \mathcal{A})$. This is a sheaf supported on the diagonal $\Delta \subset X \times_S X$, and for every open subset $U \subset X$ we have $\mathcal{A}_\Delta(U \times_S U) = \mathcal{A}(U)$ with the natural $\mathcal{A}(U)$-bimodule structure.

Lemma 3.1. *The functor $L^\bullet\pi^*_X R^\bullet\pi^X_* : D^-(X,\mathcal{A}) \to D^-(X,\mathcal{A})$ is isomorphic to the kernel functor $F(\mathcal{M})$ with the kernel $\mathcal{M}^\bullet = \mathcal{M}^\bullet(X/S,\mathcal{A}) = L^\bullet\pi^*_{X\times_S X}A$, where A is equipped with the natural A-bimodule structure.*

Proof.(Compare [Ku, Appendix D].) For any A-bimodule N and any sheaf $\mathcal{F} \in \operatorname{Shv}(X,\mathcal{A})$, the projection formula gives an isomorphism

$$R^\bullet\pi^X_*\mathcal{F} \otimes_A N \cong R^\bullet\pi^X_*(\mathcal{F} \otimes_{\mathcal{A}} L^\bullet\pi^*_X N),$$

while adjunction gives a map

$$L^\bullet\pi^*_X N \to R^\bullet\pi_{1*} L^\bullet\pi^*_{X\times_S X} N,$$

and the projection formula again gives an isomorphism

$$\mathcal{F} \otimes R^\bullet \pi_{1*} L^\bullet \pi^*_{X \times_S X} N \cong R^\bullet \pi_{1*}(\mathcal{F} \boxtimes \mathcal{A} \otimes_{\mathcal{A} \boxtimes \mathcal{A}^{opp}} L^\bullet \pi^*_{X \times_S X} N).$$

Composing all three, we obtain a map

$$R^\bullet \pi^X_* \mathcal{F} \otimes_A N \to \mathbb{R}^\bullet \pi^{X \times_S X}_* (\mathcal{F} \boxtimes \mathcal{A} \otimes_{\mathcal{A} \boxtimes \mathcal{A}^{opp}} L^\bullet \pi^*_{X \times_S X} N),$$

which gives by adjunction a base-change map

$$L^\bullet \pi^*_X \left(R^\bullet \pi^X_* (\mathcal{F}) \overset{\mathsf{L}}{\otimes}_A N \right) \to R^\bullet \pi_{2*} \left((\mathcal{F} \boxtimes \mathcal{A}) \overset{\mathsf{L}}{\otimes}_{\mathcal{A} \boxtimes \mathcal{A}^{opp}} L^\bullet \pi^*_{X \times_S X} N \right).$$

We have to prove that it is an isomorphism for $N = A$. More generally, we will prove that it is an isomorphism for every A-bimodule N. Indeed, since every A-bimodule has a free resolution, it suffices to consider the free bimodule $N = A^{opp} \otimes_B A$. Then the left-hand side is isomorphic to $H^\bullet(X, \mathcal{F}) \otimes_B \mathcal{A}$, and the right-hand side is isomorphic to $R^\bullet \pi_{2*}(\mathcal{F} \boxtimes \mathcal{A})$, which are the same $\mathcal{A}$-module by the Künneth formula. $\square$

We see that the two functors that we want to compare are kernel functors, with kernels $\mathcal{A}_\Delta, \mathcal{M}^\bullet(X/S, \mathcal{A}) \in D^-(X \times X, \mathcal{A}^{opp} \boxtimes \mathcal{A})$. The adjunction map is given by a map $\mathcal{M}^\bullet(X, \mathcal{A}) \to \mathcal{A}_\Delta$.

Definition 3.2. Denote by $\mathcal{K}^\bullet(X/S, \mathcal{A}) \in D^-(X \times_S X, \mathcal{A}^{opp} \boxtimes \mathcal{A})$ the cone of the adjunction map

$$\mathcal{M}^\bullet(X, \mathcal{A}) = L^\bullet \pi^* A \to \mathcal{A}_\Delta.$$

Thus the adjunction map is an isomorphism if and only if $\mathcal{K}^\bullet(X/S, \mathcal{A}) = 0$; if this happens, $R^\bullet \pi^X_*$ and $L^\bullet \pi^*_X$ are mutually inverse equivalences of categories. We will need one refinement of this statement.

Lemma 3.3. *Assume that $H^i(X, \mathcal{A}) = 0$ for $i \geq 1$, and that the category* $\mathrm{Shv}(X \times_S X, \mathcal{A}^{opp} \boxtimes \mathcal{A})$ *has homological dimension k. Then if the complex $\mathcal{K}^\bullet(X/S, \mathcal{A})$ has no homology in degrees $\geq -(k+1)$, we have $\mathcal{K}^\bullet(X/S, \mathcal{A}) =$* 0.

Proof. The kernel $\mathcal{A}_\Delta$ is actually a sheaf placed in degree 0, while the kernel $\mathcal{M}^\bullet(X/S, \mathcal{A})$ is trivial in positive degrees by construction. Therefore under the assumptions of the Lemma, the connecting map in the exact triangle

$$\mathcal{K}^\bullet(X/S, \mathcal{A}) \longrightarrow \mathcal{M}^\bullet(X/S, \mathcal{A}) \longrightarrow \mathcal{A}_\Delta \longrightarrow$$

must be equal to 0, and we have $\mathcal{M}(X/S,\mathcal{A}) = \mathcal{A}_\Delta \oplus \mathcal{K}(X/S,\mathcal{A})$. In other words, the functor $F = L^\bullet\pi^*R^\bullet\pi_*$ splits into a direct sum of the identity functor Id and some other functor F', in such a way that the natural adjunction map $\mathsf{ad} : F \to \mathsf{Id}$ becomes the projection $\mathsf{Id} \oplus F' \to \mathsf{Id}$ onto the first summand. Since $H^i(X,\mathcal{A}) = 0$ for $i \geq 1$, we have $R^\bullet\pi_*^X L^\bullet\pi_X^* \cong \mathsf{Id}$, so that the natural map

$$F = L^\bullet\pi_X^*R^\bullet\pi_*^X \to L^\bullet\pi_X^*R^\bullet\pi_*^X L^\bullet\pi_X^*R^\bullet\pi_*^X \cong F \circ F$$

is an isomorphism. Moreover, the composition

$$F \longrightarrow F \circ F \xrightarrow{\mathsf{ad}\circ\mathsf{id}} \mathsf{Id} \circ F \cong F$$

tautologically coincides with the identity map. Thus

$$\mathsf{ad}\circ\mathsf{id} : F \circ F \to \mathsf{Id} \circ F \cong F$$

must be an isomorphism, too. But it vanishes on the direct summand

$$F' \cong F' \circ \mathsf{Id} \subset (\mathsf{Id} \oplus F') \circ (\mathsf{Id} \oplus F') = F \circ F.$$

We conclude that $F' = 0$. $\square$

Remark 3.4. In all of the above, the scheme structure on X is really irrelevant; in particular, if we have a finite scheme map $p : X \to X_1$ which is identical on the level of points – for example, a Frobenius map – then $p_*\mathcal{K}(X/S,\mathcal{A})$ tautologically coincides with $\mathcal{K}(X_1/S, p_*\mathcal{A})$. In fact, one can develop the same theory for general ringed topological spaces, or for ringed toposes, with some appropriate coherence conditions. We do not do this since we do not need this; however, it might be useful in some applications (see e.g. Remark 4.5). Ringed topological spaces $\langle X,\mathcal{A}\rangle$ such that π_* and π^* are equivalences of categories are sometimes called $\mathcal{A}$-*affine*. A well-known example is the flag variety $X = G/B$ associated to a reductive algebraic group G; in this example, $\mathcal{A}$ is the algebra of differential operators on X (see [BB]).

3.2 Critical lines. Assume now, as in Section 2.2, that the base field k has odd positive characteristic $\mathsf{char}\, k = p$. For any $a \in \mathbb{Z}/p\mathbb{Z}$, denote by $S_a \subset S_h = \operatorname{Spec} k[[h,t]]$ the line $t = ah$. Moreover, consider the polynomial

$$P(h,t) = t(t^{p-1} - h^{p-1}) = \prod_{a\in\mathbb{Z}/p\mathbb{Z}} (t - ah),$$

and denote by $S_s = \operatorname{Spec} k[[h,t]]/P(h,t)$ the union of all lines $S_a \subset S_h$, $a \in \mathbb{Z}/p\mathbb{Z}$. Equivalently, $S_s \subset S_h$ is the fiber of the map $s : S_h \to S$ over the special point $o \in S = \operatorname{Spec} k[[t]]$. The projection $\sigma : S_h \to \operatorname{Spec} k[[h]]$ is in particular flat on $S_s \subset S_h$; for any $a \in \mathbb{Z}/p\mathbb{Z}$, it induces an isomorphism $\sigma : S_a \to \operatorname{Spec} k[[h]]$. The scheme S_s is a local scheme with special point o, and the open complement $\widetilde{S}_s = S_s \setminus \{o\}$ decomposes

$$\widetilde{S}_s = \coprod_{a \in \mathbb{Z}/p\mathbb{Z}} \widetilde{S}_a,$$

where we denote $\widetilde{S}_a = S_a \setminus \{o\}$. The map σ identifies $\widetilde{S}_a$ with the point $\operatorname{Spec} k((h))$.

Assume given a symplectic scheme X/k projective over a normal affine scheme Y/k. Fix an ample line bundle L on X, and assume given a twistor deformations $\mathfrak{X}$ of the pair $\langle X, L\rangle$. Moreover, assume that the twistor deformation $\mathfrak{X}$ is exact in the sense of Definition 2.3, and that $H^i(\mathfrak{X}, \mathcal{O}_X) = 0$ for $i \geq 1$. Let $\mathfrak{X}_h^{(1)}$, $\mathcal{Y}_h^{(1)}$ and $\mathfrak{X}_o^{(1)}$ be as in Subsection 2.2.

For any scheme Z/k and any complete local k-algebra B, we will denote by $Z \widehat{\times} \operatorname{Spec} B$ the completed product,

$$Z \widehat{\times} \operatorname{Spec} B = \mathcal{S}pec_Z\, \mathcal{O}_Z \widehat{\otimes}_k B,$$

where the product on the right-hand side is completed with respect to the adic topology on $\mathcal{O}_Z \otimes_k B$ induced from the adic topology on B. In particular, consider the schemes $\mathfrak{X}_h^{(1)}$, $\mathcal{Y}_h^{(1)}$ defined in (2.1) and (2.2) of Subsection 2.2. Then we have

$$\begin{aligned} \mathfrak{X}_h^{(1)} &= \mathfrak{X}^{(1)} \widehat{\times}_{S^{(1)}} S_h, \\ \mathcal{Y}_h^{(1)} &= S_h \widehat{\times}_{S^{(1)}} \mathcal{Y}^{(1)}, \end{aligned}$$

where in the first equation, S_h is mapped to $S^{(1)}$ by the map s. The scheme $\mathfrak{X}_h^{(1)}$ is projective over $\mathcal{Y}_h^{(1)}$, and both are flat over S_h. Let $\mathfrak{X}_o^{(1)} \subset \mathfrak{X}_h^{(1)}$ be as in Subsection 2.2,

$$\mathfrak{X}_o^{(1)} = \mathcal{S}pec_{\mathfrak{X}^{(1)}}\, \mathcal{O}_{\mathfrak{X}}^p \otimes_{k[[t^p]]} k[[t]].$$

Moreover, let $X_s^{(1)} = \mathfrak{X}_h^{(1)} \times_{S_h} S_s$. We note that in fact $X_s^{(1)} \cong X^{(1)} \widehat{\times} S_s$; in particular, it does not depend on the twistor deformation $\mathfrak{X}$. The flat projection $\sigma : S_s \to \operatorname{Spec} k[[h]]$ defines a flat projection $\sigma : X_s^{(1)} \to X_h^{(1)}$ of degree p, where we denote $X_h^{(1)} = X^{(1)} \widehat{\times} \operatorname{Spec} k[[h]]$. The map σ is ramified over the special point $o \in \operatorname{Spec} k[[h]]$; in fact the fiber $X^{(1)}_t = X^{(1)}_s \times_{\operatorname{Spec} k[[h]]} \{o\}$ – or equivalently, the scheme-theoretic intersection $X^{(1)}_t = X^{(1)}_s \cap \mathfrak{X}^{(1)}_o$ – is easily seen to coincide with $X^{(1)} \times \operatorname{Spec} k[t]/t^p$, and the map σ simply projects this to $X^{(1)}$. The complement $\widetilde{X}^{(1)}_s = X^{(1)}_s \setminus X^{(1)}_t$ decomposes

$$\widetilde{X}^{(1)}_s = \coprod_{a \in \mathbb{Z}/p\mathbb{Z}} \widetilde{X}^{(1)}_a, \tag{3.1}$$

where we denote $X^{(1)}_a = X^{(1)}_h \times_{S_h} S_a$, $\widetilde{X}^{(1)}_a = \mathfrak{X}^{(1)}_a \setminus \mathfrak{X}^{(1)}$. For any $a \in \mathbb{Z}/p\mathbb{Z}$, the projection σ identifies the scheme $X^{(1)}_a$ with $X^{(1)}_h$, while $\widetilde{X}^{(1)}_a$ is identified with $\widetilde{X}^{(1)}_h = X^{(1)}_h \setminus X^{(1)}_h$. Denote also $Y^{(1)}_h = Y^{(1)} \widehat{\times} \operatorname{Spec} k[[h]]$ and $\widetilde{Y}^{(1)}_h = Y^{(1)}_h \setminus Y^{(1)}$; then $X^{(1)}_h$ – and therefore, $X^{(1)}_a$ for any $a \in \mathbb{Z}/p\mathbb{Z}$ – is projective over $Y^{(1)}_h$, and $\widetilde{X}^{(1)}_a \cong \widetilde{X}^{(1)}_h$ is projective over $\widetilde{X}^{(1)}_h$.

Remark 3.5. We must caution the reader that although the scheme $\widetilde{X}^{(1)}_h$ maps to $X^{(1)} \otimes_k k((h))$, this map is not an isomorphism, due to the completion involved in the definition of the scheme $X^{(1)}_h$. Explicitly, we have

$$\widetilde{X}^{(1)}_h = \mathcal{S}pec_{X^{(1)}}\, \mathcal{O}_{X^{(1)}}((h)),$$

where $\mathcal{O}_{X^{(1)}}((h)) = (\mathcal{O}_{X^{(1)}} \widehat{\otimes}_k k[[h]])(h^{-1})$ is the algebra of Laurent series in h with coefficients in $\mathcal{O}_{X^{(1)}}$. This algebra is much larger than $\mathcal{O}_{X^{(1)}} \otimes_k k((h))$. Since X is only projective over an affine scheme Y, all our construction will really be only defined over this large algebra. In particular, the map $\widetilde{X}^{(1)}_h \to X^{(1)} \otimes_k k((h))$ is not even bijective on the level of points (we note, for example, that every $k((h))$-valued point $x : \operatorname{Spec} k((h)) \to \widetilde{X}^{(1)}_h$ extends to a map $\operatorname{Spec} k[[h]] \to X^{(1)}_h$).

Fix now a sheaf $\mathcal{O}_h$ on $\mathfrak{X}^{(1)}_h$ provided by Proposition 2.6, denote by $\mathcal{O}_s$ its restriction to $X^{(1)}_s \subset \mathfrak{X}^{(1)}_h$, and for any $a \in \mathbb{Z}/p\mathbb{Z}$, denote by $\mathcal{O}_a$ its restriction to $X^{(1)}_a \subset X^{(1)}_s \subset \mathfrak{X}^{(1)}_h$. By Proposition 2.6, $\mathcal{O}_h$ restricts to $\mathcal{O}_{\mathfrak{X}}$ on $\mathfrak{X}^{(1)}_o \subset \mathfrak{X}^{(1)}_h$, which in turn restricts to $\mathsf{Fr}_*\mathcal{O}_X$ on $X^{(1)} \subset \mathfrak{X}^{(1)}_o$, where $\mathsf{Fr} : X \to X^{(1)}$ is the Frobenius map. Hence $\mathcal{O}_a$ also restricts to $\mathsf{Fr}_*\mathcal{O}_X$ on $X^{(1)} = X^{(1)}_a \cap \mathfrak{X}^{(1)}_o \subset \mathfrak{X}^{(1)}_h$.

To help the reader to visualize various schemes that we have introduced, we note that they fit together into a diagram

$$\begin{array}{ccccc} \widetilde{X}_a^{(1)} & \longrightarrow & \widetilde{X}_s^{(1)} & \longrightarrow & \widetilde{\mathfrak{X}}_h^{(1)} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ X_a^{(1)} & \longrightarrow & X_s^{(1)} & \longrightarrow & \mathfrak{X}_h^{(1)} \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ X^{(1)} & \longrightarrow & X_t^{(1)} & \longrightarrow & \mathfrak{X}_o^{(1)}, \end{array} \tag{3.2}$$

where all the squares are Cartesian, all the vertical arrows in the bottom row are closed embeddings, and all the vertical arrows in the top row are the complementary open embeddings. The whole diagram is obtained from $\mathfrak{X}_h^{(1)}/S_h$ by base change with respect to the similar diagram

$$\begin{array}{ccccc} \widetilde{S}_a & \longrightarrow & \widetilde{S}_s & \longrightarrow & \widetilde{S}_h \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ S_a & \longrightarrow & S_s & \longrightarrow & S_h \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ \{o\} & \longrightarrow & S_t & \longrightarrow & S. \end{array}$$

Here $S_h = \operatorname{Spec} k[[t,h]]$ is the (formal) two-dimensional affine plane, $S \subset S_h$ is the line $h = 0$, $S_a \subset S_h$ is the line $ah = t$, $S_t = \operatorname{Spec} k[t]/t^p$, and $S_s \subset S_h$ is the union of the lines S_a for all $a \in \mathbb{Z}/p\mathbb{Z}$. We also have a version of (3.2) with $\mathcal{Y}$, Y instead of $\mathfrak{X}$, X.

All in all, the have the following schemes equipped with algebra sheaves: X/k with $\mathcal{O}_X$, $\mathfrak{X}_o^{(1)}/S$ with $\mathcal{O}_{\mathfrak{X}}$, $X_s^{(1)}/S_s$ with $\mathcal{O}_s$, $X_t^{(1)}/S_t$ with $\mathcal{O}_{\mathfrak{X}}/t^p$, and for any $a \in \mathbb{Z}/p\mathbb{Z}$, $X_a^{(1)}/S_a$ and $\widetilde{X}_a^{(1)}/\widetilde{S}_a$ with $\mathcal{O}_a$. By assumptions on $\mathfrak{X}$ and by Lemma 2.7, all these algebra sheaves have no higher cohomology. As in Definition 3.2, denote

$$\begin{aligned} \mathcal{K}^\bullet &= \mathcal{K}(X/k, \mathcal{O}_X), & \mathcal{K}_o^\bullet &= \mathcal{K}(\mathfrak{X}_o^{(1)}/S, \mathcal{O}_{\mathfrak{X}}), \\ \mathcal{K}_t^\bullet &= \mathcal{K}(X_t^{(1)}/S_t, \mathcal{O}_{\mathfrak{X}}/t^p), & \mathcal{K}_s^\bullet &= \mathcal{K}(\mathfrak{X}_s^{(1)}/S_s, \mathcal{O}_s), \\ \mathcal{K}_a^\bullet &= \mathcal{K}(X_a^{(1)}/S_a, \mathcal{O}_a), & \widetilde{\mathcal{K}}_a^\bullet &= \mathcal{K}(\widetilde{X}_a^{(1)}/\widetilde{S}_a, \mathcal{O}_a), \end{aligned}$$

By construction, all these kernels are complexes of coherent sheaves bounded from above (in fact trivial in positive degrees). We note that we tautologically have

$$\mathcal{K}^\bullet(X^{(1)}, \mathsf{Fr}_*\mathcal{O}_X) \cong \mathsf{Fr}_*\mathcal{K}^\bullet$$

(see Remark 3.4); moreover, since the construction of the kernel $\mathcal{K}^\bullet(X/S,\mathcal{A})$ is compatible with flat base change with respect to S, and $\mathcal{O}_a$ restricts to $\mathsf{Fr}_*\mathcal{O}_X$ on $X^{(1)} \subset X_a^{(1)}$, the kernel $\mathcal{K}_a^\bullet$ restricts to $\mathsf{Fr}_*\mathcal{K}^\bullet$ on $X^{(1)} \times X^{(1)} \subset X_a^{(1)} \times_{S_a} X_a^{(1)}$. Analogously, $\mathcal{K}_s^\bullet$ and $\mathcal{K}_o^\bullet$ both restrict to $\mathcal{K}_t^\bullet$.

Definition 3.6. A line $S_a \subset S_h$ is called *critical* if the complex $\widetilde{\mathcal{K}}_a^\bullet \in \mathcal{D}^-(\widetilde{X}_h^{(1)} \times_{k((h))} \widetilde{X}_h^{(1)})$ is non-trivial. For any point $y \in Y$, a line S_a is called *critical at y* if it becomes critical after replacing Y with the localization $\operatorname{Spec}\mathcal{O}_{Y,y}$.

3.3 Bounds. Our goal in this Subsection is to bound the number of critical lines. We do it in the following way. Firstly, assume that the line bundle L on X is very ample. Then the line bundle $L \boxtimes L$ on $X \times X$ is also very ample and defines a closed embedding $X \times X \to \mathbf{P}_Y = \mathbb{P}^{N_Y} \times Y \times Y$ for some integer $N_Y > 0$. Using this embedding, we can consider $\mathcal{K}^\bullet$ as a complex of sheaves on $\mathbf{P}_Y$. Analogously, we have closed embeddings $X^{(1)} \times X^{(1)} \to \mathbf{P}_{Y^{(1)}} = \mathbb{P}^{N_Y} \times Y^{(1)} \times Y^{(1)}$ and $\widetilde{X}_h \times_{k((h))} \widetilde{X}_h \to \widetilde{\mathbf{P}}_h = \mathbb{P}^{N_Y} \times \widetilde{Y}_h \times_{k((h))} \widetilde{Y}_h$.

Choose now a closed point $y \in Y^{(1)}$, let $\mathbf{P}_y \subset \mathbf{P}_{Y^{(1)}}$ be the fiber over the point $y \times y \in Y \times Y$, and denote its embedding by $i_y : \mathbf{P}_y \to \mathbf{P}_{Y^{(1)}}$. Denote also $\widetilde{\mathbf{P}}_y = \rho^{-1}(\mathbf{P}_y) \subset \widetilde{\mathbf{P}}_h$, where $\rho : \widetilde{Y}_k \to Y^{(1)}$ is the natural projection; by abuse of notation, keep the notation $i_y : \widetilde{\mathbf{P}}_y \to \widetilde{\mathbf{P}}_h$. For any integer m and any $a \in \mathbb{Z}/p\mathbb{Z}$, denote

$$C(y,a,m) = \sum_{\substack{0 \geq l \geq -2\,\mathsf{dim}\,X,\\ 0 \leq q \leq N_Y}} \binom{N_Y}{q} \mathsf{dim}_{k((h))} \mathbb{H}^{l+q}\left(\widetilde{\mathbf{P}}_y, i_y^*\widetilde{\mathcal{K}}_a^\bullet(m-q)\right). \tag{3.3}$$

Lemma 3.7. *For any line S_a which is critical at the point y, we have*

$$C(y,a,m) \geq p^{\mathsf{dim}\,X}.$$

Proof. Replacing Y with an open neighborhood of $y \in Y$ if necessary, we may assume that S_a is critical everywhere. Then for some point $x \in \widetilde{\mathbf{P}}_y$ with embedding $i_x : x \to \widetilde{\mathbf{P}}_y$, the restriction $i_x^* i_y^* \widetilde{\mathcal{K}}_a^\bullet$ of the complex $i_y^*\widetilde{\mathcal{K}}_a^\bullet$ to the point x is non-trivial. Moreover, since by construction $\mathcal{O}_a$ is an Azumaya algebra on $\widetilde{X}_a^{(1)}$, the category of sheaves of $\mathcal{O}_a$-modules on $\widetilde{X}_a^{(1)} \times_{k((h))} \widetilde{X}_a^{(1)}$ has homological dimension $2\,\mathsf{dim}\,X$. Therefore by Lemma 3.3 we may assume that this restriction $i_x^* i_y^* \widetilde{\mathcal{K}}_a^\bullet$ is non-trivial in some degree l with $0 \geq l \geq -2\,\mathsf{dim}\,X$. The same is true for any twist $\widetilde{\mathcal{K}}_a^\bullet(m)$. Therefore

$$\mathsf{dim}_{k((h))} \mathbb{H}^l(\widetilde{\mathbf{P}}_y, i_{x*} i_x^* i_y^* \widetilde{\mathcal{K}}_a^\bullet(m)) > 0 \tag{3.4}$$

for some l, $0 \geq l \geq -2\,\mathsf{dim}\,X$, and any integer m. However, the complex $\mathcal{K}_a^\bullet$ is by construction a complex of sheaves of modules over $\mathcal{O}_a^{opp} \boxtimes \mathcal{O}_a$, and on $\widetilde{X}_a^{(1)} \subset X_a^{(1)}$, the algebra $\mathcal{O}_a$ is a matrix algebra: we have $\mathcal{O}_a \mid_{\widetilde{X}_a^{(1)}} \cong \mathrm{End}(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on $\widetilde{X}_a^{(1)}$. Therefore by Morita, $\widetilde{\mathcal{K}}_a^\bullet \cong \mathcal{F}^\bullet \otimes (\mathcal{E}^* \boxtimes \mathcal{E})$ for some complex $\mathcal{F}^\bullet$, and the left-hand side of (3.4), being greater than zero, must be at least $\mathsf{rk}\,\mathcal{E}^* \boxtimes \mathcal{E} = p^{\mathsf{dim}\,X}$. On the other hand, the skyscraper sheaf $i_{x*}k((h))$ concentrated at the point $x \in \widetilde{\mathbf{P}}_y$ admits a Koszul resolution, whose terms are the sheaves $\mathcal{O}(-q)^{\oplus\binom{N_Y}{q}}$, $0 \leq q \leq N_Y$. By the projection formula, $i_{x*}i_x^*i_y^*\widetilde{\mathcal{K}}_a^\bullet(m) \cong i_{x*}k((h)) \otimes (i_y^*\widetilde{\mathcal{K}}_a^\bullet(m))$. Therefore there exists a spectral sequence which converges to $\mathbb{H}^\bullet(\widetilde{\mathbf{P}}_y, i_{x*}i_x^*i_y^*\widetilde{\mathcal{K}}_a^\bullet)$ and starts with

$$\bigoplus_{0 \leq q \leq N_Y} \mathbb{H}^\bullet\left(\widetilde{\mathbf{P}}_y, i_y^*\widetilde{\mathcal{K}}_a^\bullet(m-q)[q]\right)^{\oplus\binom{N_Y}{q}}.$$

Comparing this to (3.3), we conclude that

$$\mathsf{dim}_{k((h))}\,\mathbb{H}^l(\widetilde{\mathbf{P}}_y, i_{x*}i_x^*i_y^*\widetilde{\mathcal{K}}_a^\bullet(m)) \leq C(y,a,m),$$

which proves the claim. □

Denote now

$$C(y,m) = \sum_{\substack{0 \geq l \geq -2\,\mathsf{dim}\,X,\\ 0 \leq q \leq N_Y}} \binom{N_Y}{q} \mathsf{dim}_k\,\mathbb{H}^{l+q}\left(\mathbf{P}_y, i_y^*\mathsf{Fr}_*\mathcal{K}^\bullet(m-q)\right), \tag{3.5}$$

and let $C(y)$ be the number of lines S_a, $a \in \mathbb{Z}/p\mathbb{Z}$, which are critical at $y \in Y$ in the sense of Definition 3.6.

Proposition 3.8. *Assume that $\mathcal{K}_o^\bullet$ is annihilated by t^M for some integer M. Then for any closed point $y \in Y$ and for any integer m, we have*

$$C(y)p^{\mathsf{dim}\,X} \leq MC(y,m) \tag{3.6}$$

Proof. By Lemma 3.7, we have

$$C(y)p^{\mathsf{dim}\,X} \leq \sum_{a \in \mathbb{Z}/p\mathbb{Z}} C(y,a,m).$$

Recall that we have a map $\sigma : \widetilde{X}_o \to \widetilde{X}_h$, and by (3.1), the left-hand side is the disjoint union of $\widetilde{X}_a$, $a \in \mathbb{Z}/p\mathbb{Z}$. Comparing (3.3) and (3.5), we see that it suffices to prove that

$$\mathsf{dim}_{k((h))}\,\mathbb{H}^l\left(\widetilde{\mathbf{P}}_y, i_y^*\sigma_*\widetilde{\mathcal{K}}_s^\bullet(m)\right) \leq M\,\mathsf{dim}_k\,\mathbb{H}^l\left(\mathbf{P}_y, i_y^*\mathsf{Fr}_*\mathcal{K}^\bullet(m)\right)$$

for any integers m and l.

Now, by definition $i_y^*\sigma_*\mathcal{K}_s^\bullet$ is a complex of coherent sheaves on the projective space $\mathbb{P}^{N_Y}_{k[[h]]}$ over $\operatorname{Spec} k[[h]]$; $\widetilde{\mathbf{P}}_y$ is the fiber of this projective space over the generic point $\operatorname{Spec} k((h)) \subset \operatorname{Spec} k[[h]]$, while $\mathbf{P}_y$ is its fiber over the special point $o \in \operatorname{Spec} k[[h]]$. Moreover, the complex $i_y^*\sigma_*\mathcal{K}_s^\bullet$ is bounded from above. Therefore we can apply semicontinuity and deduce that

$$\mathsf{dim}_{k((h))} \mathbb{H}^l \left(\widetilde{\mathbf{P}}_y, i_y^*\sigma_*\widetilde{\mathcal{K}}_s^\bullet(m)\right) \leq \mathsf{dim}_k \mathbb{H}^l \left(\mathbf{P}_y, i_y^*\sigma_* i_o^*\mathcal{K}_s^\bullet\right),$$

where $i_o : \mathfrak{X}_o^{(1)} \times_S \mathfrak{X}_o^{(1)} \subset \mathfrak{X}_h^{(1)} \times_{S_h} \mathfrak{X}_h^{(1)}$ is the embedding of the fiber over the special point $o \in \operatorname{Spec} k[[h]]$. By base change, $i_o^*\mathcal{K}_s^\bullet \cong \mathcal{K}_t^\bullet$, so that the right-hand side is equal to $\mathsf{dim}_k \mathbb{H}^l(\mathbf{P}_y, i_y^*\sigma_*\mathcal{K}_t^\bullet)$. This can be computed by the spectral sequence associated to the t-adic filtration. This spectral sequence starts with

$$\mathbb{H}^l(\mathbf{P}_y, i_y^*\mathsf{Fr}_*\mathcal{K}^\bullet(m)) \otimes_k k[t]/t^p.$$

By assumption $\mathcal{K}_o^\bullet$ is annihilated by t^M; hence so does $\mathcal{K}_t^\bullet$, being isomorphic to its restriction to $X_t^{(1)} \times_{S_t} X_t^{(1)} \subset \mathfrak{X}_o^{(1)} \times_S \mathfrak{X}_o^{(1)}$. To finish the proof, it suffices to apply inductively the following obvious linear-algebraic fact.

Lemma 3.9. *For any field k, any integer l, and any complex $E^\bullet$ of finitely generated $k[[t]]$-modules, we have* $\mathsf{dim}_k H^l(E^\bullet)/t \leq \mathsf{dim}_k E^l/t$.

Proof. Changing E^l and E^{l-1} if necessary, we may insure that $H^l(E^\bullet)$ and $\mathsf{dim}_k E^l/t$ stay the same, but the module E^l is flat over $k[[t]]$. Then

$$\begin{aligned} \mathsf{dim}_k H^l(E^\bullet)/t \leq \mathsf{dim}_k \mathsf{Ker}\, d_l/t = \mathsf{dim}_{k((t))} \mathsf{Ker}\, d_l(t^{-1}) \\ \leq \mathsf{dim}_{k((t))} E^l(t^{-1}) = \mathsf{dim}_k E^l/t, \end{aligned}$$

where $d_l : E^l \to E^{l+1}$ is the differential in the complex $E^\bullet$. □

4 Proofs.

4.1 Reduction. Assume now that again, the base field k is of characteristic 0, and we are given a smooth symplectic scheme X of finite type over k equipped with a projective birational map $\pi : X \to Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$. Fix a closed point $y \in Y$. Replacing Y and $\operatorname{Spec} k$ with finite Galois covers, we can assume that the residue field of the point $y \in Y$ is exactly k. Fix an ample line bundle L on X. Choose a projective compactification $\overline{Y} \supset Y$ of the scheme Y, a projective compactification $\overline{X}$ of the scheme X, and a projective birational map $\pi : \overline{X} \to \overline{Y}$ which extends the given map $\pi : X \to Y$. Moreover, choose these compactifications in such a way that L extends to an ample line bundle on $\overline{X}$.

By [K2, Corollary 2.8], the symplectic form Ω_X on X is exact in the formal neighborhood of the fiber $\pi^{-1}(y)$ – or equivalently, Ω_X becomes exact after passing to $\widehat{X} = X \times_Y \widehat{Y}$, where $\widehat{Y}$ is the completion of Y at the point y. In other words, there exist a 1-form $\widehat{\alpha}$ on $\widehat{X}$ such that $\Omega_X = d\widehat{\alpha}$. Since $H^0(\widehat{X}, \Omega^1_{\widehat{X}}) = H^0(\widehat{Y}, \widehat{\pi_*\Omega^1_X})$, we can choose a 1-form $\alpha_X \in H^0(X, \Omega^1_X)$ which approximates $\widehat{\alpha}$ to arbitrarily high order at $y \in Y$. In particular, we can insure that $d\alpha_X$ is non-degenerate at every point $x \in \pi^{-1}(y) \subset X$. Therefore $d\alpha_X$ is non-degenerate on $\pi^{-1}(U)$ for some Zariski open neighborhood $U \subset Y$ of the point $y \in Y$. Replacing Y with U and Ω_X with $d\alpha_X$, we can assume that Ω_X is exact.

Remark 4.1. In fact, in [K2, Corollary 2.8] it is erroneously claimed that the original Ω_X itself must be exact over a Zariski-open neighborhood of the point $y \in Y$. This is not true.

Fix an exact twistor deformation $\langle \mathfrak{X}, \mathcal{L} \rangle$ associated to the pair $\langle X, L \rangle$ which is provided by Lemma 2.2. Let $\mathcal{Y} = \operatorname{Spec} H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Note that the canonical bundle $K_{\mathfrak{X}}$ is trivial, so that $H^i(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = H^i(\mathfrak{X}, K_{\mathfrak{X}}) = 0$ for $i \geq 1$ by the Grauert-Riemenschneider Vanishing Theorem. Moreover, by Lemma 2.5 $\mathcal{Y}$ is the t-adic completion of a scheme $\widetilde{Y}$ of finite type over $k[t]$, and $\widetilde{Y} \otimes_{k[t]} k$ is an étale neighborhood of $y \in Y$. Therefore, replacing Y with this étale neighborhood, we can find a subalgebra $O \subset k$ of finite type over $\mathbb{Z}$, an affine variety $\mathcal{Y}_O$ of finite type over $O[t]$, and a variety $\mathfrak{X}_O$ which is projective over $\mathcal{Y}_O$ so that the t-adic completion of $\mathcal{Y}_O \otimes_O k$ coincides with $\mathcal{Y}$, we have an isomorphism $\mathfrak{X} \cong \mathfrak{X}_O \times_{\mathcal{Y}_O} \mathcal{Y}$, and $Y \cong Y_O \otimes_O k$, where $Y_O = \mathcal{Y}_O \otimes_{O[t]} O$. Replacing $\mathcal{Y}_O$ with a dense open subset, we may additionally assume that $\mathcal{Y}_O$ and $\mathfrak{X}_O$ are flat over O, X_O is smooth symplectic over O, $\mathcal{Y}_O$ is normal, the image $W_O \subset Y_O$ of the exceptional locus of the map $X_O \to Y_O$ is also flat over $\operatorname{Spec} O$, and both the 1-form α_X and the line bundle $\mathcal{L}$ come from a 1-form and a line bundle on $\mathfrak{X}_O$. Localizing O and possibly shrinking $\mathcal{Y}_O$ even further, we may assume as well that the point $y : \operatorname{Spec} k \to \mathcal{Y}$ extends to a section $y_O : \operatorname{Spec} O \to \mathcal{Y}_O$ of the projection $\mathcal{Y}_O \to \operatorname{Spec} O$, and that $H^i(\mathfrak{X}_O, \mathcal{O}_{\mathfrak{X}_O}) = 0$ for $i \geq 1$. Finally, shrinking $\operatorname{Spec} O$ and $\mathcal{Y}_O$ even further, we may assume that the compactifications $\overline{Y}$, $\overline{X}$ come from schemes $\overline{Y}_O$, $\overline{X}_O$ which are projective over O and contain Y_O and $X_O = \mathfrak{X}_O \otimes_{\mathcal{Y}_O} Y_O$ as open dense subsets, while the map $\pi : \overline{X} \to \overline{Y}$

comes from a map $\pi : \overline{X}_O \to \overline{Y}_O$.

For any maximal ideal $\mathfrak{m} \subset O$ with residue field $k(\mathfrak{m}) = O/\mathfrak{m}$, we can consider the corresponding special fibers $X_\mathfrak{m} = X_O \times_O k(\mathfrak{m})$, $\mathfrak{X}_\mathfrak{m} = \mathfrak{X}_O \times_O k(\mathfrak{m})$ of $X_O/\operatorname{Spec} O$, $\mathfrak{X}_O/\operatorname{Spec} O$. Then no matter what is the characteristic of the field $k(\mathfrak{m})$, $X_\mathfrak{m}$ is a smooth symplectic variety over $k(\mathfrak{m})$ equipped with a projective birational map $\pi : X_\mathfrak{m} \to Y_\mathfrak{m} = Y_O \times_O O/\mathfrak{m}$, and $\mathfrak{X}_\mathfrak{m}$ is an exact twistor deformation of $X_\mathfrak{m}$. Moreover, the section $y_O : \operatorname{Spec} O \to Y_O$ defines a closed point $y_\mathfrak{m} \in Y_\mathfrak{m}$.

Consider the scheme X_O/Y_O, and let

$$\mathcal{K}^\bullet_O = \mathcal{K}^\bullet(X_O, \mathcal{O}_{X_O}) \in D^b(X_O \times_O X_O)$$

be as in Definition 3.2. By assumption, the line bundle L on $\overline{X}_O$ is ample. Replacing it with a multiple, we may assume that it is very ample. Then $L \boxtimes L$ is very ample on $\overline{X}_O \times_O \overline{X}_O$, and in particular, it defines a closed embedding $\overline{X}_O \times_O \overline{X}_O \to \overline{\mathbf{P}}_Y = \mathbb{P}^{N_Y}_O \times_O \overline{Y}_O \times_O \overline{Y}_O$ for some integer $N_Y > 0$. Moreover, it also defines a global projective embedding $\overline{X}_O \times_O \overline{X}_O \to \mathbf{P}_O = \mathbb{P}^N_O$ for some $N > N_Y$, so that we have a chain of embeddings

$$\overline{X}_O \times_O \overline{X}_O \longrightarrow \overline{\mathbf{P}}_Y \longrightarrow \mathbf{P}_O.$$

By pushforward, we can treat the complex $\mathcal{K}^\bullet_O$ as a complex of coherent sheaves on $\mathbf{P}_Y = \mathbb{P}^{N_Y} \times_O Y_O \times_O Y_O$. Extend it to a complex $\overline{\mathcal{K}^\bullet}_O$ on $\mathbf{P}_O$ in such a way that for every l, the support of the l-th homology sheaf of the complex $\overline{\mathcal{K}^\bullet}_O$ is the closure of the support of the l-th homology sheaf of the complex $\mathcal{K}^\bullet_O$ (no new components of the support appear at infinity). Denote by $\mathcal{H}^\bullet_O$ the homology sheaves of the complex $\overline{\mathcal{K}}^\bullet_O$, and denote by $\mathsf{rk}^l_O = \mathsf{dim}(\mathsf{Supp}\,\mathcal{H}^l_O) - \mathsf{dim}\operatorname{Spec} O$ the dimension of the support of the sheaf $\mathcal{H}^l_O$, relative over O. Moreover, let $\mathbf{P}_y \subset \mathbf{P}_Y \subset \mathbf{P}_O$ be the preimage of the O-valued point $y_O \times y_O \in Y_O \times_O Y_O$ under the projection $\mathbf{P}_Y \to \overline{Y}_O \times_O \overline{Y}_O$. Then $\mathbf{P}_y \subset \mathbf{P}_X$ is a linear subspace in a projective space; therefore the structure sheaf of $\mathbf{P}_y \subset \mathbf{P}_X$ has a Koszul resolution $\mathcal{F}^\bullet$ by sheaves of the form $\mathcal{F}^r = \mathcal{O}(-r)^{\oplus\binom{N-N_Y}{r}}$, $0 \leq r \leq N - N_Y$.

Now, in order to prove Theorem 1.4 below in Subsection 4.2, we want to show that for almost all maximal ideals $\mathfrak{m} \subset O$ with residue field $k(\mathfrak{m})$ of positive characteristic, the canonical quantization of the exact twistor deformation $\mathfrak{X}_\mathfrak{m}$ contains at least one line which is not critical in the sense of Definition 3.6. To do this, we will use estimates from Section 3. However, we need to re-do them in a way independent of $\mathfrak{m}$ and $\mathsf{char}\, k(\mathfrak{m})$.

We first note that shrinking $\operatorname{Spec} O$ if necessary, we may assume that all the sheaves $\mathcal{H}^l_O$, $0 \geq l \geq -4\operatorname{\mathsf{dim}} X$, are flat over O. Moreover, there exists an integer m_0 such that for any $m_1 > m$, the sheaves $\mathcal{H}^l_O(m_1)$ have no cohomology, and all global section modules $H^0(\mathbf{P}_X, \mathcal{H}^l_O(m_1))$ are flat over O. Increasing m_0, we may assume that for any $m_1 \geq m_0$ we have

$$2a_l m_1^{\mathsf{rk}^l_O} \geq \mathsf{rk}_O\, H^0(\mathbf{P}_X, \mathcal{H}^l_O(m_1)), \tag{4.1}$$

where a_l is the leading coefficient in the Hilbert polynomial of the sheaf $\mathcal{H}^l_O$. We note that by assumption $\mathcal{H}^l_O$ is flat over O, so that $\mathsf{rk}^l_O = \mathsf{dim}\,\mathsf{Supp}\,\mathcal{H}^l_O \otimes_O k$; moreover, we have

$$\mathsf{dim}\,\mathsf{Supp}\,\mathcal{H}^l_O \otimes_O k = \mathsf{dim}\,\mathsf{Supp}\,\mathcal{H}^l_O \otimes_O k \mid_{\mathbf{P}_Y} \leq \mathsf{dim}\, X \times_Y X,$$

since $\mathcal{K}^{\bullet}(X, \mathcal{O}_X)$ is by construction supported in $X \times_Y X$. But by [K2, Lemma 2.11] the map $X \to Y$ is semismall – in other words, $\mathsf{dim}\, X \times_Y X = \mathsf{dim}\, X$. We conclude that $\mathsf{rk}^l_O \leq \mathsf{dim}\, X$ for any l.

Secondly, we note that by Lemma 2.5, the map $\pi : \mathfrak{X} \to \mathcal{Y}$ is one-to-one outside of the special fiber $X \subset \mathfrak{X}$. In the language of kernels, this means that $\mathcal{K}^{\bullet}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is supported on the special fiber $X \times X \subset \mathfrak{X} \times_S \mathfrak{X})$ – or, in algebraic language, $\mathcal{K}^{\bullet}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is annihilated by t^M for some integer $M \gg 0$ (here, as before, t is the coordinate on the base $S = \operatorname{Spec} k[[t]]$ of the twistor deformation $\mathfrak{X}/S$). Shrinking $\operatorname{Spec} O$ and $\mathcal{Y}_O$ if necessary, we may assume that $\mathcal{K}^{\bullet}(\mathfrak{X}_O, \mathcal{O}_{\mathfrak{X}_O})$ is annihilated by t^M everywhere, not just over the generic point $\operatorname{Spec} k \subset \operatorname{Spec} O$.

We are now ready to prove the following result.

Proposition 4.2. *In the assumptions and notations above, there exists an integer C such that for any maximal ideal $\mathfrak{m} \subset O$ with residue field $k(\mathfrak{m}) = O/\mathfrak{m}$ of odd positive characteristic $p = \mathsf{char}\, k(\mathfrak{m})$, the canonical quantization of the twistor deformation $\mathfrak{X}_{\mathfrak{m}}$ provided by Proposition* 2.6 *contains at most C lines S_a, $a \in \mathbb{Z}/p\mathbb{Z}$ which are critical at $y_{\mathfrak{m}} \in Y_{\mathfrak{m}}$ in the sense of Definition* 3.6.

Proof. We will use the estimate of Proposition 3.8; the reader will easily check that all of its assumptions are satisfied. The only thing to do is to bound the right-hand side of (3.6) – that is, the integer $C(y, m)$ defined in (3.5) – in a way independent of p. Indeed, use the resolution $\mathcal{F}^{\bullet}$ of the structure sheaf of $\mathbf{P}_y \subset \mathbf{P}_O$ to compute the functor i_y^*. Then for any q, $0 \leq q \leq N_Y$, the homology sheaves of the complex $i_y^*(\mathsf{Fr}_*\mathcal{K}^{\bullet})(m-q)$ in (3.5) can be computed by a spectral sequence which starts with $(\mathsf{Fr}_*\mathcal{H}^\bullet_{\mathfrak{m}})\otimes\mathcal{F}^\bullet(m-q)$. But we have $\mathcal{F}^r=\mathcal{O}(-r)^{\oplus\binom{N-N_Y}{r}}$, and

$$H^\bullet(\overline{\mathbf{P}}_{\mathfrak{m}},(\mathsf{Fr}_*\mathcal{H}^l_{\mathfrak{m}})\otimes\mathcal{F}^r(m-q))=H^\bullet\left(\overline{\mathbf{P}}_{\mathfrak{m}},\mathcal{H}^l_{\mathfrak{m}}\otimes\mathsf{Fr}^*\mathcal{F}^r(m-q)\right)$$
$$=H^\bullet\left(\overline{\mathbf{P}}_{\mathfrak{m}},\mathcal{H}^l_{\mathfrak{m}}(pm-pq-pr)\right)^{\oplus\binom{N-N_Y}{r}}.$$

Now fix m to be any integer such that $m>m_0+2\dim X+N_Y$. Then by assumption, $\mathcal{H}^l_O(p(m-q-r)))$ has no higher cohomology for any $r\leq N$. Moreover, by construction we have $q\leq N_Y$ and $r\leq N-N_Y$; therefore by (4.1)

$$\dim H^0(\overline{\mathbf{P}}_{\mathfrak{m}},\mathcal{H}^l_{\mathfrak{m}}(pm-pq-pr))\leq 2a_lp^{\mathsf{rk}^l_O}(m-q-r)^{\mathsf{rk}^l_O},$$

where, since $\mathcal{H}^l_O$ is flat over O, the highest coefficient a_l of the Hilbert polynomial of the sheaf $\mathcal{H}^l_{\mathfrak{m}}$ is the same for all $\mathfrak{m}$. Collecting all this together, we see that

$$\dim H^l(\mathbf{P}_{\mathfrak{m}},i_y^*(\mathsf{Fr}_*\mathcal{K}^\bullet)(m-q))\leq$$
$$\leq p^{\mathsf{rk}^l_O}\sum_{0\leq r\leq l}2a_{l-r}\binom{N-N_Y}{r}(m-q-r)^{\mathsf{rk}^l_O}.$$

The sum in the right-hand side is a well-defined integer independent of the maximal ideal $\mathfrak{m}\subset O$. It remains to compare this to (3.6) and (3.5), and to notice that, since $\mathsf{rk}^l_O\leq\dim X$, we obtain a bound on the number $C(y)$ of critical lines independent of $\mathfrak{m}$. Summing over all l, $0\geq l\geq -2\dim X$, and over all q, $0\leq q\leq N_Y$, gives the desired constant C. □

Remark 4.3. It is perhaps useful to sum up briefly the essential points of the above proof. We start with a trivial observation: if for some finitely generated S_o-module E, the quotient E/h is annihilated by t^M, then, when we consider the decomposition

$$E(h^{-1})=\bigoplus_{a\in\mathbb{Z}/p\mathbb{Z}}E_a,$$

the module E_a is non-trivial for at most M values of $a\in\mathbb{Z}/p\mathbb{Z}$. To apply this, we need to pass from modules to coherent sheaves on $X_o\times_{S_o}X_o/S_o$, which requires two main estimates:

(i) Since $\widetilde{\mathcal{K}}^\bullet_a$ is a complex of sheaves of modules over a matrix algebra of rank $p^{2\dim X}$, every non-trivial fiber $i_x^*\widetilde{\mathcal{K}}^\bullet_a$ of this complex has homology spaces of dimension at least $p^{\dim X}$ (this is essentially Proposition 3.8).

(ii) Since the kernel $\mathcal{K}^\bullet$ is supported on the fibered product $X \times_Y X \subset X \times X$, and $X \to Y$ is semismall, the dimensions of the vector spaces $\mathbb{H}^\bullet(\widetilde{\mathbf{P}}_h, i_y^* \mathsf{Fr}_* \mathcal{K}^\bullet(m))$ are bounded from above by $Ap^{\mathsf{dim}\, X}$, where the constant A does not depend on $\mathfrak{m} \subset O$ (this is the key part of Proposition 4.2).

The rest of the argument is straightforward homological algebra: we show that the spaces whose dimension is bounded from below in (i) are obtained as an E^∞-term of a spectral sequence, and the dimensions of the spaces in the E^0-term of this sequence are bounded from above in (ii).

4.2 Lifting. Proposition 4.2 essentially shows that in the assumptions of Theorem 1.4, and possibly after replacing Y with an étale neighborhood of the point $y \in T$, one can construct a tilting generator $\mathcal{E}$ on a generic specialization $X_{\mathfrak{m}}$ of the smooth symplectic variety X. To prove Theorem 1.4, it remains to lift this generator back to characteristic 0.

Proof of Theorem 1.4. We are given a smooth symplectic scheme X of finite type over a field k of characteristic 0; we assume that it is equipped with a projective birational map $\pi : X \to Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$. We fix a closed point $y \in Y$, an ample line bundle L on X, and a twistor deformation $\mathfrak{X}$ of the pair $\langle X, L\rangle$ provided by Lemma 2.2.

As explained in the last Subsection, we may choose a subalgebra $O \subset k$ of finite type over $\mathbb{Z}$, an affine scheme Y_O smooth and of finite type over O, a scheme X_O, smooth and symplectic over O and equipped with a projective birational map $X_O \to Y_O$, a map $\tau : Y_O \otimes_O k \to Y$ and an O-valued point $y_O : \operatorname{Spec} O \to Y_O$ so that

(i) the map $\tau : Y_O \otimes_O k \to Y$ is étale and maps the point $y_O \otimes_O k$ to $y \in Y$, and

(ii) we have $X_O \cong X \times_Y Y_O$, this isomorphism is compatible with symplectic forms, and the form on X_O is exact.

Moreover, we may assume that X_O is embedded into a scheme $\mathfrak{X}_O$ so that for any maximal ideal $\mathfrak{m} \subset O$ with residue field $k(\mathfrak{m}) = O/\mathfrak{m}$ of odd positive characteristic, the special fiber $\mathfrak{X}_{\mathfrak{m}} = \mathfrak{X}_O \otimes_O k(\mathfrak{m})$ is an exact twistor deformation of the special fiber $X_{\mathfrak{m}} = X_O \otimes_O O/\mathfrak{m}$, and that $H^i(\mathfrak{X}_O, \mathcal{O}_{\mathfrak{X}_O}) = H^i(\mathfrak{X}_{\mathfrak{m}}, \mathcal{O}_{\mathfrak{X}_{\mathfrak{m}}}) = 0$ for $i \geq 1$. By Proposition 2.2, the twistor deformation $\mathfrak{X}_{\mathfrak{m}}$ has a canonical quantization. By Proposition 4.2, we can assume that for a fixed constant C and any $\mathfrak{m} \subset O$ with $p = \mathsf{char}\, k(\mathfrak{m}) > 2$, at most C of the lines S_a, $a \in \mathbb{Z}/p\mathbb{Z}$ are critical for the canonical quantization of the deformation $\mathfrak{X}_{\mathfrak{m}}$ in the sense of Definition 3.6.

Fix an arbitrary maximal ideal $\mathfrak{m} \subset O$ such that $p = \operatorname{char} k(\mathfrak{m}) > C$, and the field $k(\mathfrak{m})$ is perfect (since O is of finite type over $\mathbb{Z}$, we can even assume that $k(\mathfrak{m})$ is finite). Then for some $a \in \mathbb{Z}/p\mathbb{Z}$, the line S_a is not critical for the quantization of $\mathfrak{X}_{\mathfrak{m}}$. This means that the vector bundle $\mathcal{E}$ on

$$\widetilde{X}_{\mathfrak{m}} = \mathcal{S}pec(X_{\mathfrak{m}}^{(1)}, \mathcal{O}_{X_{\mathfrak{m}}^{(1)}}((h))),$$

corresponding to the line S_a in the canonical quantization of the twistor deformation $\mathfrak{X}_{\mathfrak{m}}$ is a tilting generator for the derived category $D^b_{coh}(\widetilde{X}_{\mathfrak{m}}^{(1)})$ in the sense of Definition 1.1. Since $k(\mathfrak{m})$ is perfect, we can invert its Frobenius map, so that $X_{\mathfrak{m}}^{(1)} \cong X_{\mathfrak{m}}$, and $\mathcal{E}$ is actually a tilting generator of the category $D^b_{coh}(\widetilde{X}_{\mathfrak{m}})$. In the language of kernels, this means that $\mathcal{K}^\bullet(\widetilde{X}_{\mathfrak{m}}/k(\mathfrak{m})((h)), \mathcal{E}nd(\mathcal{E})) = 0$.

Denote by $\widehat{O}$ the completion of the Laurent series algebra $O((h))$ with respect to the ideal $\mathfrak{m}((h)) \subset O((h))$, denote by $\widehat{Y}_{\mathfrak{m}}$ the completion of the affine variety

$$\widetilde{Y}_O = \mathcal{S}pec(Y_O, \mathcal{O}_{Y_O}((h)))$$

along $\widetilde{Y}_{\mathfrak{m}} \subset \widetilde{Y}_O$, and denote $\widehat{X}_{\mathfrak{m}} = \widehat{Y}_{\mathfrak{m}} \times_{Y_{\mathfrak{m}}} X_{\mathfrak{m}}$. Then $\widehat{X}_{\mathfrak{m}}$ is projective over $\widehat{Y}_{\mathfrak{m}}$. Therefore by Grothendieck Algebraization Theorem [EGA, III, Théorème 5.4.5], the vector bundle $\mathcal{E}$ extends from $\widetilde{X}_{\mathfrak{m}}$ to the whole $\widehat{X}_{\mathfrak{m}}$ if and only if it extends to the formal scheme neighborhood of $\widetilde{X}_{\mathfrak{m}}$ in $\widehat{X}_{\mathfrak{m}}$. But since $\mathcal{E}$ is a tilting object – that is, $\operatorname{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ for $i \geq 1$ – the latter is automatic: by standard deformation theory, $\operatorname{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$ implies that $\mathcal{E}$ extends to the infinitesimal neighborhood $\widetilde{X}_O \otimes_O O/\mathfrak{m}^l$ of arbitrary order $l \geq 1$, and $\operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) = 0$ means that such an extension is unique. We conclude that $\mathcal{E}$ extends to a vector bundle $\widehat{\mathcal{E}}$ on $\widehat{X}_{\mathfrak{m}}$. By flat base change, we have $\operatorname{Ext}^i(\widehat{\mathcal{E}}, \widehat{\mathcal{E}}) = 0$ for $i \geq 0$.

Now, we know that $\widehat{X}_{\mathfrak{m}}$ is projective and of finite type over $\widehat{Y}_{\mathfrak{m}}$. Therefore by [A, Theorem 1.10] there exists an étale neighborhood Y_h of the point $\widetilde{y}_{\mathfrak{m}} \in \widetilde{Y}_O$ and a scheme X_h projective over Y_h such that $\widehat{X}_{\mathfrak{m}} = X_h \times_{Y_h} \widehat{Y}_{\mathfrak{m}}$ and the tilting generator $\widehat{\mathcal{E}}$ comes from a vector bundle $\mathcal{E}_h$ on X_h – more precisely, $\widehat{E} \cong \tau^*\mathcal{E}_h$, where $\tau : \widehat{X}_{\mathfrak{m}} \to X_h$ is the natural projection map. Moreover, we can choose an O-subalgebra $O' \subset \widehat{O}$, flat and finite over O and with fraction field k', so that Y_h and X_h are defined over $O'((h))$, $\widehat{O}$ is faithfully flat over $O'((h))$, the point $y_{\mathfrak{m}} \subset Y_{\mathfrak{m}}$ extends to an $O'((h))$-valued point $y_h : \operatorname{Spec} O'((h)) \to Y_h$, and the natural map $Y_h \times_{O'((h))} k'((h)) \to Y \times_k k'((h))$ sends $y_h \times_{O'} k'$ to $y \times_k k'((h))$ and is étale near $y_h \times_{O'((h))} k'$. Replacing

O' and Y_h with étale covers of open subsets, we may in addition assume that the vector bundle $\mathcal{E}_h$ on X_h satisfies $\operatorname{Ext}^i(\mathcal{E}_h, \mathcal{E}_h) = 0$, $i \geq 1$. Therefore the kernel $\mathcal{K}^\bullet = \mathcal{K}^\bullet(X_h/O'((h)), \mathcal{E}nd(\mathcal{E}_h))$ on $X_h \times_{O'((h))} X_h$ is well-defined, see Definition 3.2. Since $\mathcal{K}^\bullet \otimes_{O'} \widehat{O} = \mathcal{K}^\bullet(\widehat{X}_{\mathfrak{m}}/\widehat{O}, \mathcal{E}nd(\widehat{E}))$ reduces to $\mathcal{K}^\bullet(\widetilde{X}_{\mathfrak{m}}/k(\mathfrak{m})((h)), \mathcal{E}nd(\mathcal{E})) = 0$ over $\operatorname{Spec} k(\mathfrak{m})((h)) \subset \operatorname{Spec} \widehat{O})((h))$, we have $\mathcal{K}^\bullet \otimes_{O'} \widehat{O} = 0$ by Nakayama Lemma, and since $\widehat{O}$ is faithfully flat over O', this implies that $\mathcal{K}^\bullet = 0$. In other words, $\mathcal{E}$ is a tilting generator for $D^b_{coh}(X_h)$.

Finally, we need to remove the quantization parameter h. To do this, we note that as before, there exists a subalgebra $O_h \subset k'((h))$ of finite type over k', a scheme Y'_h of finite type over O_h, an O_h-valued point y'_h : $\operatorname{Spec} O_h \to Y_h$, and a scheme X'_h smooth and projective over Y'_h such that $Y'_h \otimes_{O_h} k'((h)) \cong Y_h$, $y'_h \times_{O_h} k'((h)) = y_h$, and $X'_h \times_{Y'_h} Y_h \cong X_h$. Moreover, possibly passing to étale covers of open subsets and localizing O_h, we may assume that the natural maps $Y'_h \to \operatorname{Spec} O_h$ and $Y'_h \to Y$ are smooth, and that the tilting generator $\mathcal{E}$ comes from a tilting generator on X'_h. Take a closed point $x \in \operatorname{Spec} O_h$. Its residue field K is then a finite extension of the field k', hence of the original field k. The fiber Y_K of the scheme Y'_h over $x \in O_h$ is an étale neighborhood of $y \in Y$ and satisfies the conditions of Theorem 1.4. $\square$

Remark 4.4. Artin approximation is used two times in our proof of Theorem 1.4: firstly, in Lemma 2.5 (to show that the twistor deformation $\mathcal{Y}$ is algebraic), and secondly, in the proof itself (to lift the tilting generator from positive characteristic to characteristic 0). The second instance is standard, and there is no way around it. The first one is less standard. We note, however, that it can in fact be avoided. Indeed, our argument does not really need the full twistor deformation $\mathcal{Y}$; at the cost of a slight modification of the notion of a twistor deformation, one can work just as well with an arbitrary truncation $\mathcal{Y}/t^N$. After quantization, this only gives the sheaves $\mathcal{O}_a$ with $a = 0, \dots, N \in \mathbb{Z}/p\mathbb{Z}$; but if N is high enough, this is sufficient to find a non-critical line.

Remark 4.5. Fedosov quantization was defined originally in characteristic 0, not in positive characteristic (see e.g. [BK1]). With an argument similar to ours, one should be able to prove a result parallel to our Theorem 1.4: for a generic value of quantization parameter, the manifold X is $\mathcal{O}_h(h^{-1})$-affine, where $\mathcal{O}_h$ is the quantized structure sheaf. This might even be easier to prove than the corresponding positive characteristic statement. On the other hand, in examples such as $X = T^*G/B$, where G/B is the flag variety of a reductive algebraic group G, a stronger statement is true: not only is X $\mathcal{O}_h(h^{-1})$-affine in the derived category sense, but the global sections functor is exact, so that even the corresponding abelian categories are equivalent (see [BB]). It would be interesting to try to prove a similar statement for general X. It would be also interesting to find exactly the critical values of the quantization parameter a.

5 Addenda.

5.1 ***D*-equivalence.** We will now prove all the additional statements given in Section 1. We start with the comparison between two resolutions.

Proof of Theorem 1.6. Assume given two smooth resolutions X, X' of a normal irreducible affine variety Y with trivial canonical bundles K_X, $K_{X'}$. Assume in addition that X admits a closed non-degenerate 2-form Ω_X. Fix a closed point $y \in Y$. We note right away that since K_X is trivial, Y has rational singularities; in particular, it is a Cohen-Macaulay scheme.

As in the proof of Theorem 1.4, we may replace Y by an étale neighborhood of the point y so that it admits an *exact* non-degenerate form $\Omega_X = d\alpha_X$. Since X and X' are smooth birational varieties, the form α_X extends to a 1-form $\alpha_{X'}$ on X'. Its top power $(d\alpha_{X'})^{\operatorname{dim} X'}$ is a section of the canonical bundle $K_{X'}$. Since $K_{X'}$ is trivial, the map $X' \to Y$ is one-to-one over the smooth locus $Y^{sm} \subset Y$, and $(d\alpha_{X'})^{\operatorname{dim} X'}$ is a global function on X' which is invertible on Y^{sm}. Since Y is normal, $(d\alpha_{X'})^{\operatorname{dim} X'}$ is non-zero everywhere on X', and $d\alpha_{X'}$ is therefore a non-degenerate 2-form on X'.

Thus both X and X' are symplectic resolutions of Y. By [K2, Lemma 2.11], this means that the maps $\pi : X \to Y$, $\pi' : X' \to Y$ are semismall. In particular, there exists a closed subvariety $Z \subset Y$ such that $\operatorname{codim} Z \geq 4$, and the fibers of the maps π, π^{-1} over points in the complement $U = Y \setminus Z$ are of dimension at most 1. Since K_X and $K_{X'}$ are trivial, this implies that the graph $\widetilde{X} \subset X \times_Y X'$ of the rational map $X \dashrightarrow X'$ projects bijectively both to X and to X' over $U \subset Y$. Thus $X^o = \pi^{-1}(U) \cong \pi'^{-1}(U)$ is a common open subscheme in X and X'. Moreover,

$$H^i(X^o, \mathcal{O}_{X^o}) = \mathbb{H}^i(U, R^{\bullet}\pi_*\mathcal{O}_{X^o}) = H^i(U, \mathcal{O}_U), \tag{5.1}$$

and since Y is Cohen-Macaulay, this group vanishes for $i = 1, 2$.

Let us now trace the steps of the proof of Theorem 1.4 for symplectic schemes X/Y, X'/Y. First of all, we find a subalgebra $O \subset k$ of finite type over $\mathbb{Z}$ and O-models Y_O, X_O, X'_O. We obviously can arrange the construction so that the same Y_O serves for both X and X'. Moreover, we can arrange so that X^o admits a model X^o_O, and $H^i(X^o, \mathcal{O}_{X^o}) = 0$ for $i = 1, 2$ (we can arrange $R^{\geq 1}\pi_*\mathcal{O}_X = 0$, and the model Y_O is still a Cohen-Macaulay scheme, so that (5.1) applies). Then we fix an ample line bundle L on X, choose a maximal ideal $\mathfrak{m} \subset O$ with residual characteristic $p = \mathsf{char}(O/\mathfrak{m})$ and an element $a \in \mathbb{Z}/p\mathbb{Z}$, and take the deformation $\mathcal{O}_a$ of the structure sheaf $\mathcal{O}_{X_\mathfrak{m}}$ which corresponds to a non-critical parameter $a \in \mathbb{Z}/p\mathbb{Z}$. By Lemma 2.7, we have $\mathcal{O}_a(h^{-1}) \cong \mathcal{E}nd(\mathcal{E})$ for some tilting vector bundle $\mathcal{E}$ on $\widetilde{X_\mathfrak{m}}$.

Now, by [BK3, Proposition 1.22], there is a family of one-parameter deformations $\mathcal{O}_h$ of the structure sheaf $\mathcal{O}_X$ which are numbered by elements of the cohomology group $Q(X) = H^1_{et}(X, \mathcal{O}^*_X/\mathcal{O}^{*p}_X)$; in [BK3], they are called Frobenius-constant quantizations on X. In this language, the deformation $\mathcal{O}_a$ corresponds to the image of $a[L] \in \mathsf{Pic}(X)$ under the natural map $\mathsf{Pic}(X) = H^1_{et}(X, \mathcal{O}^*_X) \to Q(X)$. A similar family of deformations exists for the scheme $X'_\mathfrak{m}$. Since the complement to $X^o_\mathfrak{m} \subset X'_\mathfrak{m}$ is of codimension at least 2, the line bundle L extends to a line bundle L' on $X'_\mathfrak{m}$; denote by $\mathcal{O}'_a$ the deformation which corresponds to the image of $a[L'] \in \mathsf{Pic}(X')$ under the map $\mathsf{Pic}(X') \to Q(X')$. Then the same argument as in Lemma 2.7 shows that $\mathcal{O}'_a(h^{-1}) \cong \mathcal{E}nd(\mathcal{E}')$ for some tilting vector bundle $\mathcal{E}'$ on $\widetilde{X'_\mathfrak{m}}$. On the other hand, since $H^i(X^o_\mathfrak{m}, \mathcal{O}_{X^o}) = 0$ for $i = 1, 2$, the deformations $\mathcal{O}_a$, $\mathcal{O}'_a$ coincide on X^o. Indeed, while [BK3, Definition 1.21] in fact requires control over the cohomology for $i = 1, 2, 3$, but as the more precise [BK3, Proposition 1.18] shows, this is only needed to insure the existence of quantizations; to show that two quantizations with the same parameter are isomorphic, it is enough to require that the first and the second cohomology groups are trivial. Thus we obtain vector bundles $\mathcal{E}$ on $\widetilde{X}_\mathfrak{m}$ and $\mathcal{E}'$ on $\widetilde{X'}_\mathfrak{m}$ such that $\operatorname{Ext}^i(\mathcal{E}, \mathcal{E}) = \operatorname{Ext}^i(\mathcal{E}', \mathcal{E}') = 0$ for $i \geq 1$ and $\mathcal{E} \cong \mathcal{E}'$ on the common open subset $X^o_\mathfrak{m}$. Moreover, since $\mathcal{E}nd\,\mathcal{E}$ is isomorphic to $\mathcal{O}_a(h^{-1})$, we also have $H^i(X^o_\mathfrak{m}, \mathcal{E}nd\,\mathcal{E}) = \operatorname{Ext}^i_{X^o_\mathfrak{m}}(\mathcal{E}, \mathcal{E}) = 0$ for $i = 1, 2$.

Then we lift $\mathcal{E}$ and $\mathcal{E}'$ to tilting vector bundles on $\widehat{X}_\mathfrak{m}$ and $\widehat{X}'_\mathfrak{m}$, and, possibly replacing Y with an étale cover, obtain tilting vector bundles $\mathcal{E}$, $\mathcal{E}'$ on X, X'. Since $\operatorname{Ext}^i_{X^o_\mathfrak{m}}(\mathcal{E}, \mathcal{E}) = 0$ on $X^o_\mathfrak{m}$ for $i = 1, 2$, the lifting is unique on $\widehat{X^o}_\mathfrak{m}$, and the resulting tilting vector bundles $\mathcal{E}$, $\mathcal{E}'$ are isomorphic on X^o. Since the complements $X \setminus X^o$, $X' \setminus X^o$ have codimension at least 2 this implies that

$$R' = \operatorname{End}(\mathcal{E}') = H^0(X', \mathcal{E}nd(\mathcal{E}')) = H^0(X^o, \mathcal{E}nd(E)) = \operatorname{End}(E) = R.$$

Now, we note that as in the proof of Theorem 1.4, we can choose $\mathfrak{m} \subset O$ and

$a \in \mathbb{Z}/p\mathbb{Z}$ so that the resulting tilting vector bundle $\mathcal{E}$ generates the derived category $D^b(X)$. Since the line bundle L' is *not* ample on X', we cannot insure the same for X'. However, since $D^b(X) \cong D^b(R\text{-mod}^{\mathfrak{fg}})$, we do know that $R' = R$ has finite global homological dimension, and we do have a pair of adjoint functors $D^b(R'\text{-mod}^{\mathfrak{fg}}) \to D^b(X)$, $D^b(X) \to D^b(R'\text{-mod}^{\mathfrak{fg}})$. Moreover, their composition $D^b(R'\text{-mod}^{\mathfrak{fg}}) \to D^b(X') \to D^b(R'\text{-mod}^{\mathfrak{fg}})$ is the identity functor, so that the functor $D^b(R'\text{-mod}^{\mathfrak{fg}}) \to D^b(X')$ is a fully faithful embedding with admissible image. To finish the proof, it suffices to use the following standard trick.

Lemma 5.1. *Assume given an irreducible smooth variety X with trivial canonical bundle K_X equipped with a birational projective map $\pi : X \to Y$ to an affine variety Y. Then any non-trivial admissible full triangulated subcategory in $D^b(X)$ coincides with the whole $D^b(X)$.*

For the proof we refer the reader, for instance, to [BK2, Section 2]. □

5.2 Positive weights. Next, we turn to the situation of a positive-weight $\mathbb{G}_m$-action.

Proof of Theorem 1.8. To prove (i), we first note that since the map $\pi : X \to Y$ is semismall by [K2, Lemma 2.11], the differential ξ of the $\mathbb{G}_m$-action on Y extends to a vector field on X by [GK, Lemma 5.3]. Moreover, fix a relatively very ample line bundle L on X; then since $H^1(X, \mathcal{O}_X) = 0$ by the Grauert-Riemenschneider Vanishing, the line bundle L admits an action of the vector field $\xi \in H^0(X, \mathcal{T}(X))$. Replacing Y with its normalization, we may assume that Y is normal. Then every global section $s \in H^0(X, L^{-1})$ gives an embedding $L \hookrightarrow \mathcal{O}_X$, which identifies $H^0(X, L)$ with an ideal $I_s \subset A = H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$. Since L is very ample, the algebra $\bigoplus_i H^0(X, L^{\otimes i})$ is generated by $H^0(X, L)$, so that we also have $H^0(X, L^{\otimes n}) = I_s^n \subset A$, and X is isomorphic to the blow-up of Y in the ideal I. Thus to lift the $\mathbb{G}_m$-action on Y to X, it suffices to show that $I_s \subset A$ is $\mathbb{G}_m$-invariant for an appropriate choice of $s \in H^0(X, L^{-1})$. Since a subspace in A is $\mathbb{G}_m$-invariant if and only if it is preserved by ξ, it suffices to assume that s is an eigenvector of the vector field ξ: $\xi(s) = \lambda s$ for some $\lambda \in k$.

Indeed, denote by $\mathfrak{m} \subset A$ the maximal ideal of the point $y \in Y$, consider the A-module $B = H^0(X, L^{-1})$, and let λ be an eigenvalue of the operator ξ on the finite-dimensional k-vector space $B/\mathfrak{m}B$. Then since the $\mathbb{G}_m$-action on Y is positive-weight, there exists an integer N such that for any $l \geq N$, the λ-eigenspace of ξ on $(\mathfrak{m}^l/\mathfrak{m}^{l+1}) \otimes (B/\mathfrak{m}B)$ is trivial. This means that $\xi - \lambda\,\mathsf{id}$ is invertible on $\mathfrak{m}^N B \subset B$. Therefore the map $B \to B/\mathfrak{m}^N B$ is isomorphic on λ-eigenspaces, and every λ-eigenvector s_0 of ξ on the finite-dimensional k-vector space $B/\mathfrak{m}^N B$ lifts to a λ-eigenvector $s \in B$.

To prove (ii), note that by assumption, the tilting vector bundle $\mathcal{E}$ is defined on $\widehat{X} = \widehat{Y} \times_Y X$, where $\widehat{Y}$ is the completion of Y at $y \in Y$. Moreover, since $\mathcal{E}$ is tilting, we have $H^1(\widehat{X}, \mathcal{E}nd\,\mathcal{E}) = 0$, so that $\mathcal{E}$ admits an action of the vector field ξ. Therefore the complete Noetherian module $M = H^0(\widehat{X}, \mathcal{E})$ over the complete Noetherian algebra $\widehat{A} = H^0(\widehat{Y}, \mathcal{O}_{\widehat{Y}})$ admits an action of the derivation $\xi : \widehat{A} \to \widehat{A}$. In order to lift it to a $\mathbb{G}_m$-action, we first have to correct it.

Lemma 5.2. *Assume given an Artin k-algebra B with ideal $I \subset B$. Then for every $\mathbb{G}_m$-action $\mathbb{G}_m \to \mathsf{Aut}(B_0)$ on the quotient $B_0 = B/I$ whose differential lifts to a derivation of B, the composition $\mathbb{G}_m \to \mathbb{G}_m \to \mathsf{Aut}(B_0)$ with some non-trivial map $\mathbb{G}_m \to \mathbb{G}_m$, $\lambda \mapsto \lambda^N$ lifts to a $\mathbb{G}_m$-action on B. Moreover, if B is local with maximal ideal $\mathfrak{m} \subset B$, and $I \subset \mathfrak{m}^2$, then we can take $N = 1$, and the lifting is unique.*

Proof. (Compare [K2, Section 4].) To prove uniqueness, note that the difference between two liftings is a $\mathbb{G}_m$-action on B which is trivial on $\mathfrak{m}/\mathfrak{m}^2$; therefore its differential $\xi \in \operatorname{End}_k(B)$ is nilpotent, and being semisimple it must be trivial. To prove the rest, let ξ_s be the semisimple part in the Jordan decomposition of the endomorphism $\xi \in \operatorname{End}_k(B')$. The Lie algebra $\operatorname{End}_k(B)$ acts on the tensor product $B^* \otimes B^* \otimes B$, a map $a \in \operatorname{End}_k(B)$ is derivation if and only if $a(m) = 0$, where $m \in B^* \otimes_k B^* \otimes B$ is multiplication $B \otimes_k B \to B$ in B. Since the Jordan decomposition is universal, $\xi(m) = 0$ implies $\xi_s(m) = 0$, so that $\xi_s : B \to B$ is also a derivation. Moreover, ξ_s preserves the line $\Lambda^{\mathsf{dim}\, I} I \subset \Lambda^{\mathsf{dim}\, I}(B)$, which means that ξ_s preserves $I \subset B$. Since ξ is by assumption semisimple on B_0, the difference $\xi - \xi_s$ maps B into $I \subset B$, and $\xi_s = \xi$ on $B_0 = B/I$. Denote by $T \subset \mathsf{Aut}(B)$, $T_0 \subset \mathsf{Aut}(B_0)$ the minimal algebraic subgroups whose Lie algebras contain ξ_s. Then $T_0 \cong \mathbb{G}_m$, T is an algebraic torus, and to finish the proof, it remains to show that the natural map $T \to T_0$ is surjective. This immediately follows from the minimality of $T_0 \subset \mathsf{Aut}(B_0)$. $\square$

Denote by $\mathfrak{m} \subset \widehat{A}$ the maximal ideal of the local k-algebra $\widehat{A}$. Applying this Lemma inductively to the split square-zero extensions $(A/\mathfrak{m}^n A) \oplus (M/\mathfrak{m}^n M)$, we see that the given $\mathbb{G}_m$-action on $\widehat{A}$ extends to a $\mathbb{G}_m$-action on the module $\widehat{M}$. To finish the proof, we apply the following standard general fact.

Lemma 5.3. *Assume given a finitely-generated k-algebra A equipped with* *a positive-weight $\mathbb{G}_m$-action, let $\mathfrak{m} \subset A$ be the ideal of elements of strictly positive weights, and let $\widehat{A}$ be the completion of A with respect to the $\mathfrak{m}$-adic topology. Then the $\mathfrak{m}$-adic completion functor is an equivalence between the category of finitely-generated $\mathbb{G}_m$-equivariant A-modules and the category of complete Noetherian $\mathbb{G}_m$-equivariant $\widehat{A}$-modules.*

Proof. Assume given a complete Noetherian $\mathbb{G}_m$-equivariant module N. Since N is Noetherian, the quotient $N/\widehat{\mathfrak{m}}^l N$ is a finite-dimensional k-vector space for every l. For finite-dimensional spaces, a $\mathbb{G}_m$-action is the same as a grading, so that $N/\widehat{m}^l N$ splits into a sum of components $(N/\widehat{\mathfrak{m}}^l N)^p$ of weight p, $p \in \mathbb{Z}$. Consider the space

$$N^f = \bigoplus_p N^p = \bigoplus_p \lim_{\leftarrow} (N/\widehat{\mathfrak{m}}^{\bullet} N)^p \tag{5.2}$$

of $\mathbb{G}_m$-finite vectors in N. Then N^f is a $\mathbb{G}_m$-equivariant A-module, and we claim that $N \mapsto N^f$ is an equivalence inverse to $M \mapsto \widehat{M}$. Indeed, since for any finitely generated $\mathbb{G}_m$-equivariant A-module M we obviously have $\widehat{M}/\widehat{\mathfrak{m}}^l \widehat{M} \cong M/\mathfrak{m}^l M$, we have $\widehat{M}^f \cong M$. Conversely, for any Noetherian complete $\mathbb{G}_m$-equivariant module N and any l, we have a natural map $N^f/\mathfrak{m}^l N^f \to \widehat{N}/\mathfrak{m}^l \widehat{N}$. This map is surjective by definition. Moreover, since $\mathfrak{m}$ has positive weights, for every p the inverse limit in (5.2) actually stabilizes at some finite level. Therefore every $a \in N$ is uniquely represented as $a = \lim_{l \to \infty} a^{\leq l}$, where $a^{\leq l} \in N^f \subset N$ is a finite sum of vectors $a^i \in N^i$ of weights $i \leq l$. Since N is Noetherian, for some fixed constant $q \in \mathbb{Z}$ we have $N^l = 0$, $l < q$. Then for every $a \in N$, $a \in \widehat{\mathfrak{m}}^p N$ means by definition

$$a = \sum_i m_i a_i$$

for some $a_i \in N$, $m_i \in \widehat{\mathfrak{m}}^p$; if a is $\mathbb{G}_m$-finite, so that $a = a^{\leq l}$ for some l, this implies

$$a = \sum_i m_i^{\leq (l-q)} a_i^{\leq l},$$

so that $a \in \mathfrak{m}^p N^f$. Therefore $N^f/\mathfrak{m}^p N^f \to N/\widehat{\mathfrak{m}}^p N$ is bijective. In particular, the k-vector space $N^f/\mathfrak{m} N^f$ is finite-dimensional, so that N^f is finitely-generated over A, and moreover, we have $\widehat{N^f} \cong N$. □

5.3 Resolution of the diagonal. Finally, we turn to cohomological results.

Proof of Theorem 1.9. First of all, by Artin approximation [A] it suffices to prove the claim under the additional assumption that $A = H^0(Y, \mathcal{O}_Y)$ is a *complete* local k-algebra. Since $\mathcal{E}$ is a tilting generator for X, so it the dual vector bundle $\mathcal{E}^*$. The product $\mathcal{E}^* \boxtimes \mathcal{E}$ is then a tilting generator for $X \times X$, so that $D^b(X \times X) \cong D^b((R^{op} \otimes R)\text{-mod}^{\mathrm{fg}})$. Under this equivalence, the diagonal sheaf goes to the tautological R-bimodule R. Since $X \times X$ has finite homological dimension, so does the algebra $R^{op} \times R$. Therefore the bimodule R admits a finite projective resolution, and to prove the claim, it suffices to shows that

(i) every indecomposable projective R-bimodule is of the form $P' \boxtimes P$, where P' is a projective right R-module, and P is a projective left R-module, and

(ii) every projective left R-module P corresponds to a vector bundle on X under the equivalence $D^b(X) \cong D^b(R\text{-mod}^{\mathrm{fg}})$.

The second claim is immediate: every projective module is a direct summand of R^N for some N, and R^N corresponds to $\mathcal{E}^N$. To prove (i), note that since A is complete, so is R. Therefore every indecomposable projective module is a projective cover of a unique simple R-module, the same is true for R-bimodules, and it suffices to prove that every simple R-bimodule is of the form $M' \boxtimes M$, M' a simple right R-module, M a simple left R-module. In other words, we may replace R with the semisimple quotient R/I, where $I \subset R$ is the radical of algebra R. Then the claim becomes obvious. $\square$

Proof of Corollary 1.10. We first prove that the natural map $H^\bullet(X, \Lambda) \to H^\bullet(F, \Lambda)$ is an isomorphism (here Λ is the coefficient field, such as $\mathbb{Q}_l$ or $\mathbb{Q}$). Indeed, since the $\mathbb{G}_m$-action on Y lifts to an action of X compatible with map $\pi : X \to Y$, the direct image complex $R^\bullet \pi_* \Lambda$ is $\mathbb{G}_m$-equivariant. By proper base change, it suffices to use the following.

Lemma 5.4. *Assume given a positive-weight $\mathbb{G}_m$-action on an affine algebraic variety Y with fixed point $y \in Y$. Denote by $i_y : y \to Y$ the embedding. Then for any complex $\mathcal{F}^\bullet$ of constructible sheaves on Y, the natural map $H^\bullet(Y, \mathcal{F}^\bullet) \to i_y^* \mathcal{F}^\bullet$ is an isomorphism*

Proof. We have to show that $H^\bullet(Y, j_! j^* \mathcal{F}^\bullet) = 0$, where $j : U \to Y$ is the embedding of the complement $U = Y \setminus \{y\}$. By assumption, the algebra $A = H^0(Y, \mathcal{O}_Y)$ of functions on Y is positively graded. Let $E = \operatorname{Proj} A^\bullet$, and let $\widetilde{Y}$ be the total space of the line bundle $\mathcal{O}(1)$ on E. Then we have a proper map $\tau : \widetilde{Y} \to Y$ and an embedding $\widetilde{j} : U \to \widetilde{Y}$. By proper base change, $\tau_* \widetilde{j}_! \cong \tau_! \widetilde{j}_! \cong j_!$, so that it suffices to prove that $H^\bullet(\widetilde{Y}, \widetilde{j}_! j^* \mathcal{F}^\bullet) = 0$. But since $\mathcal{F}^\bullet$ is by assumption $\mathbb{G}_m$-equivariant, we have $j^*\mathcal{F}^\bullet \cong \tau^*\mathcal{G}^\bullet$ for some complex $\mathcal{G}^\bullet$ on E, and it suffices to prove that the natural map $H^\bullet(\widetilde{Y}, \tau^*\mathcal{G}^\bullet) \to H^\bullet(E, \mathcal{G}^\bullet)$ is an isomorphism. By the projection formula, it suffices to prove that $\tau_*\Lambda_{\widetilde{Y}} \cong \Lambda_E$. This claim is local on E, so that we may assume $\widetilde{Y} \cong E \times \mathbb{A}^1$, and the claim immediately follows from the Künneth formula. □

Now, X is smooth; therefore for every integer l, all weights in the cohomology group $H^l(X, \Lambda)$ are $\geq l$. On the other hand, F is proper; therefore weights in $H^l(F, \Lambda)$ are $\leq l$. We conclude that for every l, $H^l(X, \Lambda) = H^l(F, \Lambda)$ is pure of weight l.

Consider now the $\mathbb{G}_m$-equivariant cohomology groups $H^\bullet_{\mathbb{G}_m}(X, \Lambda)$ – that is, the cohomology groups of the simplicial scheme $X_\bullet$, $X_l = X \times {\mathbb{G}_m}^l$ with face and degeneracy maps coming from the $\mathbb{G}_m$-action on X (for the theory of equivariant cohomology, see [BL]). Then we have a spectral sequence which converges to $H^\bullet_{\mathbb{G}_m}(X, \Lambda)$ and starts with $H^\bullet(X, \Lambda)[u]$, where u is a free generator of degree 2. Since $H^\bullet(X, \Lambda)$ is pure, this spectral sequence collapses, so that $H^\bullet_{\mathbb{G}_m}(X, \Lambda) \cong H^\bullet(X, \Lambda)[u]$. Analogously, for cohomology with compact support we have $H^\bullet_{\mathbb{G}_m,c}(X, \Lambda) \cong H^\bullet_c(X, \Lambda)[u]$. But since all the $\mathbb{G}_m$-fixed points in X lie within proper subvariety $F \subset X$, the standard localization theorem (see e.g. [AB]) shows that the natural map

$$\tau : H^\bullet_{\mathbb{G}_m,c}(X, \Lambda) \to H^\bullet_{\mathbb{G}_m}(X, \Lambda)$$

becomes an isomorphism after we invert the parameter u. Therefore it suffices to prove that the image of the map τ lies in the $k[u]$-submodule generated by classes of algebraic cycles. This immediately follows from the existence of a resolution of the diagonal provided by Theorem 1.9. Indeed, by construction all terms $\mathcal{E}_i$, $\mathcal{F}_i$ in this resolution are direct summands of the bundle $\mathcal{E}^N$ for some integer N. By Theorem 1.8 we may assume that the tilting generator $\mathcal{E}$ is $\mathbb{G}_m$-equivariant. For any idempotent $P \in \operatorname{End}(\mathcal{E}^N)$, its degree-0 component P^0 with respect to the $\mathbb{G}_m$-action is also an idempotent, and $\mathsf{Im}\, P^0 = \mathsf{Im}\, P$; therefore we may assume that all the $\mathcal{E}_i$ and $\mathcal{F}_i$ are $\mathbb{G}_m$-equivariant vector bundles. Then we have

$$\tau(a) = \sum_i (-1)^i \langle [\mathcal{E}_i], a \rangle [\mathcal{F}_i],$$

where $[\mathcal{E}_i]$, $[\mathcal{F}_i]$ are Chern characters of the bundles $\mathcal{E}_i$, $\mathcal{F}_i$, and $\langle -, - \rangle$ is the Poincaré pairing. □

Remark 5.5. It is natural to expect that the statement of Corollary 1.10 holds for Y sufficiently small – say, local Henselian – but without any additional structures such as a group action. Unfortunately, we have not been able to prove it. In particular, the above proof does not work: the natural map $H^\bullet_c(X,\Lambda) \to H^\bullet(X,\Lambda)$ only becomes bijective after we pass to the equivariant cohomology and localize, and in ordinary cohomology, it is often 0 outside of the middle degree. In practice, a $\mathbb{G}_m$-action always exists in the symplectic case, see [K2, Section 4]; but in general it is not possible to prove that it is positive-weight.

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