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ADVANCES IN Mathematics
Advances in Mathematics 198 (2005) 847–862
www.elsevier.com/locate/aim

A p-adic Simpson correspondence

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Received 28 May 2004; accepted 25 May 2005

Communicated by Johan De Jong

Dedicated to M. Artin on the occasion of his 70th birthday

Abstract

For curves over a p-adic field we construct an equivalence between the category of Higgs-bundles and that of "generalised representations" of the étale fundamental group. The definition of "generalised representations" uses p-adic Hodge theory and almost étale coverings, and it includes usual representations which form a full subcategory. The equivalence depends on the choice of an exponential function for the multiplicative group.

Keywords: Almost étale extensions; Higgs-bundles; p-adic Hodge theory

1. Introduction

The purpose of this note is to construct Higgs-bundles associated to representations of the geometric fundamental group of a curve over a p-adic field K. It thus can be considered a p-adic analogue of the results of Simpson and Corlette (see [12]). The functor is fully faithful but it is difficult to characterise its image: namely the resulting Higgs-bundles are semistable of slope zero, but we do not know whether any such Higgs-bundle lies in the image (this is true for line-bundles on curves over p-adic local fields). Conversely we can construct for all Higgs-bundles so-called "generalised representations", which form a category containing the usual representations as full subcategory. However, we do not know which of those come from genuine representations.

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doi:10.1016/j.aim.2005.05.026

Over a local field K and for line-bundles one can check that one gets all line-bundles of degree zero, so one can hope that over local fields all semistable Higgs-bundles of degree zero lie in the image.

Recall that a Higgs-bundle on an algebraic manifold X is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a vectorbundle on X and θ a global section of $\mathcal{E}nd(E) \otimes \Omega_X$ satisfying $\theta \wedge \theta = 0$ (that is in local coordinates the components of θ commute). We also use variants where X is only logsmooth, or where θ has coefficients in some Tate-twist. The latter corresponds (I assume) to a factor $2\pi i$ in the classical complex setup.

We should note that our functors depend on certain choices, the most important being that of an exponential function for the multiplicative group. This induces exponential functions on all commutative group schemes over K. Another choice is that we have to choose lifts to certain types of dual numbers, and different lifts amount to twists by "Higgs-line-bundles". That is we do not obtain just one functor but a whole family of them, all related by such twists.

The method used in the proofs is the theory of almost étale extensions (see [6], for a more systematic treatment [11]) which was developed by the author for applications in p-adic Hodge-theory. We develop a nonabelian Hodge–Tate-theory. What is still missing is an appropriate role for connections and Frobenius, which might result in a more powerful theory generalising Fontaine's ideas.

This work was inspired by the workshop on nonabelian Hodge-theory at MSRI Berkeley, at Easter 2002. We also mention the preprint [9] which uses similar techniques in a different setting.

2. Generalised representations

We denote by V a complete discrete valuation-ring with perfect residue-field k of characteristic $p > 0$ and fraction-field K of characteristic 0. $\bar{K}$ is the algebraic closure of K and $\bar{V}$ the integral closure of V in $\bar{K}$. X is a proper V-scheme which has toroidal singularities (as explained in [6, Chapter 2, Appendix 1]), for example X could be smooth or have semistable singularities. Further more $D \subset X$ is a divisor which satisfies the conditions in [6]. Especially the generic fibre X_K is smooth and D_K a divisor with normal crossings. As in [6] $X^\circ = X - D$.

We have a topos $\mathcal{X}^\circ$ of sheaves on the situs whose objects consists of finite étale coverings of the generic fibres U_K° of schemes $U \to X$ which are étale over X. The normalisation of $\mathcal{O}_U$ in such covers defines a sheaf $\bar{\mathcal{O}}$ on $\mathcal{X}^\circ$. Furthermore if $\mathbb{L}$ is the locally constant sheaf on $\mathcal{X}_K^\circ$ associated to a representation of the fundamental group $\pi_1(X_K^\circ)$ on a finite V-module the associated maps

$$H^i\left(X_{\bar{K}}^\circ, \mathbb{L}\right) \otimes \bar{V} \to H^i\left(\mathcal{X}_{\bar{K}}^\circ, \mathbb{L} \otimes \bar{\mathcal{O}}\right)$$

are almost isomorphisms, that is their kernels and cokernels are annihilated by the maximal ideal of $\bar{V}$ (see [6, Chapter 4, Theorem 9] for curves also [4]). This remains true if $\mathbb{L}$ is only a locally constant étale sheaf on the geometric fibre $X_{\bar{K}}^\circ$ because such a sheaf is already defined over a finite extension of K. It follows that the functor $\mathbb{L} \otimes \bar{\mathcal{O}}$ is fully faithful as a functor from continuous representations of $\pi_1(X^\circ_{\bar{K}})$ on finitely presented torsion $\bar{V}$-modules, to $\bar{\mathcal{O}}$-modules up to almost isomorphisms. (In the first category almost maps coincide with usual maps). Also the essential image is closed under extensions and deformations. Deformations arise if we have a family of representations over a complete local ring, and consider various base-changes to $\bar{V}/(p^s)$. If one of them lies in the essential image so do all. In the following we restrict to representations on free modules over $\bar{V}/(p^s)$ which correspond to vectorbundles over $\bar{O}/(p^s)$. We call the latter generalised representations.

3. The local structure of generalised representations

Now assume given an affine $U = Spec(R) \subset X$ which is small, that is R is étale over a toroidal model. By adjoining roots of characters of the torus we obtain a subextension R_∞ of $\bar{R}$ (the integral closure of R in the maximal étale cover of U°_K) which is Galois over $R_1 = R \otimes_V \bar{V}$ with group $\Delta_\infty = \hat{\mathbb{Z}}(1)^d$. We write it as the union of algebras R_n which have themselves toroidal singularities. Furthermore $\bar{R}$ is almost étale over R_∞ ([6, Section 2, Theorem 4]). This implies that each $\bar{R}$-module with a continuous semilinear action of $\Delta = Gal(\bar{R}/R)$ is almost induced from an R_∞-module with Δ_∞-action. Also we can compute the Galois-cohomology $H^i(\Delta, \bar{R}/(p^s))$ (with s any positive rational number): namely it is almost isomorphic to $H^i(\Delta_\infty, R_\infty/(p^s))$ which in turn is the direct sum of $H^i(\Delta_\infty, R \otimes_V \bar{V}/(p^s))$ and of a direct summand annihilated by $p^{1/(p-1)}$ (see [6, p. 206]). This results from the decomposition of R_∞ into Δ_∞-eigenspaces where the contributions from nontrivial eigenspaces are annihilated by $\zeta - 1$, ζ a nontrivial root of unity. Finally the (interesting) first summand is canonically identified with the logarithmic differentials $\tilde{\Omega}^i_{R/V} \otimes_V \bar{V}/(p^s)$. Examples of such (locally defined) generalised representations are given by homomorphisms

$$\Delta \twoheadrightarrow \Delta_\infty \to GL(r, R \otimes_V \bar{V}/(p^s)),$$

we show that many others are close to these.

Lemma 1. *Suppose $\alpha > 1/(p-1)$ is a rational number, and $\bar{M} \cong \bar{R}^r/(p^s)$ a generalised representation (it admits a semilinear Δ-operation).*

(i) *Suppose that $\bar{M}$ is trivial modulo $p^{2\alpha}$. Then its reduction modulo $p^{s-\alpha}$ is given by a representation $\Delta_\infty \to GL(r.R \otimes_V \bar{V}/(p^{s-\alpha}))$, and this representation is trivial modulo p^α.*

(ii) *Suppose given two representations $\Delta_\infty \to GL(r_i, R \otimes_V \bar{V}/(p^s))$, trivial modulo p^α, and an $\bar{R} - \Delta$-linear map between the associated generalised representations. Then its reduction modulo $p^{s-\alpha}$ is given by an $R_1 - \Delta_\infty$-linear map of representations.*

Proof. For (ii) we consider the representation on the Hom-space, which we call M, and have to show that Δ-invariants in $\bar{M} = M \otimes \bar{R}$ come from Δ_∞-invariants in M. This follows because $H^0(\Delta, \bar{M})$ is almost isomorphic to $H^0(\Delta_\infty, M \otimes R_\infty)$, and the latter decomposes into a direct sum corresponding to the eigenspace decomposition of R_∞. On nontrivial eigenspaces some element of Δ_∞ operates as the sum of a nontrivial root of unity ζ and of an endomorphism divisible by p^α and the corresponding contribution is annihilated by p^α. For (i) we choose a positive rational ε with $\alpha > 3\varepsilon + 1/(p-1)$, and show by induction over n that the assertion holds for the representation modulo $p^{2\alpha+n\varepsilon}$. We may assume that $s \geqslant 3\alpha + n\varepsilon$.

For $n = 0$ $\bar{M}$ modulo $p^{2\alpha}$ is by assumption induced from a (trivial) M. Assume we have found such an M modulo $p^{2\alpha+n\varepsilon}$, inducing $\bar{M}$. If we try to lift M to a representation modulo $p^{3\alpha+n\varepsilon}$ we encounter an obstruction in $H^2(\Delta_\infty, End(M)/(p^\alpha))$ whose image in $H^2(\Delta, End(\bar{M})/(p^\alpha))$ vanishes because $\bar{M}$ lifts. As the induced map is almost injective (over R_∞ it is a direct summand) the obstruction vanishes after multiplication by p^ε, that is M modulo $p^{2\alpha+(n-1)\varepsilon}$ lifts to a representation modulo $p^{3\alpha+(n-1)\varepsilon}$. Over $\bar{R}$ the induced generalised representation differs from $\bar{M}$ by a class in $H^1(\Delta, End(\bar{M})/(p^\alpha))$. Again this cohomology is almost isomorphic to the direct sum of $H^1(\Delta_\infty, End(M)/(p^\alpha))$ and terms annihilated by $p^{1/(p-1)}$. Hence our class becomes "constant" after multiplication by $p^{\alpha-2\varepsilon}$ and vanishes after modifying the lift of M, which is now a lift from coefficients modulo $p^{\alpha+(n+1)\varepsilon}$ to coefficients modulo $p^{2\alpha+(n+1)\varepsilon}$. This finishes the proof. □

Remarks. (i) The result extends to p-adic representations: this follows from the inductive method of liftings.

(ii) For $\alpha > 1/(p-1)$ the exponential and logarithmic series converge for arguments divisible by p^α. Applying the logarithm to the images of generators of Δ_∞ then defines endomorphisms of M or an element of $End(M) \otimes \tilde{\Omega}_{R/V} \otimes \bar{V}(-1)$, divisible by p^α, and with commuting components. We shall see that this element is independent of the choices involved in the construction of R_∞, by defining an inverse functor which associates to such "Higgs-bundles" a generalised representation.

Namely consider Fontaine's rings $A_{inf}(R)$ and $A_{inf}(V)$ associated to R and V (see [8]). $A_{inf}(V)$ surjects onto the p-adic completion $\hat{\bar{V}}$ and the kernel is a principal ideal with generator ξ. Here we only need the quotient $A_2(V) = A_{inf}(V)/(\xi^2)$, which is an extension of $\hat{\bar{V}}$ by $\hat{\bar{V}}\xi$. The latter contains canonically $\hat{\bar{V}}(1) = p^{1/(p-1)}\hat{\bar{V}}\xi$. Similar assertions (with $\bar{V}$ replaced by $\bar{R}$) hold for $A_2(R)$. Also $A_2(V)$ and $A_2(R)$ have natural toroidal (or logarithmic) structures.

Now we first lift $R \otimes_V \hat{\bar{V}}$ to a log-smooth algebra $\tilde{R}$ over $A_2(V)$. Two such lifts are isomorphic but not canonically isomorphic. Namely the automorphism group of a lift is the group of logarithmic derivations $Hom_R(\tilde{\Omega}_{R/V}, R \otimes_V \hat{\bar{V}}\xi)$. Also an étale map from $Spec(R)$ to a toroidal model defines such an $\tilde{R}$ induced from the toroidal model.

Next we lift $R \subset \hat{\bar{R}}$ to an $A_2(V)$-linear $\tilde{R} \to A_2(R)$. Again two lifts differ by a logarithmic derivation into $\hat{\bar{R}}\xi$. If $M \cong R \otimes_V \bar{V}^r/(p^s)$ denotes a free module together with an endomorphism $\theta \in End(M) \otimes_R \tilde{\Omega}_{R/V}$ such that $\theta \wedge \theta = 0$ (that is θ defines a Higgs-bundle, or the components of θ commute), and θ is divisible by p^α for some $\alpha > 1/(p-1)$, then the two pushforwards of M via different lifts $\tilde{R} \to A_2(R)$ are canonically isomorphic:

Use the fact that θ behaves like a connection. For example if R is smooth over V, $t_i \in R$ form local coordinates with associated derivations $\partial_i = \partial/\partial t_i$, and if the two lifts differ on t_i by $u_i\xi \in \hat{\bar{R}}\xi$, then an isomorphism is given by the Taylor-series

$$\sum_I \theta(\partial)^I(m)/I! \otimes u^I.$$

Here $m \in M$, the sum is over all multi-indices $I = (i_1, \ldots, i_d)$, and there is an obvious divided power structure to explain how to divide by the factorials.

For more general logarithmic coordinates one has to use the logarithmic Taylor-series: for one variable t with derivative $\tilde{\partial} = t\partial/\partial t$ and two lifts differing by $tu\xi$ one obtains

$$\sum \theta(\tilde{\partial})^n(m)/n! \otimes u^n,$$

and the same for several variables. Note that the pushforward is always the same module $M \otimes \hat{\bar{R}}$ but that different lifts of $\tilde{R}$ induce nontrivial automorphisms of this module. Also θ should be considered as an element of $End(M) \otimes \hat{\bar{R}}\xi^{-1}$. Finally everything extends to p-adic Higgs-bundles, by passing to a p-adic limit.

Now the Galois-group Δ acts on lifts $\tilde{R} \to A_2(R)$ and thus semilinearly on $M \otimes \hat{\bar{R}}$. If we choose an étale map from $Spec(R)$ to a toroidal model we obtain a lift $\tilde{R}$ mapping to $A_2(R)$ by extracting p-power roots out of characters of the torus (which map to elements of R). Also we have an R_∞ and Δ acts on the preferred lifting via its quotient $\Delta_\infty = \hat{\mathbb{Z}}(1)^d$. That group acts in turn on the ith coordinate via its ith projection to $\hat{\mathbb{Z}}(1)$ and the inclusion $\hat{\bar{R}}(1) \subset \hat{\bar{R}}\xi$. It follows that θ induces the previous element of $End(M) \otimes_R \tilde{\Omega}_{R/V}(-1)$ via $\hat{\bar{R}}\xi^{-1} \subset \hat{\bar{R}}(-1)$. As conclusion we get that for a Higgs-bundle modulo p^s with θ divisible by p^α ($\alpha > 1/(p-1)$) we get an associated generalised representation modulo p^s which will be trivial modulo p^β for any $\beta < \alpha + 1/(p-1)$. Conversely, if we start with a generalised representation modulo (p^s) which is trivial modulo p^α we obtain a bundle with endomorphism modulo $(p^{s-\alpha})$, and dividing this endomorphism by $p^{1/(p-1)}$ gives a Higgs-bundle modulo $p^{s-\alpha-1/(p-1)}$. These procedures are inverse up to a loss of exponents s which can be bounded by any $\beta > 2/(p-1)$. Passing to the limit $s \to \infty$ we obtain an equivalence of categories between generalised p-adic representations which are trivial modulo p^β for some $\beta > 2/(p-1)$, and p-adic Higgs-bundles with θ divisible by p^α for some $\alpha > 1/(p-1)$. We formalise this as follows:

Definition 2. A Higgs-module (M, θ) (M a finitely generated free R-module, $\theta \in End_R(M) \otimes_R \Omega_{R/V} \otimes_R \hat{\bar{R}}\xi^{-1}$) is called small if θ is divisible by p^α for some $\alpha > 1/(p-1)$. A generalised representation $\bar{M}$ is called small if it is trivial modulo $p^{2\alpha}$.

Thus our local theory gives a bijection between small Higgs-modules and small generalised representations.

There also exists a $\mathbb{Q}_p$-theory where we consider continuous Δ-representations on finitely generated projective $\hat{\bar{R}}[1/p]$-modules and projective $\hat{R}_1[1/p]$-modules with an endomorphism θ with coefficients $\tilde{\Omega}_{X/V}(-1)$, $\theta \wedge \theta = 0$. For simplicity we do this only for the case of curves, that is for relative dimension $d = 1$.

Suppose first that we are given a finitely generated projective $R\hat{\otimes}_V \bar{V}[1/p]$-module M with an $\tilde{\Omega}_{X/V}(-1)$-valued endomorphism θ. Then the coefficients λ_i of the characteristic polynomial of θ (that is the traces of $\bigwedge^i \theta$ on $\bigwedge^i M$) are elements of $\tilde{\Omega}^{\otimes i}_{X/V} \otimes R\hat{\otimes}_V \bar{V}[1/p]$. If we assume that they are integral and divisible by $p^{2i\alpha}$ for some $\alpha > 1/(p-1)$ we can find a finitely generated $R\hat{\otimes}_V \bar{V}$-submodule $M^\circ \subset M$ which is stable under $\theta/p^{2\alpha}$ and which generates M. The previous constructions (using local lifts of R to $A_2(V)$ and to $A_2(R)$) then define a representation of Δ on the p-adic completion of $\bar{R} \cdot M$. Less canonically the quotient $\Delta_\infty = \hat{\mathbb{Z}}(1)$ acts by exponentiating θ. Thus we get a functor from small $\mathbb{Q}_p$-Higgs-bundles ("small" is now defined in terms of divisibility of the coefficients λ_i) to generalised $\mathbb{Q}_p$-representations.

For the converse assume Δ operates semilinearly on a projective $\hat{\bar{R}}[1/p]$-module $\bar{M}$ and assume it is generated by a finitely generated $\hat{\bar{R}}$-submodule $\bar{M}^\circ$ such that $\bar{M}^\circ$ is generated by elements which are Δ-invariant modulo $p^{2\alpha}\bar{M}^\circ$, for some $\alpha > 1/(p-1)$. Replacing α by a slightly smaller α' we may replace $\bar{M}^\circ$ by its invariants under $Gal(\bar{R}/R_\infty)$ and consider the problem with coefficients $\hat{R}_\infty$. Then lifting the action of a generator of $\Delta_\infty = \hat{\mathbb{Z}}(1)$ on generators of $\bar{M}^\circ$ we find a small action of Δ_∞ on some $\hat{R}^n_\infty$ and a Δ_∞-linear surjection of this module onto $\bar{M}^\circ$. That is $\bar{M}$ becomes the quotient of the $\mathbb{Q}_p$-object defined by an integral small generalised representation. Applying the same reasoning to the kernel we get a resolution by integral small generalised representations. The cokernel of the induced map on associated Higgs-bundles then defines the inverse functor.

Both functors are fully faithful: using resolutions and duals we reduce to the previous results for integral objects. Thus:

Theorem 3. *The construction above defines, for small toroidal affines, an equivalence of categories between small generalised representations and small Higgs-bundles, or generalised $\mathbb{Q}_p$-representations and Higgs-bundles (both assumed small) on the generic fibre.*

We can use these methods to compare cohomologies, first in a locals setting but in such a way that it will globalise later. Suppose M is a finitely generated projective $\hat{R}_1$-module with a Higgs-field θ such that all invariants λ_i are divisible by $p^{2i\alpha}$, $\alpha > 1/(p-1)$. Then we can find a sublattice $M^0 \subset M$ with $\theta(M^0) \subseteq p^{2\alpha}M^0 \otimes \tilde{\Omega}_{R/V}\xi^{-1}$, and a representation of Δ on M^0 which induces a generalised representation $\bar{M}$. The cohomology-groups $H^i(\Delta, \bar{M})$ can be, up to almost isomorphism, computed by reduction to R_∞ and the action of $\Delta_\infty = \hat{\mathbb{Z}}^d(1)$. Decomposing R_∞ into eigenspaces we see that the contribution of nontrivial eigenspaces is annihilated by $p^{1/(p-1)}$, while the trivial eigenspace has the same cohomology as the Koszul-complex $\bigwedge^*(\tilde{\Omega}_{R/V} \otimes \hat{R}_0\xi^{-1}) \otimes M^0$, with exterior multiplication by θ as differential. A map of complexes inducing this isomorphism can be constructed as follows:

Consider the symmetric algebra $\mathcal{S} = \bigoplus_{n\geqslant 0} S^n(\tilde{\Omega}_{R/V} \otimes \hat{R}_0\xi^{-1})$. It has a Higgs-field $\theta = -\sum \partial_i \otimes \omega_i$, where the ω_i are a local basis of $\tilde{\Omega}_{R/V}$ and the ∂_i the dual derivations (of degree -1) on the symmetric algebra. The Koszul-complex of $-\theta_{\mathcal{S}}$ is a resolution of R_0. Also the Higgs-field $\theta_{\mathcal{S}}$ has divided powers $\theta^n_{\mathcal{S}}/n!$, and the associated exponential series is finite when applied to elements of $\mathcal{S}$.

The completed tensorproduct $M^0\hat{\otimes}\hat{\mathcal{S}}$ also has a Higgs-field and by the usual procedure we obtain Δ-representations on the p-adic completion of the tensorproduct with $\bar{R}$. This gives a resolution of $\bar{M}$ in the category of continuous Δ-representations, and as for any such resolution its Δ-invariants map in the derived category to the complex representing cohomology (in fact the analogue also holds for all coefficients M^0/p^sM^0, thus one can avoid continuous cohomology with $\mathbb{Q}_p$-coefficients). Finally the Koszul-complex for M^0 maps into the complex of invariants, by sending m to the sum $\sum_{n\geqslant 0}\theta^n(m)/n!$ in the p-adic completion of $M^0 \otimes \mathcal{S}$.

For general $\mathbb{Q}_p$-representations we denote $R_0 = R \otimes_V \bar{V}$ and by R_n the result of adjoining all roots of order $n!$ of the toroidal coordinates, so that $R_0 \subset R_n \subset R_\infty$, and the R_n are still toroidal over $\bar{V}$. We now pass to an R_n where our representation becomes small, thus obtaining invariants λ_i in the p-adic completion of $\tilde{\Omega}^{\otimes i}_{R_n/V}\xi^{-i}$. These are invariant under $Gal(R_n/R_0)$ and thus define elements in $\hat{\tilde{\Omega}}^{\otimes i}_{R/V} \otimes_R R_0[1/p]\xi^{-i}$. We claim that they are independent of the choice of the R_n:

Lemma 4. *The invariants λ_i do not depend on the choice of R_∞ and commute with basechange.*

Proof. We use that we have found a lift of R to $A_2(V)$ and to $A_2(R)$ such that for each $\delta \in \Delta$ and a in the lift we have $\delta(a) - a \in \hat{R}_0\xi$, and for δ in a subgroup Δ_n of finite index this is divisible by $p^{n+\alpha}$. Assume more generally that we have a finite subextension $R_0 \subset S \subset \bar{R}$ such that $\delta(a) - a \in p^{n+\alpha}\hat{S}\xi$ if δ lies in the subgroup Δ_S fixing S, and an element $\theta \in p^{-n}\tilde{\Omega}_{R/V} \otimes End(S^r)\xi^{-1}$. Then we can define an action of Δ_S on $\hat{S}^r$ and on $\hat{\bar{R}}^r$ by the Taylor-series above, which converges because of the assumptions on divisibility. We also assume that there are subgroups $\Delta_{S,m} \subset \Delta$, of finite index, such that $\delta(a) - a$ is divisible by $p^{m+n+\alpha}$ if $\delta \in \Delta_{S,m}$.

Now assume that two such data define representations which become isomorphic after tensoring with $\mathbb{Q}_p$. We then claim that the two θ's have the same coefficients λ_i in their characteristic polynomials. To show this we are allowed to enlarge the S's and may assume that they coincide. The two lifts of R to $A_2(R)$ then differ by a logarithmic derivation $R \to \hat{\bar{R}}\xi$. Its reduction modulo $p^{m+n+\alpha}$ is invariant under $\Delta_{S,m}$. As the discriminant of the compositum of S and R_s over R_s (our original sequence) divides p for big s we can pass to this compositum and then have that after multiplication by p our $\Delta_{S,m}$-invariants are S-linear combinations of invariants under

$Gal(\bar{R}/R_s)$. However, from the explicit decomposition of R_∞ into Δ_∞-eigenspaces it follows that such invariants modulo $p^{m+n+\alpha}$ are almost sums of elements of $R_s\xi$ and elements annihilated by $p^{1/(p-1)}$. Thus multiplying again by (say) p^2 we see that our invariant lifts. That means after replacing Δ_S by a smaller subgroup we may assume that the two lifts differ by the sum of a derivation with values in $\hat{S}\xi$ and of a derivation with values in $p^{n+\alpha}\hat{\bar{R}}\xi$. Changing one of the lifts we get rid of the first term.

But then previous arguments imply that the two lifts give isomorphic representations, so we may assume that the two lifts coincide and only the θ's might differ. But then one concludes that an isomorphism between the two representations has coefficients in $\hat{S}\otimes\mathbb{Q}_p$ and conjugates the two θ's. Namely to show that the coefficients are invariant under Δ_S localise and reduce to R a discrete valuation-ring, then replace R by S. This finally shows the claim. $\square$

For a generalised $\mathbb{Q}_p$-representation over R we can pass to some R_n where it is given by θ acting on a projective $\hat{R}_n[1/p]$-module, then add a direct summand (with trivial θ) to make it free of rank r. Thus our reasoning applies also to such modules and shows that the λ_i are canonical.

Finally the same type of reasoning shows that the definition of λ_i's commutes with basechange: assume $R \to R'$ is a logarithmic map, and we are given a generalised $\mathbb{Q}_p$-representation over R which induces such a representation over R'. Then we can choose S and S' as above, and assume that the map extends to $S \to S'$. Next we compare the two maps from an $A_2(V)$-lift of R to $A_2(R')$ obtained by either mapping to $A_2(R) \to A_2(R')$, or first to a lift of R' and then to $A_2(R')$. For any element a of the lift we obtain two 1-cocycles on $\Delta'_{S'}$ with values in $\hat{S}'\xi$ which are divisible by p^n, and whose difference is a boundary. As before we then can assume that it is the boundary of an element divisible by p^n.

The restriction to $\Delta'_{S'}$ of the pushforward is defined by the first cocycle, and a Higgs-field $\theta \in End(S^r)\otimes\tilde{\Omega}_{R/V}\xi^{-1}$, and (after basechange $S \to S'$) isomorphic to the representation given by the second cocycle. However, this second cocyle is defined for all lifts of elements of R', not just of R, and we get the representation defined by the pushforward of θ. But this is our claim.

4. Globalisation

Again X is proper over $Spec(V)$, with toroidal singularities. A small generalised representation of its fundamental group is defined as a compatible system of small generalised representations on a covering of X by small affines. For example, it can be defined by a representation of $\pi_1(X^\circ \otimes_V \bar{K})$ on $\hat{\bar{V}}^r$ which is trivial module $p^{2\alpha}$, $\alpha > 1/(p-1)$. However, a representation may induce locally small generalised representations without being trivial modulo some p-power. A small Higgs-bundle on X is a vector-bundle $\mathcal{E}$ on X together with an endomorphism $\theta \in End(\mathcal{E})\otimes\tilde{\Omega}_{X/V}\otimes\hat{\bar{V}}\xi^{-1}$ which is divisible by p^α for some $\alpha > 1/(p-1)$, such that $\theta\wedge\theta = 0$. Also $\hat{X}$ and

$\hat{\mathcal{E}}$ denote p-adic completions over $\hat{\bar{V}}$, that is $\hat{X}$ is a formal scheme and $\hat{\mathcal{E}}$ a formal vector-bundle on it. For any open $\hat{U} \subset \hat{X}$ and any formal vectorfield $\vartheta \in \Gamma(\hat{U}, \tilde{\mathcal{T}}_{X/V})\xi$ we obtain an automorphism $\exp(\theta(\vartheta))$ of $\hat{\mathcal{E}}$ on $\hat{U}$. Hence for any open covering $\hat{U}_i$ of $\hat{X}$ and any 1-Cech-cocycle $\vartheta_{ij} \in \Gamma(\hat{U}_i \cap \hat{U}_j, \tilde{\mathcal{T}}_{X/V})\xi$ we get a functor $\exp(\theta(\vartheta_{ij}))$ by formal Higgs-bundles by twisting $\hat{\mathcal{E}}$ by the cocycle $\exp(\theta(\vartheta_{ij}))$. These functors are multiplicative in $\{\vartheta_{ij}\}$, up to canonical isomorphism. Furthermore if $\vartheta_{ij} = \vartheta_i - \vartheta_j$ is a boundary the $\exp(\theta(\vartheta_i))$ define an isomorphism between the identity and the functor given by $\{\vartheta_{ij}\}$.

If we choose a covering U_i of X by small affines we have over each U_i equivalences of categories between small Higgs-bundles and small generalised representations. However, these depend on a choice of lifting U_i to $A_2(V)$ since automorphisms of such a lift act on small Higgs-bundles via our $\exp(\theta(\vartheta))$-construction. Only if we choose a lift of X to $A_2(V)$ we get a global equivalence of categories which depends on the lift: for two lifts we can choose local isomorphisms over each U_i. On the overlaps they differ by a vectorfield ϑ_{ij}, and $\exp(\theta(\vartheta_{ij}))$ describes the difference between the associated equivalences of categories. Furthermore change of local isomorphisms results in modifying $\{\vartheta_{ij}\}$ by a coboundary. We also remark that after inverting p K lifts to $A_2(V)$, and thus if X is defined over K we can lift it if we invert p. We then could extend the theory to any such X if we only consider Higgs-bundles with θ divisible by a sufficiently high p-power.

Similarly for functoriality: if $f : X \to Y$ denotes a (logarithmic) map we have natural pullbacks f^* for Higgs-bundles as well as for generalised representations. If we lift X and Y to $A_2(V)$ and choose local lifts f_i of f these will differ on the overlaps by vectorfields $\vartheta_{ij} \in f^*(\tilde{\mathcal{T}}_{Y/S})$ which act on the pullbacks $f^*(\mathcal{F}, \theta)$ of small Higgs-bundles on Y. Then the two functors f^* differ by twisting by the corresponding cocycle.

All in all we get:

Theorem 5. *This procedure defines* (*for liftable schemes*) *an equivalence of categories between small generalised representations and small Higgs-bundles. This also extends to the* $\mathbb{Q}_p$*-theory, and there the functor induces an isomorphism on cohomology* (*we have shown this locally, but the construction gives a global map*).

We cannot drop the adjective “small” as is demonstrated by explicit calculations for rank one bundles on curves:

Namely suppose X is a smooth proper (geometrically connected) curve over V. Then the abelianised fundamental group of $X \otimes_V \bar{K}$ is free and isomorphic to $\hat{\mathbb{Z}}^{2g}$ where g denotes the genus of X. Its continuous $\hat{\bar{K}}$-representations are parametrised by the images of the generators which are elements of $\hat{\bar{V}}^*$ whose reduction modulo the maximal ideal is torsion, that is lie in the algebraic closure of $\mathbb{F}_p$. This is always the case if k is finite, that is if K is a local field. The logarithm maps such elements surjectively onto $\hat{\bar{K}}$, with kernel the roots of unity $\mu(\bar{K})$, and we obtain an exact sequence

$$0 \to Hom(\pi_1(X_{\bar{K}}), \mu(\bar{K})) \to Hom(\pi_1(X_{\bar{K}}), \hat{\bar{K}}^*) \to Hom(\pi_1(X_{\bar{K}}), \hat{\bar{K}}) \to 0.$$

The first term coincides with the torsion-points in the Jacobian J of X, and the third with the homomorphisms from the Tate-module $T_p(J)$ into $\hat{\bar{K}}$. By Hodge–Tate theory [3] the latter coincides with the direct sum $Lie(J) \otimes_V \hat{\bar{K}} \oplus \Gamma(X, \Omega_{X/V}) \otimes_V \hat{\bar{K}}(-1)$. On the other hand from the logarithm-sequence for the Jacobian we obtain an exact sequence

$$0 \to J(\bar{K})_{tors} \to J(\hat{\bar{K}})' \to Lie(J) \otimes_V \hat{\bar{K}} \to 0,$$

where the middle term is the preimage of the torsion in $J(\bar{k})$. This exact sequence turns out to be the restriction of the first sequence to the direct summand $Lie(J) \otimes_V \hat{\bar{K}}$ (proofs will follow from the considerations below. They amount to functoriality of the logarithm map for the homomorphism $J[p^\infty] \to \mu_{p^\infty}$ defined by an element of the Tate-module of J, or better of its dual which, however, coincides with J).

Now assume that K is a local field, that is its residue-field is finite. If we choose a splitting of this exact sequence over the second summand $\Gamma(X, \Omega_{X/V}) \otimes_V \hat{\bar{K}}(-1)$ we obtain an isomorphism between one-dimensional representations of $\pi_1(X_{\bar{K}})$ and the product of $J(\hat{\bar{K}})$ and $\Gamma(X, \Omega_{X/V}) \otimes_V \hat{\bar{K}}(-1)$, that is a bijection (in rank one) between representations and Higgs-bundles. Unfortunately there is no canonical splitting because the first exact sequence realises the universal extension of $T_p(J)[1/p]$ by $\hat{\bar{K}} - Gal(\bar{K}/K)$-modules, so it cannot be trivial on a direct summand (see [10]). That is there exists no continuous $Gal(\bar{K}/K)$-invariant bijection between representations and Higgs-bundles. Nevertheless we can construct such bijections if we choose an exponential map for $\hat{\bar{K}}$:

Namely the logarithm defines an exact sequence

$$0 \to \mu(\bar{K}) \to \mathbb{G}_m(\hat{\bar{V}})' \to \hat{\bar{K}} \to 0,$$

where $\mathbb{G}_m(\hat{\bar{V}})'$ denotes the elements of $\hat{\bar{V}}^*$ which are torsion modulo the maximal ideal. An exponential map is a continuous right inverse of the logarithm, or a continuous splitting of this extension. It induces such a splitting for all commutative algebraic groups over K:

Suppose G is such an algebraic group. Then the exponential and logarithm induce isomorphisms between sufficiently small open subgroups of $G(\hat{\bar{K}})$ and its Lie-algebra $\mathfrak{g} \otimes_K \hat{\bar{K}}$. If $G(\hat{\bar{K}})' \subseteq G(\hat{\bar{K}})$ denotes the subgroup of elements g for which some multiple g^n ($n \in \mathbb{N}$ nonzero) lies in this neighbourhood, we can extend the logarithm to $G(\hat{\bar{K}})'$ and obtain an exact sequence

$$0 \to G(\hat{\bar{K}})_{tors} \to G(\hat{\bar{K}})' \to \mathfrak{g} \otimes_K \hat{\bar{K}}.$$

We show that the last map is surjective and construct a right inverse, as follows:

First of all the logarithm is an isomorphism for additive or unipotent groups. We thus may divide the connected component of the identity of G by its maximal unipotent subgroup and may assume that G is semiabelian, using Rosenlicht's theorem that G is an extension of an abelian variety by a linear group. Namely this operation does not change the torsion-points and thus allows us to lift splittings. Then passing to a finite extension of K we may assume that G has split semistable reduction and is over V the quotient of a semiabelian $\tilde{G}$ by a group of periods $\iota : Y \to \tilde{G}(K)$. We may replace G by $\tilde{G}$ and may assume that G is semiabelian over V, that is G is an extension

$$0 \to T \to G \to A \to 0,$$

with A an abelian variety and T a split torus. If $T_p(G)$ denotes its Tate-module, then any element $\rho \in Hom(T_p(G), \mathbb{Z}_p(1))$ in the Tate-module of its dual group defines over $\hat{\bar{V}}$ a homomorphism of p-divisible groups $G[p^\infty] \to \mu_{p^\infty}$ and thus also a map from the kernel of the reduction map $G(\hat{\bar{K}}) \to G(\bar{k})$ to the 1-units in $\hat{\bar{V}}^*$. We thus obtain a transformation from the kernel of the reduction-map to the kernel of the reduction map on $T_p(G)(-1) \otimes \hat{\bar{V}}^*$, and this induces an isomorphism on p-torsion. On the level of Lie-algebras we get a map from $\mathfrak{g} \otimes_K \hat{\bar{K}}$ to $T_p(G) \otimes \hat{\bar{K}}(-1)$. These maps are compatible via the logarithm, by naturality.

Now by Hodge–Tate theory $T_p(G) \otimes \hat{\bar{K}}(-1)$ is isomorphic to $\mathfrak{g} \otimes_K \hat{\bar{K}} \oplus Lie(A^t) \otimes_K \hat{\bar{K}}(-1)$. Strictly speaking this has been shown as it stands only if G is defined over a finite extension of V, not if G is defined over $\hat{\bar{V}}$. In general ($G = A$ only defined over $\hat{\bar{K}}$) we obtain a subspace $\mathfrak{g} \otimes_K \hat{\bar{K}}$ with quotient $Lie(A^t) \otimes_K \hat{\bar{K}}(-1)$, without a canonical lift. This follows from the proof in [3] by applying it to a universal family of abelian schemes A, and then specialising. By Galois-invariance (or for many other reasons) our map lands in the first factor and it is known to be an isomorphism onto the first direct summand. As the logarithm-map is surjective for G (it is p-divisible) and as we have an isomorphism on torsion-groups we get all in all an isomorphism from the kernel of reduction on $G(\hat{\bar{K}})$ onto the preimage of $\mathfrak{g} \otimes_K \hat{\bar{K}}$ in the kernel of reduction of $T_p(G)(-1) \otimes \hat{\bar{V}}^*$. Thus finally the exponential map on the multiplicative group induces such a map on G. These maps are functorial.

We also remark that we obtain a certain uniformity in G, as follows: call an element of the Lie-algebra $\mathfrak{g}$ small if there exists a rigid analytic homomorphism from the closed rigid unit disk (with addition) into G, with derivative at the origin the given element. Such a map is unique, has values in $G(\hat{\bar{K}})'$, and one checks that for the multiplicative group $\mathbb{G}_m$ an element is small if and only if it has valuation $> 1/(p-1)$ (the exponential series has to converge). It then follows that there exists an integer n such that the exponential map is rigid analytic on any product of p^n with a small element: reduce to G semiabelian over $\hat{\bar{V}}$. Then an analytic homomorphism from the closed unit disk into G coincides with a map from the p-adic completion of the additive group into G, is trivial on the special fibre, the map from $G(\hat{\bar{V}})'$ to $T_p(G)(-1) \otimes \hat{\bar{V}}^*$ is analytic, and everything reduces to the case of the multiplicative group.

As an application we can exponentiate cohomology-classes to line-bundles: consider a semistable proper curve X over V and a sequence of elements $\alpha_i \in \Gamma(X, \tilde{\Omega}^{\otimes i}_{X/V}) \otimes \hat{\bar{K}}(-i)$, for $1 \leqslant i \leqslant r$. For example the α_i could be the coefficients of $det(Tid - \theta)$ for a Higgs-field θ, on a vectorbundle of rank r on the generic fibre of X. Associated to these invariants is an algebra $\mathcal{A}$ over $X_{\hat{\bar{K}}}$, commutative and locally free of rank r. Namely it is the quotient of the symmetric algebra in the logarithmic tangent-bundle $\tilde{\mathcal{T}}_{X/V} \otimes \hat{\bar{K}}(1)$, under the monic polynomial with coefficients λ_i. As a module $\mathcal{A}$ is the direct sum of powers $\tilde{\mathcal{T}}^{\otimes i}_{X/V} \otimes \hat{\bar{K}}(i)$, for $0 \leqslant i \leqslant r-1$. For example if the λ_i come from a Higgs-field on $\mathcal{E}$ then $\mathcal{E}_{\hat{\bar{K}}}$ becomes naturally an $\mathcal{A}$-module.

Now the Picard-group $H^1(X_{\hat{\bar{K}}}, \mathcal{A}^*)$ consists of the $\hat{\bar{K}}$-points of a smooth algebraic group over $\hat{\bar{K}}$, with tangent-space $H^1(X_{\hat{\bar{K}}}, \mathcal{A})$, so we can exponentiate classes in the latter to get line-bundles. To avoid the complications due to automorphisms we first consider line-bundles trivialised at finitely many $\hat{\bar{V}}$-points of X, up to isomorphism. If the number of points is positive and $> (r-1)(2-2g)$ then $\mathcal{A}$ has no global sections vanishing at all these points, so these objects have no automorphisms, and the moduli-problem is representable by a smooth locally algebraic group over $\hat{\bar{K}}$ with tangent space the first cohomology of $X_{\hat{\bar{K}}}$ with coefficients in the local sections of $\mathcal{A}$ which vanish in the prescribed points. Let G denote the connected component of the identity of this moduli-space.

The product of the fibres of $\mathcal{A}^*$ maps to G (via change of trivialisations) and has as image a closed subgroup H (the kernel is given by the global sections of $\mathcal{A}^*$). If $\mathfrak{g}/\mathfrak{h} = H^1(X_{\hat{\bar{K}}}, \mathcal{A})$ denotes the Lie-algebra of the quotient G/H, choose a section $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ of the projection, and the exponential map for G, to define a family of $\mathcal{A}$-line-bundles on $X_{\hat{\bar{K}}}$ parametrised by $\mathfrak{g}/\mathfrak{h}$. This family is p-adically continuous and additive in the sense that the line-bundle corresponding to a sum is canonically isomorphic to the tensorproduct of the line-bundles corresponding to the factors, and these isomorphisms are compatible with associativity.

Finally the difference between two sections of $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ lifts to a linear map from $\mathfrak{g}$ into the product of the fibres of $\mathcal{A}$ in the prescribed points, and then exponentiates to a coherent family of isomorphisms between the associated line-bundles. Finally this coherent family of isomorphisms is canonical up to a multiplicative family of invertible global sections of $\mathcal{A}$. Also for p-adically small elements of $\mathfrak{g}/\mathfrak{h}$ the exponential coincides with the map defined by the exponential for some model for $\mathcal{A}$ on the p-adic formal scheme defined by X, by continuity.

This construction allows us to construct a twisted pullback for Higgs-bundles on $X_{\hat{\bar{K}}}$: namely for a map $f: X \to Y$ and a Higgs-bundle $(\mathcal{E}, \theta_Y)$ on $Y_{\hat{\bar{K}}}$ the usual pullback $f^*(\mathcal{E}), f^*(\theta_Y)$ admits an action of the algebra $f^*(\mathcal{A}_Y)$ ($\mathcal{A}_Y$ is defined like the previous $\mathcal{A}$, with X replaced by Y). Also the obstruction to lift f to $A_2(V)$ is an element of $H^1(X, f^*(\tilde{\mathcal{T}}_{Y/V}))$ which exponentiates to an $f^*(\mathcal{A}_Y)$-line-bundle. The twisted pullback $f^\circ(\mathcal{E}, \theta_Y)$ is defined as the tensorproduct of $f^*(\mathcal{E}, \theta_Y)$ with this line-bundle. This functor depends on certain choices (trivialisations in suitable points) but different families give isomorphic functors. Also for compositions $(fg)^\circ$ is naturally isomorphic to $g^\circ f^\circ$, and we leave it to the reader to check the various compatibilities involved (later we consider actions of a finite group on X where compatibilities become easy to check because one can make all arbitrary choices invariant under this group). Finally there exists an integer $n \geqslant 2$, independent of X, such that if the λ_i extend to a semistable model of X over $\hat{\bar{V}}$ (so $\mathcal{A}$ has an extension to this model) and if a class in $H^1(X, \tilde{\mathcal{T}}_{X/V}) \otimes_V \hat{\bar{V}}(1)$ is divisible by p^n, then our exponential coincides with the usual exponential of the class on the formal scheme defined by X.

Now we come to our main result.

Theorem 6. *There exists an equivalence of categories between Higgs-bundles and generalised representations, if we allow $\hat{\bar{K}}$-coefficients.*

Proof. For this we use the equivalence of categories for small representations and a descent argument. The latter uses the following construction of coverings of X.

After passing to an extension of V choose a V-rational point in X and use it to embed the generic fibre X_K into its Jacobian J. Then multiplication by p^n on J induces a covering $X_{n,K}$ of X_K which has (after finite extension of K) a semistable model X_n mapping to X. If $J_{n,K}$ denotes the Jacobian of $X_{n,K}$ the induced map $J_{n,K} \to J_K$ is divisible by p^n, and so is by duality the pullback $J_K \to J_{n,K}$. Next consider the induced map on differentials on Néron-models which is identified with the pullback $\Gamma(X, \tilde{\Omega}_{X/V}) \to \Gamma(X_n, \tilde{\Omega}_{X_n/V})$. Thus the pullback on differentials is divisible by p^n on global sections. As global sections generate $\tilde{\Omega}_{X/V}$ on the generic fibre there exists an integer n_0 such that the subsheaf of $\tilde{\Omega}_{X/V}$ generated by global sections contains $p^{n_0}\tilde{\Omega}_{X/V}$, and so the pullback-map on differentials is divisible by p^{n-n_0}. Thus we can make generalised representations or Higgs-bundles small by pullback along such a covering.

Now if we have a generalised representation on X we choose a finite Galois covering $Y \to X$ such that it becomes small on Y, that is it is given by a small Higgs-bundle there. The covering group acts via the f°-action on this Higgs-bundle, and via the usual action if we twist by the obstruction to lift $Y \to X$ to $A_2(V)$. By descent we get a Higgs-bundle on X. Conversely given such a Higgs-bundle on X we choose Y such that it becomes small on Y, then twist the pullback by the inverse of the obstruction-class, and get a generalised representation on Y which descends to X. □

Things are slightly more complicated if we allow open curves, that is the divisor at infinity D is nonempty: namely then the covering may ramify over D, and an equivariant action of the covering group does not always allow descent. We need that the (finite) stabiliser of a point acts trivially on the fibre in this point. In the $\mathcal{A}$-linebundles used for the twisting, the action of inertia is given by the exponential of its derivative (the inertia-group is $\hat{\mathbb{Z}}(1)$), and to get descent we need the same for the action of inertia on the fibre of our generalised representation. The same reasoning gives the converse, so we get an equivalence between Higgs-bundles and representations satisfying the following condition on inertia.

The action of the inertia group $\hat{\mathbb{Z}}(1)$ on the fibre at a point in the boundary has a derivative $Res(\theta)$ (it is the residue of the associated θ). It then must be equal to the exponential (in the multiplicative group of the algebra generated by $Res(\theta)$) of its derivative. Especially the action factors over $\mathbb{Z}_p(1)$.

Our functors induce isomorphisms between the $\mathcal{X}^0$-cohomology of generalised representations and Higgs-cohomology of Higgs-bundles. This has been shown for small objects. The general result follows by descent once we show that Higgs-cohomology is invariant under our twists by $\mathcal{A}[1/p]$-line-bundles. We only use that these line-bundles come from twists by vectorfields, and that they are in the image of the exponential map. The latter is used as follows.

$\mathcal{A}$ is a quotient of the symmetric algebra $S(\tilde{\mathcal{T}})$ under a monic polynomial (naturally an element $P \in S(\tilde{\mathcal{T}}) \otimes \tilde{\Omega}^{\otimes r}$) with coefficients λ_i. Denote by $\mathcal{B}$ the algebra (of rank $r+1$) defined by multiplying this polynomial by T. Then our line-bundles lift to compatible systems of $\mathcal{B}[1/p]$-line-bundles, trivialised on $\mathcal{A}/(\theta) = \mathcal{O}_X$. We claim that the Koszul-complex $\mathcal{E} \to \mathcal{E} \otimes \tilde{\Omega}_{X/V}$ is up to quasi-isomorphism invariant under tensoring $\mathcal{E}$ with $\mathcal{B}$-line-bundles $\mathcal{M}$ which are trivialised modulo θ:

Proposition 1. *For two $\mathcal{B}$-line-bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ any isomorphism $\mathcal{M}_1/(\theta) \cong \mathcal{M}_2/(\theta)$ induces a quasi-isomorphism on Koszul-complexes for $\mathcal{E} \otimes_{\mathcal{B}} \mathcal{M}_i$.*

Proof. Denote by $\mathcal{N} \subset \mathcal{M}_1 \oplus \mathcal{M}_2$ the elements which lie modulo θ in the graph of the isomorphism. Then multiplication by θ induces a map from $\mathcal{M}_1 \oplus \mathcal{M}_2$ to $\mathcal{N} \otimes \tilde{\Omega}_{X/V}$, and after tensoring (over $\mathcal{B}$) with $\mathcal{A}$ the resulting complex becomes locally free over $\mathcal{A}$, and the two projections induce quasi-isomorphisms with the Koszul-complexes of the $\mathcal{M}_i$. Tensoring with $\mathcal{E}$ gives the desired zigzag of quasi-isomorphisms. Also the construction is transitive if we have three bundles $\mathcal{M}_i$ and compatible isomorphisms between their reductions modulo θ (do the construction above with three summands). □

5. Examples and open questions

A natural question to ask is which Higgs-bundles come from actual representations of the fundamental group. If this representation is trivial modulo a sufficiently high p-power the associated Higgs-bundle has a model which is trivial modulo some positive p-power, so its restriction to the generic fibre is semistable (even without the Higgs-field) of slope zero. By descent it follows that in general the associated Higgs-bundle is semistable of slope zero (now we need the Higgs-field to construct the twisted pullback f°). Furthermore if a Higgs-bundle (on the generic fibre) has an integral model which is the trivial bundle modulo some p^{α} for $\alpha > 0$ we can (by pullback) assume that a Higgs-field is divisible by a high p-power. It then follows by deformation-theory that it is associated to an π_1-representation: for some small ε (to take care of the "almost") we first get this modulo $p^{2\alpha-\varepsilon}$, by considering extensions of trivial representations or generalised representations. Then continue. This generalises a result in [1,2].

Also if for a dominant (i.e. nonconstant) map $Y_K \to X_K$ the pullback to Y of a generalised representation comes from a real representations, then the same holds already on X: we may assume that the cover is Galois, with group G. Then G operates on the locally constant system of Y, and the inertia acts trivially on the fibres at fixed-points (by checking for the generalised representations). Thus the system descends.

This suffices to show that all rank-one Higgs-bundles (L, θ) with $\mathcal{L} \in J(\hat{\bar{K}})'$ come from π_1-representations: if $\mathcal{L}$ is torsion it can be trivialised by pullback. Thus we reduce to $\mathcal{L}$ which is trivial on the special fibre.

The resulting homomorphism

$$J(\hat{\bar{K}})' \times \Gamma(X, \Omega_X) \otimes_K \hat{\bar{K}}(-1) \to Hom(\pi_1(X_{\bar{K}}), \hat{\bar{V}}^*)$$

coincides on the first factor with the previously defined map: namely this is true on the torsion, and this determines the map by $Gal(\bar{K}/K)$-invariance. If X is only defined over $\hat{\bar{K}}$ the same follows by an approximation argument. For the second factor one can show that there the map is the exponential of a $\hat{\bar{K}}$-linear map into the Lie-algebra of the group of representations, that is into $H^1(X, \mathcal{O}_X) \otimes_K \hat{\bar{K}} \oplus \Gamma(X, \Omega_X) \otimes_K \hat{\bar{K}}(-1)$. Furthermore the second component of our map is the identity, while the first one depends on the choice of lifting X to $A_2(V)$.

Example. Sometimes $\mathbb{Q}_p$-representations $\mathbb{L}$ of the fundamental group are associated to filtered Frobenius-crystals $\mathcal{E}$, that is there are functorial isomorphisms $B_{crys}(R) \otimes \mathcal{L} \cong B_{crys}(R) \otimes \mathcal{E}$ respecting Galois-action, filtration, and Frobenius (for small R's). The induced isomorphism on gr_F^0 is $\hat{\bar{R}} \otimes \mathbb{L} \cong \oplus_i \hat{\bar{R}}[1/p](-i) \otimes gr_F^i(\mathcal{E})$ shows that $\mathbb{L}$ corresponds to the graded Higgs-bundle $gr_F(\mathcal{E})$ "with Tate-twists".

Unipotent representations of the fundamental group correspond to unipotent Higgs-bundles. Here "unipotent" means in both cases that the object is a successive extension of the trivial representation, respectively, Higgs-bundle. That follows easily from the comparison-theorem for cohomology as extensions are classified by H^1.

In fact using the method in [5] (proof of Theorem 5) one can even show that unipotent representations are (logarithmically) crystalline (see also [13]): for simplicity assume that our curve X has two V-rational points O and ∞. We consider representations of the fundamental-group of $X - \infty$ which has the advantage of being free. If $T_p(J)$ denotes the Tate-module of the Jacobian J of X we construct by induction unipotent smooth p-adic systems $\mathbb{L}_n$ on the generic fibre $(X - \{\infty\})_K$ such that we have exact sequences

$$0 \to T_p(J)^{\otimes n} \to \mathbb{L}_n \to \mathbb{L}_{n-1} \to 0.$$

Furthermore in the fibre over 0 the quotient $\mathbb{L}_0 = \mathbb{Z}_p$ lifts to a subsheaf of $\mathbb{L}_n$, the projection $\mathbb{L}_n \to \mathbb{L}_0 = \mathbb{Z}_p$ induces an isomorphism on homomorphisms into $\mathbb{Z}_p$ (over $(X - \{\infty\})_{\bar{K}}$), and the inclusion $T_p(J)^{\otimes n} \to \mathbb{L}_n$ an isomorphism on extensions by $\mathbb{Z}_p$.

All these hold for $\mathbb{L}_0 = \mathbb{Z}_p$. If we have constructed $\mathbb{L}_{n-1}$ we know that its extensions (over $X_{\bar{K}}$) by the constant sheaf $T_p(J)^{\otimes n}$ are classified by the corresponding H^1 which is isomorphic to

$$H^1((X - \{\infty\})_{\bar{K}}, \mathcal{H}om(T_p(J)^{\otimes n-1}, T_p(J)^{\otimes n}))$$
$$= Hom(T_p(J)^{\otimes n}, T_p(J)^{\otimes n}),$$

and $\mathbb{L}_n$ is the class corresponding to the identity. From the long exact sequence of Ext-groups one derives the assertion about $Ext^i((X - \{\infty\})_{\bar{K}}; \mathbb{L}_n, \mathbb{Z}_p)$. Finally, the automorphisms of the extension are $T_p(J)^{\otimes n}$ and they act simply transitively on the splittings in 0. Thus we get that $\mathbb{L}_n$ is unique up to unique isomorphism, and thus already defined over K.

The same type of argument works on the crystalline side and constructs filtered (with degrees $\leqslant 0$) crystals $\mathcal{E}_n$, on the logarithmic crystalline site of X relative to $\mathbb{Z}_p$, with splittings at 0 as before. These are unique, either filtered or unfiltered, and thus Frobenius-crystals. Finally, the two objects correspond via the comparison theory. As any unipotent representation of the fundamental-group is a quotient of a direct sum of $\mathbb{L}_n$'s the assertion follows.

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