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INTEGRAL CHARACTERISTIC NUMBERS FOR WEAKLY ALMOST COMPLEX MANIFOLDS

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§1. INTRODUCTION

LET M be a closed weakly almost complex manifold of real dimension $2k$ and let $\tau : M \to BU$ be its tangential map. For an element v of $H^{2k}(BU; Q)$ we let correspond the value of $\tau^*(v)$ evaluated on the fundamental class of M. This correspondence defines a homomorphism

$$\alpha(M) : H^{2k}(BU; Q) \to Q.$$

Let I^{2k} be the subgroup of $H^{2k}(BU; Q)$ consisting of those elements which are mapped by $\alpha(M)$ into the integer group Z for any closed weakly almost complex manifold M.

The purpose of the present paper is to determine explicitly the group I^{2k}. To state the result, we denote by $ch(BU)$ the image of Atiyah–Hirzebruch group $K(BU)$ in $H^{**}(BU; Q)$ by the character homomorphism. Also, the universal Todd class will be denoted by $\mathscr{T}$. Then $2k$-dimensional components of elements of the form $y \cdot \mathscr{T}$ with y belonging to $ch(BU)$ constitute a subgroup I'^{2k} of $H^{2k}(BU; Q)$. Now, the differentiable Riemann-Roch theorem due to Atiyah–Hirzebruch [3] implies that the subgroup I'^{2k} is contained in I^{2k}. Our result is summarized in the following:

THEOREM I. *The group I^{2k} actually coincides with I'^{2k}.*
This answers affirmatively a conjecture of Atiyah–Hirzebruch [4].

We can restate Theorem I in a more convenient way using K-theory. Let $\mathscr{U}_{2k}$ be the complex cobordism group of real dimension $2k$. The group $\mathscr{U}_{2k}$ is canonically identified with the stable homotopy group $\pi_{2k+2N}(MU(N))$, where $MU(N)$ is the universal Thom space for complex N-dimensional vector bundles and N is large compared with k. Now the Atiyah–Hirzebruch functor K gives rise to a homomorphism

$$\rho : \tilde{K}(MU(N)) \to Hom(\pi_{2k+2N}(MU(N)), \tilde{K}(S^{2k+2N}))$$

defined by $\rho(v)(x) = x^*(v)$. In §4 it will be shown that Theorem I is equivalent to:

THEOREM II. *The homomorphism ρ is surjective.*

Theorem II is proved in §3. In §2 we discuss duality between the homology theory K_* and the cohomology theory K^* both with coefficients in the unitary spectrum†. We also

† In preparing the present paper, the author was made aware of an unpublished paper of D. W. Anderson [1] in which the duality was thoroughly exploited. We treat it here only in a special setting.

introduce a natural transformation $H: \pi_k^S(\ ) \to \tilde{K}_k(\ )$ from the stable homotopy groups to the reduced K_*-homology theory. This transformation may be viewed as an analogue of the usual Hurewicz homomorphism.

In §3, it is shown that the image of the homomorphism

$$H: \ \pi_{2k+2N}(MU(N)) \to \tilde{K}_{2k+2N}(MU(N))$$

is a direct summand, the fact which implies Theorem II in virtue of duality.

In a subsequent paper some applications will be given. In particular, integral characteristic numbers for closed oriented C^∞-manifolds will be determined.

§2. REMARKS ON K-THEORY

A *spectrum* $\mathbf{E}$ is a sequence $\{E_n; n \in Z\}$ of spaces with base points together with a sequence of maps $\varepsilon_n : SE_n \to E_{n+1}$ preserving base points. Giving a map $\varepsilon_n : SE_n \to E_{n+1}$ is equivalent to giving its adjoint $\bar{\varepsilon}_n : E_n \to \Omega E_{n+1}$; $\bar{\varepsilon}_n(x)(t) = \varepsilon_n(t, x)$. We follow G. W. Whitehead [16] for notions pertaining to the homology theory and the cohomology theory with coefficients in the spectrum $\mathbf{E}$. However, we shall have to extend them on the category of CW-pairs (not necessarily finite). Specifically, for a *finite* CW-pair (X, A), its q-th homology group $H_q(X, A; \mathbf{E})$ is defined by

$$H_q(X, A; \mathbf{E}) = \varinjlim_n \pi_{q+n}(E_n \wedge (X/A)), \tag{2.1}$$

where lim means the direct limit of the direct system of abelian groups with the homomorphisms

$$\pi_{q+n}(E_n \wedge (X/A)) \xrightarrow{S} \pi_{q+n+1}(SE_n \wedge (X/A)) \xrightarrow{\varepsilon_{n*}} \pi_{q+n+1}(E_{n+1} \wedge (X/A)).$$

Similarly the cohomology group $H^q(X, A; \mathbf{E})$ is defined by

$$H^q(X, A; \mathbf{E}) = \varinjlim_n [S^{n-q}(X/A), E_n], \tag{2.2}$$

where [,] means the set of base point preserving homotopy classes.

We now pass to the K-theory. Let U be the infinite unitary group and BU a classifying space for U. Since U is a countable CW-group we may assume that BU is a countable CW-complex (cf. [11]). There is a natural homotopy equivalence

$$h_1 : U \to \Omega(Z \times BU) = \Omega BU.$$

We also have a homotopy equivalence

$$h_2 : Z \times BU \to \Omega U$$

due to Bott [7]. The *unitary spectrum* $\mathbf{U} = \{U_n, u_n\}$ is defined as follows:

$$\begin{aligned} U_{2m} &= Z \times BU, \\ U_{2m+1} &= U, \\ \bar{u}_{2m} &= h_2 : U_{2m} \to \Omega U_{2m+1}, \\ \bar{u}_{2m+1} &= h_1 : U_{2m+1} \to \Omega U_{2m+2}. \end{aligned}$$

The corresponding homology theory and cohomology theory are denoted by K_* and K^* respectively. Since the unitary spectrum is an Ω-spectrum and moreover periodic, the definitions (2.1) and (2.2) take somewhat simple form in this case. Namely, the homology is given by

$$K_q(X, A) = \varinjlim_m \pi_{q+2m}(U_{2m} \wedge (X/A)), \qquad U_{2m} = Z \times BU, \tag{2.3}$$

since the groups $\pi_{q+2m}(U_{2m} \wedge (X/A))$ are cofinal in the direct system. Similarly, we have

$$K^q(X, A) = \varinjlim_m [S^{2m-q}(X/A), U_{2m}], \qquad U_{2m} = Z \times BU. \tag{2.4}$$

We also notice that, in the cohomology case, *the homomorphism*

$$[S^{n-q}(X/A), U_n] \xrightarrow{S} [SS^{n-q}(X/A), SU_n] \xrightarrow{u_{n*}} [S^{n+1-q}(X/A), U_{n+1}] \tag{2.5}$$

is an isomorphism.

This follows from the commutativity of the following diagram

$$\begin{tikzcd} {[S^{n-q}(X/A), U_n]} \arrow[r, "S"] \arrow[d, "\bar{u}_{n*}"] & {[SS^{n-q}(X/A), SU_n]} \arrow[d, "u_{n*}"] \\ {[S^{n-q}(X/A), \Omega U_{n+1}]} \arrow[r, phantom, "\cong"] & {[SS^{n-q}(X/A), U_{n+1}],} \end{tikzcd} \tag{2.6}$$

where the bottom row is the usual isomorphism (cf. [16; (2.15)]) and $\bar{u}_{n*}$ is an isomorphism since $\bar{u}_n$ is a homotopy equivalence. It follows from (2.5) that there are canonical identifications:

$$\begin{aligned} K^q(X, A) &= [X/A, U_q], \\ &= [S^{2m-q}(X/A), U_{2m}],\ U_{2m} = Z \times BU. \end{aligned} \tag{2.7}$$

This shows in particular that the cohomology theory K^* is identical with the Atiyah–Hirzebruch theory.

We define the *Bott isomorphism*

$$\beta : K_{q-2}(X, A) \to K_q(X, A)$$

to be induced in the expression (2.3) by the identity

$$\pi_{q-2+2(m+1)}((z \times BU) \wedge (X/A)) = \pi_{q+2m}((Z \times BU) \wedge (X/A)).$$

The *Bott isomorphism*

$$\beta : K^q(X, A) \to K^{q-2}(X, A) \tag{2.9}$$

is defined similarly. We also have a natural isomorphism

$$\alpha : K^q(X, A) = \tilde{K}^q(X/A) \to \tilde{K}^{q+1}(S(X/A)) \tag{2.10}$$

induced in the expression (2.7) by the identity

$$[S^{2m-q}(X/A), Z \times BU] = [S^{2m-(q+1)}(S(X/A)), Z \times BU].$$

Note that the inverse isomorphism α^{-1} is equal to $(-1)^{q+1}$ times the suspension isomorphism σ^* used in [16].

Let $p : S^1 \wedge (X/A) \wedge (Y/B) \to S^1 \wedge ((X/A) \wedge (Y/B))$,
$p' : S^1 \wedge (X/A) \wedge (Y/B) \to (S^1 \wedge (X/A)) \wedge (Y/B)$,
$p'' : S^1 \wedge (X/A) \wedge (Y/B) \to (X/A) \wedge (S^1 \wedge (Y/B))$
be the natural homotopy equivalences used in [16; (2.4)], where (Y, B) is also a finite CW-pair. Then the compositions

$$\alpha_L = p'^{*-1} \circ p^* \circ \alpha : \tilde{K}^r((X/A) \wedge (Y/B)) \to \tilde{K}^{r+1}(S(X/A) \wedge (Y/B)),$$
$$\alpha_R = p''^{*-1} \circ p^* \circ \alpha : \tilde{K}^r((X/A) \wedge (Y/B)) \to \tilde{K}^{r+1}((X/A) \wedge S(Y/B))$$

are isomorphisms.

The cohomology theory K^* has a product (cf. [5; §1.5])

$$\wedge : K^p(X, A) \otimes K^p(Y, B) \to K^{p+q}(X \times Y, A \times Y \cup X \times B). \tag{2.11}$$

This product is *associative and anti-commutative.* Moreover there exists an element $1 \in K^0(x_0) = \tilde{K}^0(S^0)$ which serves as the unit in the product (2.11).

The following lemma can be easily verified along the lines of [5; §1.5].

LEMMA (2.12). *Let $x \in \tilde{K}^p(X/A)$ and $y \in \tilde{K}^q(Y/B)$. Then the following identities hold.*

$$\alpha(x) \wedge y = (-1)^q \alpha_L(x \wedge y),$$
$$x \wedge \alpha(y) = \alpha_R(x \wedge y),$$
$$\beta(x) \wedge y = \beta(x \wedge y),$$
$$x \wedge \beta(y) = \beta(x \wedge y).$$

In particular, we have

$$\alpha(1) \wedge y = (-1)^q \alpha(y),$$
$$x \wedge \alpha(1) = \alpha(x),$$
$$\beta(1) \wedge y = \beta(y),$$
$$x \wedge \beta(1) = \beta(x).$$

Remark. Strictly speaking, the Bott isomorphism (2.9) may differ from the one given in [5]. The Bott isomorphism in [5] is the multiplication by $\alpha^{-2}(g)$ where g is a prescribed generator of $\tilde{K}^0(S^2) \cong Z$, while $\alpha^2\beta(1) = \pm g$. If one wants to make $\alpha^2\beta(1) = g$, one has only to replace, if necessary, the homotopy equivalence $h_2 : Z \times BU \to \Omega U$ by $-h_2$.

We now generalize the homology and cohomology theory to the category of arbitrary CW-complexes. Let (X, A) be a CW-pair. The q-th homology group $H_q(X, A; \mathbf{E})$ of (X, A) with coefficients in the spectrum $\mathbf{E}$ is defined by (2.1). It is easy to see that

$$H_q(X, A; \mathbf{E}) = \varinjlim H_q(X_\alpha, X_\alpha \cap A; \mathbf{E}) \tag{2.13}$$

where X_α ranges over all finite subcomplexes of X. From (2.13) or directly as in [16] it follows that this actually defines a homology theory satisfying the first six axioms of Eilenberg–Steenrod. The induced homomorphism f_* and the suspension isomorphism σ_* are defined as in [16].

As for the cohomology, the generalization will be limited on the cohomology theory based on the unitary spectrum. The q-th cohomology group is defined by (2.7). It is easy to verify that this defines a cohomology theory satisfying the axioms of Eilenberg–Steenrod except the dimension axiom. The suspension isomorphism σ^* is defined by $\sigma^* = (-1)^{q+1}\alpha$, where α is given by (2.10). The Bott isomorphisms (2.8) and (2.9) are also valid in this extended sense.

Moreover, from the expression (2.7) it follows that this cohomology theory is *additive* in the sense of Milnor [12]. That is, if X is the disjoint union of CW-complexes X_α, then the cohomology group $K^q(X)$ is canonically isomorphic to the direct product $\Pi_\alpha K^q(X_\alpha)$. As a special case of a theorem of Milnor on general additive cohomology theory [12] we get

LEMMA (2.14). *Let X be a CW-complex. Suppose that there is a sequence of finite subcomplexes $X_1 \subset X_2 \subset \cdots$ with union X. Then the natural homomorphism*

$$K^q(X) \to \varprojlim_n K^q(X_n)$$

is onto. The kernel vanishes whenever the natural homomorphisms $K^{q-1}(X_{n+1}) \to K^{q-1}(X_n)$ are onto for all n.

A CW-complex X will be called *K^*-admissible* if it satisfies the following condition:
(2.15) *The natural homomorphism*

$$K^*(X) \to \varprojlim_n K^*(X_\alpha)$$

is bijective. Here the inverse limit is taken over all *finite* subcomplexes of X.

It is clear that if there is a sequence $X_1 \subset X_2 \subset \cdots$ of finite subcomplexes with union X such that $K^*(X) \to \varprojlim_n K^*(X_n)$ is a bijection then X is K^*-admissible.

LEMMA (2.16). (i) *Let X be a CW-complex. If there is a sequence $X_1 \subset X_2 \subset \cdots$ of finite subcomplexes with union X such that $H^*(X_n; Z)$ are free abelian groups for all n and the natural homomorphisms $H^*(X_{n+1}; Z) \to H^*(X_n; Z)$ are surjective for all n, then the CW-complex X is K^*-admissible.*

(ii) *Let X and Y be countable CW-complexes. Suppose that both complexes X and Y satisfy the condition of* (i). *Then the product $X \times Y$ and the reduced join $X \wedge Y$ are K^*-admissible CW-complexes.*

Proof. (i). The sequence

$$0 \to H^*(X_{n+1}, X_n; Z) \to H^*(X_{n+1}; Z) \to H^*(X_n; Z) \to 0$$

is exact and the group $H^*(X_{n+1}, X_n; Z)$ is without torsion by virtue of the assumption. It follows that $K^*(X_{n+1}, X_n)$ is a free abelian group and $\mathrm{ch} : K^q(X_{n+1}, X_n) \to H^*(X_{n+1}, X_n; Q)$ is an injective homomorphism (cf. [5]). Consider the commutative diagram

$$\begin{array}{ccccc} & & K^q(X_{n+1}, X_n) & \xrightarrow{j^*} & K^q(X_{n+1}) \\ & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ 0 & \to & H^*(X_{n+1}, X_n; Q) & \xrightarrow{j^*} & H^*(X_{n+1}; Q) \end{array}$$

where the bottom row is exact. Since $j^* \circ \mathrm{ch}$ is injective. $j^* : K^q(X_{n+1}, X_n) \to K^q(X_{n+1})$ is injective. Then the exactness of the cohomology sequence for K^*-theory implies the surjectivity of $K^{q-1}(X_{n+1}) \to K^{q-1}(X_n)$. Therefore Lemma (2.14) applies and the group $K^q(X)$ is canonically isomorphic to $\varprojlim_n K^q(X_n)$. Thus X is K^*-admissible.

(ii). Let $X_1 \subset X_2 \subset \cdots$ be a sequence of finite subcomplexes of X with union X and let $Y_1 \subset Y_2 \subset \cdots$ be a sequence of finite subcomplexes of Y with union Y. If $H^*(X_n; Z)$ and $H^*(Y_n; Z)$ are both free abelian groups and if $H^*(X_{n+1}; Z) \to H^*(X_n; Z)$ and $H^*(Y_{n+1}; Z) \to H^*(Y_n; Z)$ are both surjective, then the Künneth formula implies that $H^*(X_{n+1} \times Y_{n+1}; Z) \to H^*(X_n \times Y_n; Z)$ and $H^*(X_{n+1} \wedge Y_{n+1}; Z) \to H^*(X_n \wedge Y_n; Z)$ are surjective homomorphisms and the groups considered are all free abelian. Since we have assumed the countability of the CW-complexes X and Y, the product $X \times Y$ and the reduced join $X \wedge Y$ are CW-complexes. Moreover $X_1 \times Y_1, X_2 \times Y_2, \cdots$ is a sequence of finite subcomplexes with union $X \times Y$. Similarly, $X_1 \wedge Y_1, X_2 \wedge Y_2, \cdots$ is a sequence of finite subcomplexes with union $X \wedge Y$. It follows from (i) that $X \times Y$ and $X \wedge Y$ are K^*-admissible. This completes the proof.

LEMMA (2.17). *The CW-complexes U_p, $U_p \wedge U_q$ $U_p \wedge U_q \wedge U_r$, $S^p \wedge U_q$, $U_p \wedge S^q$, $(S^p \wedge U_q) \wedge U_r$ and $U_p \wedge (S^q \wedge U_r)$ are all K^*-admissible.*

Proof. First consider the infinite unitary group $U = \bigcup_n U(n)$. It is classical that $H^*(U(n); Z)$ is free abelian and that $H^*(U(n+1); Z) \to H^*(U(n); Z)$ is surjective. Hence the CW-complex U is K^*-admissible by (2.16). As a classifying space BU we may take the limit space $\bigcup G_{n,n}$ where $G_{n,n} = U(2n)/U(n) \times U(n)$ is the complex Grassman manifold. It is also classical that the sequence $G_{1,1} \subset G_{2,2} \subset \cdots$ satisfies the condition of (i) in (2.16). Hence BU is K^*-admissible. Therefore $Z \times BU$ is also K^*-admissible. Then the complex $U_p \wedge U_q$ is K^*-admissible by (2.16) (ii). The remaining cases are treated similarly. This completes the proof.

Now we shall prescribe a specific element in $\tilde{K}^{p+q}(U_p \wedge U_q) = [U_p \wedge U_q, U_{p+q}]$. Let $X_{p,i}$ be a finite subcomplex of U_p. We denote by $\iota_p \in \tilde{K}^p(U_p) = [U_p, U_p]$ the element represented by the identity map $U_p \to U_p$. The restriction of ι_p on the subcomplex $X_{p,i}$ is denoted by $\iota_{p,i} \in \tilde{K}^p(X_{p,i})$. Consider the element

$$(\iota_{p,i} \wedge \iota_{q,j}) \in \prod_{i,j} \tilde{K}^{p+q}(X_{p,i} \wedge X_{q,j}),$$

where $X_{p,i}$ and $X_{q,j}$ range over all finite subcomplexes of U_p and U_q respectively. By the naturality of the product (2.11) the above element belongs to the inverse limit group $\varprojlim_{i,j} \tilde{K}^{p+q}(X_{p,} \wedge X_{q,j})$. Since the finite complexes $X_{p,i} \wedge X_{q,j}$ have the whole $U_p \wedge U_q$ as their union and since $U_p \wedge U_q$ is K^*-admissible, we have a canonical isomorphism

$$\tilde{K}^{p+q}(U_p \wedge U_q) \cong \varprojlim_{i,j} \tilde{K}^{p+q}(X_{p,i} \wedge X_{q,j}).$$

Let $\iota_p \wedge \iota_q$ denote the element corresponding to $(\iota_{p,i} \wedge \iota_{q,j})$ by this isomorphism. *This element $\iota_p \wedge \iota_q$ is the unique one which gives $\iota_{p,i} \wedge \iota_{q,j}$* when restricted on $X_{p,i} \wedge X_{q,j}$.

We will define the element $t_{p,q} \in [U_p \wedge U_q, U_{p+q}]$ by

$$t_{p,q} = (-1)^{pq} \iota_p \wedge \iota_q.$$

We often use the same notation $t_{p,q}$ to mean a map $U_p \wedge U_q \to U_{p+q}$ representing the homotopy class $t_{p,q}$.

PROPOSITION (2.18). *The homotopy classes $t_{p,q}$ satisfy the following relations.*

$$\alpha_L(t_{p,q}) = (u_p^* \wedge 1)t_{p+1,q},$$
$$\alpha_R(t_{p,q}) = (-1)^p(1 \wedge u_q^*)t_{p,q+1}.$$

Proof. Recall that α_L is an isomorphism

$$\tilde{K}^{p+q}(U_p \wedge U_q) = [U_p \wedge U_q, U_{p+q}] \to [SU_p \wedge U_q, U_{p+q+1}] = \tilde{K}^{p+q+1}(SU_p \wedge U_q).$$

Note also that the element $\iota_p \in [U_p, U_p] = \tilde{K}^p(U_p)$ is mapped by the isomorphism $\alpha : \tilde{K}^p(U_p) \to \tilde{K}^{p+1}(SU_p) = [SU_p, U_{p+1}]$ to the homotopy class represented by the map $u_p : SU_p \to U_{p+1}$. In other words, we have

$$\alpha(\iota_p) = u_p^*(\iota_{p+1}). \tag{2.19}$$

Take finite subcomplexes $X_{p,i} \subset U_p$ and $X_{q,j} \subset U_q$. Applying Lemma (2.12) to the element $\iota_{p,i} \wedge \iota_{q,j} \in \tilde{K}^{p+q}(X_{p,i} \wedge X_{q,j})$, we have

$$\alpha_L(\iota_{p,i} \wedge \iota_{q,i}) = (-1)^q\alpha(\iota_{p,i}) \wedge \iota_{q,j},$$

where $\alpha(\iota_{p,i}) \in \tilde{K}^{p+1}(SX_{p,i})$. Since the union of the subcomplexes $SX_{p,i} \wedge X_{q,j} \subset SU_p \wedge U_q$ is $SU_p \wedge U_q$ and the CW-complex $SU_p \wedge U_q$ is K^*-admissible, we may pass to the inverse limit to get the relation

$$\alpha_L(\iota_p \wedge \iota_q) = (-1)^q\alpha(\iota_p) \wedge \iota_q,$$
$$= (-1)^q u_p^*(\iota_{p+1}) \wedge \iota_q.$$

The first relation in (2.18) follows easily from this. The second relation is proved similarly.

Remark. Proposition (2.18) just means that *the double sequence of maps $t_{p,q} : U_p \wedge U_q \to U_{p+q}$ defines a pairing* $(\mathbf{U}, \mathbf{U}) \to \mathbf{U}$ *in the sense of G. W. Whitehead* [16]. It can be proved that *this pairing is associative and anti-commutative.* The associativity means here that the two maps $U_p \wedge U_q \wedge U_r \to U_{p+q+r}$ defined by $(x, y, z) \to t_{p+q,r}(t_{p,q}(x, y), z)$ and $(x, y, z) \to t_{p,q+r}(x, t_{q,r}(y, z))$ are homotopic, base points being preserved. The anti-commutativity means that the map $U_p \wedge U_q \to U_{p+q}$ defined by $(x, y) \to t_{q,p}(y, x)$ represents $(-1)^{pq}$ times the element $t_{p,q} \in [U_p \wedge U_q, U_{p+q}]$. These facts come from the associativity and anti-commutativity of the product (2.11) which supply the desired homotopies on finite subcomplexes. We may pass to the inverse limit in virtue of (2.17) as in the proof of (2.18).

According to a general procedure due to G. W. Whitehead [16], the above pairing defines several kinds of products in the homology theory K_* and the cohomology theory K^*. Generalizations to the category of not necessarily finite CW-complexes are easy, except that one must deal carefully with the cartesian product. However, the last point does not matter when one deals only with countable CW-complexes.

We use the *cohomology cross-product*

$$\wedge : \tilde{K}^p(X) \otimes \tilde{K}^q(Y) \to \tilde{K}^{p+q}(X \wedge Y),$$

the /-*product*

$$/ : \tilde{K}^{p+q}(X \wedge Y) \otimes \tilde{K}_q(Y) \to \tilde{K}^p(X),$$

and the *Kronecker index*

$$\langle\ ,\ \rangle : \tilde{K}_p(X) \otimes \tilde{K}^q(X) \to \tilde{K}_{p-q}(S^0),$$

where X and Y are countable CW-complexes with base point.

It is easily seen that the cohomology cross-product coincides with the product (2.11) when the CW-complexes X and Y are both finite, and that the commutation laws (2.12) hold even in the extended sense. Commutation laws between the other kinds of products and the isomorphisms α, β are obtained. In particular, the Bott isomorphisms β are compatible with all the products. For example we have the relations

(2.20) $$\langle \beta(x), u\rangle = \langle x, \beta(u)\rangle = \beta\langle x, u\rangle$$

for $x \in \tilde{K}_p(X)$ and $u \in \tilde{K}^q(X)$.

We shall identify $\tilde{K}_{2m}(S^0) = \tilde{K}^{-2m}(S^0)$ with the integer group Z as follows. First $\tilde{K}_0(S^0) = \tilde{K}^0(S^0)$ is identified with Z by corresponding the unit element 1 to the integer 1. Then $\tilde{K}_{2m}(S^0)$ is identified through iterations of β. Of course $\tilde{K}_{2m+1}(S^0) = \tilde{K}^{-2m+1}(S^0) = 0$ by the Bott periodicity.

Under the above convention the Kronecker index becomes a homomorphism

$$\langle\ ,\ \rangle : \tilde{K}_p(X) \otimes \tilde{K}^q(X) \to Z,$$

and (2.20) becomes

(2.20)′ $$\langle \beta(x), u\rangle = \langle x, \beta(u)\rangle = \langle x, u\rangle.$$

LEMMA (2.21). (i). *If* $u \in \tilde{K}^p(X)$, $v \in \tilde{K}^q(Y)$ *and* $y \in \tilde{K}_q(Y)$, *then*

$$(u \wedge v)/y = (-1)^q \langle y, v\rangle u.$$

(ii). *If* $u' \in \tilde{K}^{p-1}(X)$, $v' \in \tilde{K}^{q+1}(Y)$ *and* $y \in \tilde{K}_q(Y)$, *then* $(u' \wedge v')/y = 0$.

This Lemma can be easily proved through examination of the definitions of products, taking account of the associativity and the anti-commutativity of the pairing $(\mathbf{U}, \mathbf{U}) \to \mathbf{U}$. The details are omitted.

Next we shall specify a generator g_r of $\pi_r(U_r) = [S^r, U_r] = \tilde{K}^r(S^r)$. We put $g_0 = 1 \in \tilde{K}^0(S^0)$ and define inductively g_r by

$$g_r = \alpha(g_{r-1}).$$

It is to be noted that

(2.22) $$g_r = u_{r*}S(g_{r-1}).$$

We shall also write g_r to denote a map $S^r \to U_r$ representing the element $g_r \in \pi_r(U_r)$.

Following G. W. Whitehead [16] we define the *sphere spectrum* $\mathbf{S} = \{S^n, \varepsilon_n\}$ by taking $\varepsilon_n : SS^n \to S^{n+1}$ to be the identity map. Then (2.22) just means that

(2.23) *the sequence of maps* $g_r : S^r \to U_r$ *defines a map of spectra* $\mathbf{g} : \mathbf{S} \to \mathbf{U}$ *in the sense of* [16].

The q-th homology group $H_q(X, A; \mathbf{S})$ of a CW-pair (X, A) with coefficients in the sphere spectrum $\mathbf{S}$ is nothing but the q-th stable homotopy group $\pi_q^s(X/A)$. The map of spectra $\mathbf{g}$ induces a natural transformation

$$H : H_q(\ ,\ ; \mathbf{S}) \to H_q(\ ,\ ; \mathbf{U}) = K_q(\ ,\ ).$$

Specifically, if X is a CW-complex with a base point and if $x \in \pi_q^s(X)$ is represented by a map $f: S^{q+n} \to S^n \wedge X$, then $H(x) \in \tilde{K}_q(X)$ is represented by the map $(g_n \wedge 1) \circ f: S^{q+n} \to U_n \wedge X$.

The map of spectra $\mathbf{g}$ also induces a natural transformation

$$H: H^q(\ \ ,\ \ ; \mathbf{S}) \to H^q(\ \ ,\ \ ; \mathbf{U}) = K^q(\ \ ,\ \ ).$$

Here we work in the category of *finite* CW-complexes. If X is a finite CW-complex with a base point and if $u \in \tilde{H}^q(X; \mathbf{S})$ is represented by a map $f: S^{n-q}X \to S^n$, then $H(u) \in \tilde{K}^q(X)$ is represented by the map $g_n \circ f: S^{n-q}X \to U_n$.

LEMMA (2.24). *Under the usual identification $S^{p+q} = S^p \wedge S^q$, we have the relation*

$$g_{p+q} = (-1)^{pq} g_p \wedge g_q$$

in $\tilde{K}^{p+q}(S^{p+q})$.

Proof. Apply α_L to the identity $g_r = 1 \wedge g_r$. Noting that $\alpha_L = \alpha$ in $\tilde{K}^r(S^0 \wedge S^r) = \tilde{K}^r(S^r)$ we get from Lemma (2.12)

$$\alpha(g_r) = (-1)^r \alpha(1) \wedge g_r,$$

that is,

$$g_{1+r} = (-1)^r g_1 \wedge g_r.$$

This proves the case $p = 1$. The general case follows by induction from this and the associativity of the cross-product.

LEMMA (2.25). *The following diagram is homotopy commutative*

$$\begin{tikzcd} S^p \wedge U_q \arrow[r, "g_p \wedge 1"] \arrow[dr, "n_{p,q}"'] & U_p \wedge U_q \arrow[d, "t_{p,q}"] & U_p \wedge S^q \arrow[l, "1 \wedge g_q"'] \arrow[dl, "n'_{p,q}"] \\ & U_{p+q} & \end{tikzcd}$$

where the map $n_{p,q}$ is the composition

$$S^p \wedge U_q \xrightarrow{S^{p-1}u_q} S^{p-1} \wedge U_{q+1} \xrightarrow{S^{p-2}u_{q+1}} \cdots \to SU_{p+q-1} \xrightarrow{Su_{p+q-1}} U_{p+q},$$

and the homotopy class of $n'_{p,q}$ is represented, after being multiplied by $(-1)^{pq}$, by the map $(x, y) \to n_{q,p}(y, x)$.

Proof. The homotopy commutativity of the second triangle follows from that of the first and the anti-commutativity of the pairing $t_{p,q}: U_p \wedge U_q \to U_{p+q}$.

To prove the homotopy commutativity of the first, we proceed by induction on p. By the definition of the cohomology cross-product, the map $t_{p,q} \circ (g_p \wedge \iota_q)$ represents $(-1)^{pq} g_p \wedge \iota_q \in \tilde{K}^{p+q}(S^p \wedge U_q) = [S^p \wedge U_q, U_{p+q}]$. We denote by the same letter $n_{p,q}$ the homotopy class represented by the map. Then the desired homotopy commutativity means the relation

$$n_{p,q} = (-1)^{pq} g_p \wedge \iota_q.$$

Now

$$
\begin{aligned}
(-1)^r g_1 \wedge \iota_r &= (-1)^r \alpha(1) \wedge \iota_r \\
&= \alpha_L(1 \wedge \iota_r), \quad \text{by (2.12)}, \\
&= \alpha(\iota_r) \\
&= u_r^*(\iota_{r+1}), \text{ by (2.19)}, \\
&= n_{1,r}.
\end{aligned}
$$

This proves the case $p = 1$. Suppose inductively that the above relation holds for all q. Under the natural identification $S^{p+1} \wedge U_{q-1} = S^p \wedge (S^1 \wedge U_{q-1})$, we have

$$n_{p+1,q-1} = (1 \wedge u_{q-1})^*(n_{p,q}).$$

The inductive assumption yields then

$$
\begin{aligned}
n_{p+1,q-1} &= (-1)^{pq} g_p \wedge u_{q-1}^*(\iota_q) \\
&= (-1)^{pq} g_p \wedge \alpha(\iota_{q-1}), \quad \text{by (2.19)}, \\
&= (-1)^{pq} g_p \wedge ((-1)^{q-1} g_1 \wedge \iota_{q-1}), \quad (\text{case: } p = 1), \\
&= (-1)^{pq+q-1}(g_p \wedge g_1) \wedge \iota_{q-1} \\
&= (-1)^{(p+1)(q-1)} g_{p+1} \wedge \iota_{q-1}, \quad \text{by (2.24)}.
\end{aligned}
$$

This completes the induction.

The double sequence of maps $n_{p,q} : S^p \wedge U_q \to U_{p+q}$ forms a pairing $(\mathbf{S}, \mathbf{U}) \to \mathbf{U}$ and hence induces several products (cf. [16]). Among these we need the /-product

$$/ : \tilde{H}^{p+q}(X \wedge Y; \mathbf{S}) \otimes \tilde{K}_q(Y) \to \tilde{K}^p(X)$$

and the Kronecker index

$$\langle \ , \ \rangle : \tilde{H}_p(X; \mathbf{S}) \otimes \tilde{K}^q(X) \to \tilde{K}_{p-q}(S^0).$$

Here X and Y are CW-complexes with base point which are assumed finite in the case of /-product. As before we shall regard the Kronecker index as a homomorphism

$$\langle \ , \ \rangle : \tilde{H}_p(X; \mathbf{S}) \otimes \tilde{K}^q(X) \to Z.$$

As a direct consequence of Lemma (2.25) we have

LEMMA (2.26). *The following two diagrams are commutative.*

$$
\begin{array}{ccc}
\tilde{H}^{p+q}(X \wedge Y; \mathbf{S}) \otimes \tilde{K}_q(Y) & \xrightarrow{\ /\ } & \tilde{K}^p(X) \\
\big\downarrow{\scriptstyle H \otimes 1} & & \big\| \\
\tilde{K}^{p+q}(X \wedge Y) \otimes \tilde{K}_q(Y) & \xrightarrow{\ /\ } & \tilde{K}^p(X),
\end{array}
$$

$$
\begin{array}{ccc}
\tilde{H}_p(X; \mathbf{S}) \otimes \tilde{K}^q(X) & \xrightarrow{\langle\ ,\ \rangle} & Z \\
\big\downarrow{\scriptstyle H \otimes 1} & & \big\| \\
\tilde{K}_p(X) \otimes \tilde{K}^q(X) & \xrightarrow{\langle\ ,\ \rangle} & Z.
\end{array}
$$

For later use we deduce some consequences from Lemma (2.26). We denote by $s_p \in \tilde{H}_p(S^p, \mathbf{S}) = \pi_p^S(S^p)$ the element represented by the identity map $S^p \to S^p$. We also denote by $s_p^* \in \tilde{H}^p(S^p; \mathbf{S})$ the element represented by the identity map.

LEMMA (2.27). *Let X be a CW-complex with base point and let $f: S^p \to X$ be a base point preserving map. Let $x \in \tilde{H}_p(X, \mathbf{S}) = \pi_p^S(X)$ be represented by f. Then, for any $v \in \tilde{K}^q(X)$,*

$$\langle H(x), v\rangle = \langle s_p, f^*v\rangle.$$

In fact,

$$\begin{aligned}\langle H(x), v\rangle &= \langle x, v\rangle, \quad \text{by (2.26)},\\ &= \langle f_*(s_p), v\rangle\\ &= \langle s_p, f^*v\rangle.\end{aligned}$$

PROPOSITION (2.28). *Let X and Y be finite CW-complexes with base point, and let $u: Y \wedge X \to S^n$ be a duality map in the sense of* [14]. *Then the homomorphism*

$$u^*g_n/ : \tilde{K}_p(X) \to \tilde{K}^{n-p}(Y)$$

is an isomorphism, where $g_n \in \tilde{K}^n(S^n)$ is the generator prescribed before.

In fact, by [16; (8.2)], we know that the homomorphism

$$u^*s_n/ : \tilde{K}_p(X) \to \tilde{K}^{n-p}(Y)$$

is bijective. Now $H(s_n) = g_n$, so that the homomorphism $u^*g_n/$ coincides with $u^*s_n/$ by (2.26).

We come now to the duality theorem. Let X be a CW-complex with base point. The Kronecker index

$$\langle\ ,\ \rangle : \tilde{K}_q(X) \otimes \tilde{K}^q(X) \to Z$$

induces a homomorphism

$$\gamma : \tilde{K}^q(X) \to Hom(\tilde{K}_q(X), Z)$$

defined by

$$\gamma(v) = \langle\ , v\rangle.$$

THEOREM (2.29). *Let X be a finite CW-complex with base point. Suppose that the cohomology $K^*(X)$ is torsion free. Then the homomorphism*

$$\gamma : \tilde{K}^q(X) \to Hom(\tilde{K}_q(X), Z)$$

is a bijection.

Proof. Let A be a finite CW-complex with base point. We shall write $\underset{\sim}{K}_q(A)$ for the group $\tilde{K}_q(A)$ factored by its torsion subgroup. Similarly we set $\underset{\sim}{K}^q(A) = \tilde{K}^q(A)/\text{torsion}$. The cohomology cross-product, the /-product, and the Kronecker index induce naturally homomorphisms

$$\begin{aligned}\wedge &: \underset{\sim}{K}^p(X) \otimes \underset{\sim}{K}^q(Y) \to \underset{\sim}{K}^{p+q}(X \wedge Y)\\ / &: \underset{\sim}{K}^{p+q}(X \wedge Y) \otimes \underset{\sim}{K}_q(Y) \to \underset{\sim}{K}^p(X),\\ \langle\ ,\ \rangle &: \underset{\sim}{K}_p(X) \otimes \underset{\sim}{K}^q(X) \to Z.\end{aligned}$$

The formulas in Lemma (2.21) hold also in this sense.

To prove the theorem we may assume $q = 0$ or $q = 1$ in virtue of (2.20)′.

Let Y be a $2n$-dual of X and let $u: Y \wedge X \to S^{2n}$ be a duality map. The homomorphism

$$u^*g_{2n}/: \underset{\sim}{K}_q(X) \longrightarrow \underset{\sim}{K}^{2n-q}(Y)$$

is a bijection by (2.28). From the naturality of the Bott isomorphism with respect to the product it follows that the homomorphism

$$u^*g'_{2n}/ : \tilde{K}_q(X) \to \tilde{K}^{-q}(Y)$$

is also a bijection where $g'_{2n} \in \tilde{K}^0(S^{2n})$ is a generator. Hence the homomorphism

$$w/ : \bar{K}_q(X) \to \bar{K}^{-q}(Y)$$

is a bijection where w denotes the image of $u^*g'_{2n} \in \tilde{K}^0(Y \wedge X)$ in $\bar{K}^0(Y \wedge X)$.

Now, since $K^*(X)$ is torsion free, it follows from the Künneth formula for K-theory due to Atiyah [2] that the cross-product gives a direct sum representation

$$\tilde{K}^0(Y \wedge X) \cong \tilde{K}^0(Y) \otimes \tilde{K}^0(X) + \tilde{K}^{-1}(Y) \otimes \tilde{K}^1(X).$$

This leads to a natural isomorphism

$$\bar{K}^0(Y \wedge X) \cong \bar{K}^0(Y) \otimes \bar{K}^0(X) + \bar{K}^{-1}(Y) \otimes \bar{K}^1(X).$$

Let $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ and $\{d_i\}$ be free bases of $\bar{K}^0(Y)$, $\bar{K}^{-1}(Y)$, $\bar{K}^0(X)$ and $\bar{K}^1(X)$ respectively. Let $w \in \bar{K}^0(Y \wedge X)$ correspond to

$$\sum m_{ij}a \otimes c_j + \sum n_{kl}b_k \otimes d_l$$

by the above isomorphism. Then, for any $x \in \bar{K}_q(X)$,

$$w/x = \begin{cases} \sum m_{ij}\langle x, c_j\rangle a_i, & \text{if } q = 0, \\ -\sum n_{kl}\langle x, d_l\rangle b_k, & \text{if } q = 1, \end{cases}$$

by (2.21). Let $\{e_s\}$ be a base of $\bar{K}_q(X)$. Then the matrix corresponding to the isomorphism $w/$ with respect to these bases is

$$\begin{cases} (m_{ij}) \cdot {}^t(\langle e_s, c_j\rangle), & \text{if } q = 0 \\ -(n_{kl}) \cdot {}^t(\langle e_s, d_l\rangle), & \text{if } q = 1. \end{cases}$$

This matrix has determinant ± 1 since $w/$ is an isomorphism.

Two conclusions follow from this. First,

$$\operatorname{rank} \bar{K}^0(Y) \leqq \operatorname{rank} \bar{K}^0(X),$$
$$\operatorname{rank} \bar{K}^{-1}(Y) \leqq \operatorname{rank} \bar{K}^1(X).$$

Since X and Y are $2n$-dual to each other the roles of X and Y are reciprocal, and we must have the equalities

$$\operatorname{rank} \bar{K}^0(Y) = \operatorname{rank} \bar{K}^0(X),$$
$$\operatorname{rank} \bar{K}^{-1}(Y) = \operatorname{rank} \bar{K}^1(X).$$

Second, since the matrices (m_{ij}), $(\langle e_s, c_j\rangle)$ etc. are square matrices as remarked just above, the determinants of these matrices must be ± 1.

On the other hand the matrix corresponding to the homomorphism

$$\gamma : \bar{K}^q(X) = \bar{K}^q(X) \to Hom(\tilde{K}_q(X), Z)$$

with respect to suitable bases is

$$\begin{cases} (\langle e_s, c_j\rangle), & \text{if } q = 0, \\ (\langle e_s, d_l\rangle), & \text{if } q = 1. \end{cases}$$

Since this matrix is integral and has determinant ± 1, γ is an isomorphism. This completes the proof.

COROLLARY (2.30). *Let X be a CW-complex with base point. Suppose that there is a sequence $X_1 \subset X_2 \subset \cdots$ of finite subcomplexes with union X such that $K^*(X_n)$ is torsion free for all n. Suppose moreover that X is K^*-admissible. Then the homomorphism*

$$\gamma : \tilde{K}^q(X) \to Hom(\tilde{K}_q(X), Z)$$

is a bijection.

Proof. By the assumption and Theorem (2.29),

$$\gamma : \tilde{K}^q(X_n) \cong Hom(\tilde{K}_q(X_n); Z)$$

for all n. It is easy to see that this induces a natural isomorphism

$$\gamma : \varprojlim_n K^q(X_n) \cong Hom(\varinjlim_n \tilde{K}_q(X_n), Z).$$

Since the union of the subcomplexes X_n is X,

$$\varinjlim_n \tilde{K}_q(X_n) = \tilde{K}_q(X)$$

by (2.13). Also since X is K^*-admissible by assumption,

$$\varprojlim_n \tilde{K}^q(X_n) = \tilde{K}^q(X).$$

This completes the proof.

Finally we consider the K_*-homology of the Thom complex of a complex vector bundle. Let X be a connected CW-complex and let ξ be a complex vector bundle over X of complex dimension q. Let X^ξ denote the Thom complex of ξ. Let x be any point of X. Then the inclusion map $i : S^{2q} = x^\xi \to X^\xi$ represents a generator $s \in \tilde{H}_{2q}(X^\xi; \mathbf{S}) = \pi_{2q}^S(X^\xi) \cong Z$.

LEMMA (2.31). *With the above notation, the element $H(s) \in \tilde{K}_{2q}(X^\xi)$ is not divisible by any integer other than ± 1.*

Proof. Let X_α be a connected finite subcomplex of X. The inclusion $X_\alpha^\xi \subset X^\xi$ induces a natural isomorphism $\tilde{H}_{2q}(X_\alpha^\xi; \mathbf{S}) \cong \tilde{H}_{2q}(X^\xi; \mathbf{S})$. We regard s as an element of $\tilde{H}_{2q}(X_\alpha^\xi; \mathbf{S})$.

Now, it is known that there exists an element $v \in \tilde{K}^{2q}(X_\alpha^\xi)$ such that $i^*v = g_{2q} \in \tilde{K}^{2q}(S^{2q})$ (see for example [6]). Hence

$$\langle s_{2q}, i^*v \rangle = \pm 1.$$

Since $i_* s_{2q} = s$, Lemma (2.27) yields

$$\langle H(s), v \rangle = \pm 1.$$

This implies that $H(s) \in \tilde{K}_{2q}(X_\alpha^\xi)$ is not divisible by any integer $\neq \pm 1$. Since $\tilde{K}_{2q}(X^\xi) = \varinjlim_\alpha \tilde{K}_{2q}(X_\alpha^\xi)$, the naturality of H implies that $H(s) \in \tilde{K}_{2q}(X^\xi)$ must not be divisible by any integer $\neq \pm 1$.

Remark. If we assume moreover that the group $\tilde{K}_{2q}(X^\xi)$ is finitely generated or free abelian, then Lemma (2.31) says that the homomorphism

$$H : \tilde{H}_{2q}(X^\xi; \mathbf{S}) \to \tilde{K}_{2q}(X^\xi)$$

is a bijection onto a direct summand.

§3. PROOF OF THEOREM II

Let $BU(N)$ be a classifying space for the unitary group $U(N)$ and let $MU(N)$ be the Thom complex of the universal complex vector bundle ξ_N over $BU(N)$. We may assume that $BU(N)$ is a countable CW-complex so that $MU(N)$ is also a countable CW-complex.

LEMMA (3.1). *The homomorphism*

$$\gamma : \tilde{K}^q(MU(N)) \to Hom(\tilde{K}_q(MU(N)), Z)$$

is a bijection. The group $K_q(MU(N))$ is a free abelian group.

Proof. We may take the limit space $\bigcup_n G_{n,N}$ as a classifying space $BU(N)$, where $G_{n,N} = U(n+N)/U(n) \times U(N)$ is the complex Grassman manifold. $MU(n, N)$ will denote the Thom complex of the universal bundle ξ_N restricted on the subcomplex $G_{n,N}$. Then $MU(1, N) \subset MU(2, N) \subset \cdots$ is a sequence of subcomplexes with union $MU(N)$. $H^*(G_{n,N}; Z)$ is free abelian and the natural homomorphism $H^*(G_{n+1,N}; Z) \to H^*(G_{n,N}; Z)$ is surjective. Using the Thom isomorphism, we see that $H^*(MU(n, N); Z)$ is free abelian and the natural homomorphism $H^*(MU(n+1, N); Z) \to H^*(MU(n, N); Z)$ is surjective. Hence the group $K^*(MU(n, N))$ is also free abelian. In particular the CW-complex $MU(N)$ is K^*-admissible by Lemma (2.16) and we can apply Corollary (2.30) to conclude that

$$\gamma : \tilde{K}^q(MU(N)) \to Hom(\tilde{K}_q(MU(N)), Z)$$

is bijective.

To prove the second statement we may proceed directly using the spectral sequence connecting $H_*(MU(N); K_*(x_0))$ to $K_*(MU(N))$ (cf. [10]). However we argue here as follows. Using the spectral sequence we see that $\tilde{K}_q(MU(n, N))$ is a free abelian group of finite rank. Let $i : MU(n, N) \to MU(n+1, N)$ denote the inclusion. Then the commutativity of the diagram

$$\begin{array}{ccc}
\tilde{K}^q(MU(n+1, N)) & \xrightarrow{\gamma} & Hom(\tilde{K}_q(MU(n+1, N)), Z) \\
\Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle i^* = Hom(i_*, \quad)} \\
\tilde{K}^q(MU(n, N)) & \xrightarrow{\gamma} & Hom(\tilde{K}_q(MU(n, N)), Z),
\end{array}$$

together with the surjectivity of i^* implies that the homomorphism

$$i_* : \tilde{K}_q(MU(n, N)) \to \tilde{K}_q(MU(n+1, N))$$

is a bijection onto a direct summand. Therefore the direct limit $\varinjlim_n \tilde{K}_q(MU(n, N)) = \tilde{K}_q(MU(N))$ is a free abelian group.

Now suppose $k < N$. Then the homotopy group $\pi_{2k+2N}(MU(N))$ is stable, that is,

$$\pi_{2k+2N}(MU(N)) = \pi^S_{2k+2N}(MU(N)) = H_{2k+2N}(MU(N), \mathbf{S}).$$

THEOREM (3.2). *Suppose $k < N$. Then the homomorphism*

$$H : \pi_{2k+2N}(MU(N)) \to \tilde{K}_{2k+2N}(MU(N))$$

is a bijection onto a direct summand.

Assuming (3.2) for a moment we shall prove Theorem II. First as a corollary of (3.2), we get

COROLLARY (3.3). *Suppose $k < N$. Then the homomorphism*

$$\tilde{K}^{2k+2N}(MU(N)) \xrightarrow{\gamma} Hom(\tilde{K}_{2k+2N}(MU(N)), Z) \xrightarrow{H^\#} Hom(\pi_{2k+2N}(MU(N)), Z)$$

is surjective, where $H^\# = Hom(H, \ ) = \circ H$ is the composition by H from the right.

Indeed, γ is a bijection by (3.1) and $H^\#$ is surjective since H is a bijection onto a direct summand by (3.2). Therefore $H^\# \circ \gamma$ is surjective.

Now Theorem II is proved as follows. Let $x \in \pi_{2k+2N}(MU(N))$ and $v \in \tilde{K}^{2k+2N}(MU(N))$. Then,

$$\begin{aligned}(H^\# \circ \gamma(v))(x) &= \langle H(x), v \rangle, \\ &= \langle s_{2k+2N}, x^*(v) \rangle, \quad \text{by (2.27).}\end{aligned}$$

Therefore, if we define the homomorphism

$$\rho : \tilde{K}^{2k+2N}(MU(N)) \to Hom(\pi_{2k+2N}(MU(N)), \tilde{K}^{2k+2N}(S^{2k+2N}))$$

by

$$\rho(v)(x) = x^*(v),$$

then we get

$$(H^\# \circ \gamma(v))(x) = \langle s_{2k+2N}, \rho(v)(x) \rangle.$$

Since $H^\# \circ \gamma$ is surjective by (3.3) and since the Kronecker index $\pi_{2k+2N}(S^{2k+2N}) \otimes \tilde{K}^{2k+2N}(S^{2k+2N}) \to Z$ is a dual pairing, this relation implies that the homomorphism ρ is surjective. From the naturality of the Bott isomorphism it follows that the homomorphism

$$\rho : \tilde{K}^0(MU(N)) \to Hom(\pi_{2k+2N}(MU(N)), K^0(S^{2k+2N}))$$

defined by the same formula as above is also surjective. This completes the proof of Theorem II.

The rest of this section is devoted to the proof of Theorem (3.2).

We consider the double sequence of groups

$$\pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m)), \qquad U_{2l} = Z \times BU,$$

indexed by pairs of integers (l, m) with $0 \leqq l$, $0 \leqq m$.

Let

$$\phi_{l,m}; \pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m)) \to \pi_{2(k+N+l+1+m)}(U_{2(l+1)} \wedge MU(N+m))$$

be the composition homomorphism $u_{2l+1*} \circ S \circ u_{2l*} \circ S$.

There is a map (unique up to homotopy) $f: BU(N+m) \to BU(N+m+1)$ such that ${}^*\xi_{N+m+1} = \xi_{N+m} \oplus 1$, where 1 is the trivial complex line bundle. The Thom complex of $\xi_{N+m} \oplus 1$ is naturally homeomorphic to the double suspension $S^2 MU(N+m)$. The bundle map $\xi_{N+m} \oplus 1 \to \xi_{N+m+1}$ induces a map

$$b_m : S^2 MU(N+m) \to MU(N+m+1).$$

Let $\psi_{l,m}$ be the composition homomorphism

$$\pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m)) \xrightarrow{S^2} \pi_{2(k+N+l+m+1)}(U_{2l} \wedge S^2MU(N+m))$$
$$\xrightarrow{b_{m*}} \pi_{2(k+N+l+m+1)}(U_{2l} \wedge MU(N+m+1)).$$

Then it is easily seen that

$$\psi_{l+1,m} \circ \phi_{l,m} = \phi_{l,m+1} \circ \psi_{l,m}. \tag{3.4}$$

(3.4) implies that the double sequence of groups $\pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m))$ together with homomorphisms $\phi_{l,m}$, $\psi_{l,m}$ (and their possible compositions) forms a direct system. From the definition we see immediately that

(3.5) *the partial direct limit* $\varinjlim_{l} \pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m))$ *is nothing but the homology group* $\tilde{K}_{2(k+N+m)}(MU(N+m))$.

Let $\tilde{\mathscr{U}}_q(X)$ denote the reduced homology group of a CW-complex X having base point with coefficients in the spectrum

$$\mathbf{MU} = \{MU(0), SMU(0), MU(1), SMU(1), \cdots\} \quad \text{(cf. [9])}.$$

Then it is also immediate that

(3.6) *the partial direct limit* $\varinjlim_{m} \pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m))$ *is the homology group* $\tilde{\mathscr{U}}_{2(k+l)}(Z \times BU)$.

The groups $\varinjlim_{m} \pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m)) = \tilde{K}_{2(k+N+m)}(MU(N+m))$ form a direct system in a natural way. Also the groups $\varinjlim_{m} \pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m)) = \tilde{\mathscr{U}}_{2(k+\ )}(Z \times BU)$ form a direct system, and we have

$$\varinjlim_{m,l} \pi_{2(k+N+l+m)}(U_{2l} \wedge MU(N+m)) = \varinjlim_{m} \tilde{K}_{2(k+N+m)}(MU(N+m)), \tag{3.7}$$
$$= \varinjlim_{l} \tilde{\mathscr{U}}_{2(k+l)}(Z \times BU).$$

We shall denote the total direct limit (3.7) by $\tilde{K}_{2k}(MU)$.

At this point we turn to the multiplicative property of the complex bordism homology theory (cf. [9]). There is a map (unique up to homotopy)

$$f\colon BU(p) \times BU(q) \to BU(p+q)$$

such that $f^*\xi_{p+q} = \xi_p \times \xi_q$. The Thom complex of the vector bundle $\xi_p \times \xi_q$ being naturally homeomorphic to the reduced join $MU(p) \wedge MU(q)$, the bundle map $\xi_p \times \xi_q \to \xi_{p+q}$ induces a map

$$b_{p,q}\colon MU(p) \wedge MU(q) \to MU(p+q).$$

The maps $b_{p,q}$ give rise to a pairing $(\mathbf{MU}, \mathbf{MU}) \to \mathbf{MU}$. This pairing in turn gives rise to products in the homology and cohomology. In particular we have the homology cross-product

$$\tilde{\mathscr{U}}_p(X) \otimes \tilde{\mathscr{U}}_q(S^0) \to \tilde{\mathscr{U}}_{p+q}(X).$$

We simply write xy to denote the cross-product of $x \in \tilde{\mathcal{U}}_p(X)$ and $y \in \tilde{\mathcal{U}}_q(S^0)$. If $f : S^{p+2s} \to X \wedge MU(s)$ represents $x \in \tilde{\mathcal{U}}_p(X)$ and if $h : S^{q+2t} \to MU(t)$ represents $y \in \tilde{\mathcal{U}}_q(S^0)$, then their cross-product $xy \in \tilde{\mathcal{U}}_{p+q}(X)$ is represented by the composition

$$S^{p+q+2(s+t)} = S^{p+2s} \wedge S^{q+2t} \xrightarrow{f \wedge h} (X \wedge MU(s)) \wedge MU(t) \to X \wedge (MU(s) \wedge MU(t)) \xrightarrow{1 \wedge b_{s,t}} X \wedge MU(s+t),$$

where the second map is a natural homotopy equivalence [16; (2.4)].

Taking $X = S^0$, the group $\mathcal{U}_* = \sum_p \mathcal{U}_p$, $\mathcal{U}_p = \tilde{\mathcal{U}}_p(S^0)$, becomes a commutative, associative graded ring with unit 1. This is nothing but Milnor's complex cobordism ring. Moreover the cross-product makes the homology group $\tilde{\mathcal{U}}_*(X) = \sum_p \tilde{\mathcal{U}}_p(X)$ a graded (right) $\mathcal{U}_*$-module.

Now we see that

(3.8) *the natural homomorphism* $\mathcal{U}_{2(k+l)}(Z \times BU) \to \mathcal{U}_{2(k+l+1)}(Z \times BU)$ *is a* $\mathcal{U}_*$*-module map.*

The verification of (3.8) is immediate. It reduces to the compatibility of the cross-product with induced homomorphisms and the suspension isomorphism.

Let $g_0 : S^0 \to Z \times BU$ represent $1 \in \tilde{K}^0(S^0)$. Then, analogously to (3.8) we see that

(3.9) *the natural homomorphism* $g_{0*} : \tilde{\mathcal{U}}_{2k}(S^0) \to \tilde{\mathcal{U}}_{2k}(Z \times BU)$ *is a* $\mathcal{U}_*$*-module map.*

Suppose now $k < N$ and consider the homomorphism

$$H : \pi_{2k+2N}(MU(N)) \to \tilde{K}_{2k+2N}(MU(N)),$$

or more generally

$$H : \pi_{2k+2(N+m)}(MU(N+m)) \to \tilde{K}_{2k+2(N+m)}(MU(N+m)).$$

Since the homotopy group $\pi_{2k+2(N+m)}(MU(N+m))$ is stable the homomorphism H is simply the composition

$$\pi_{2k+2(N+m)}(MU(N+m)) \xrightarrow{g_{0*}} \pi_{2k+2(N+m)}((Z \times BU) \wedge MU(N+m)) \longrightarrow \tilde{K}_{2k+2(N+m)}(MU(N+m))$$

where the second homomorphism is the natural map to the direct limit group.

On the other hand the direct system

$$\cdots \to \pi_{2k+2(N+m)}(MU(N+m)) \to \pi_{2k+2(N+m+1)}(MU(N+m+1)) \to \cdots$$

is stable, that is, we have a canonical isomorphism

$$\pi_{2k+2(N+m)}(MU(N+m)) \cong \mathcal{U}_{2k}.$$

Therefore the homomorphism H can be interpreted as a homomorphism $H : \mathcal{U}_{2k} \to \tilde{K}_{2k+2(N+m)}(MU(N+m))$, and we have the following commutative diagram:

(3.10)
$$\begin{tikzcd}
& & & \tilde{K}_{2k+2N}(MU(N)) \arrow[d] \\
& & & \tilde{K}_{2k+2(N+m)}(MU(N+m)) \arrow[d] \\
\mathcal{U}_{2k} \arrow[r, "g_{0*}"'] \arrow[rrruu, "H"] \arrow[rrru, "H"] & \tilde{\mathcal{U}}_{2k}(Z \times BU) \arrow[r] & \tilde{\mathcal{U}}_{2(k+l)}(Z \times BU) \arrow[r] & \tilde{K}_{2k}(MU).
\end{tikzcd}$$

We now quote a theorem of Milnor on the structure of cobordism ring $\mathscr{U}_*$ (cf. [13, 15]).

THEOREM OF MILNOR (3.11). *The ring $\mathscr{U}_*$ is a polynomial ring over Z with generators $M_{2q} \in \mathscr{U}_{2q}$.*

Furthermore, according to Conner–Floyd [8, 9], we have

PROPOSITION (3.12). *$\tilde{\mathscr{U}}_*(Z \times BU)$ is a free graded $\mathscr{U}_*$-module.*

Let H_l denote the composition

$$\mathscr{U}_{2k} \xrightarrow{g_{0*}} \tilde{\mathscr{U}}_{2k}(Z \times BU) \to \tilde{\mathscr{U}}_{2(k+l)}(Z \times BU),$$

and let H_∞ denote the composition

$$\mathscr{U}_{2k} \xrightarrow{g_{0*}} \tilde{\mathscr{U}}_{2k}(Z \times BU) \to \tilde{K}_{2k}(MU).$$

H_l is a $\mathscr{U}_*$-module map by (3.8) and (3.9).

For the proof of Theorem (3.2) it suffices to show the following proposition.
(3.13) *Let $x \in \mathscr{U}_{2k}$ be an element not divisible by any integer other than ± 1. Then $H_l(x) \in \tilde{\mathscr{U}}_{2(k+l)}(Z \times BU)$ is not divisible by any integer other than ± 1 for any l.*

In fact, if (3.13) holds, then passing to the direct limit $\tilde{K}_{2k}(MU)$, $H_\infty(x)$ is not divisible by any integer $\neq \pm 1$. It follows from the commutativity of the diagram (3.10) that $H(x) \in \tilde{K}_{2k+2N}(MU(N))$ is not divisible by any integer $\neq \pm 1$. Since $\mathscr{U}_{2k}$ is a free abelian group of finite rank by (3.11) and since $\tilde{K}_{2k+2N}(MU(N))$ is a free abelian group by (3.1), this implies clearly that the homomorphism $H: \mathscr{U}_{2k} \to \tilde{K}_{2k+2N}(MU(N))$ is a bijection onto a direct summand. This proves (3.2).

We now proceed to prove (3.13). First, the case $k = 0$. $\mathscr{U}_0$ is generated by 1. By the isomorphism $\mathscr{U}_0 \cong \pi_{2k+2(N+m)}(MU(N+m))$, the element 1 corresponds to s. By Lemma (2.31), $H(s) \in \tilde{K}_{2k+2(N+m)}(MU(N+m))$ is not divisible by any integer other than ± 1. Since this is true for any m, the image $H_\infty(1)$ of $H(s)$ in the direct limit $\tilde{K}_{2k}(MU(N))$ is not divisible by any integer $\neq \pm 1$; a fortiori $H_l(1)$ is not divisible by any integer $\neq \pm 1$.

Let $\{a_i\}$ be a homogeneous $\mathscr{U}$ -base of $\tilde{\mathscr{U}}(Z \times BU)$ (cf. (3.12)). If l is fixed, then we can write uniquely $H_l(1) \in \tilde{\mathscr{U}}_{2l}(Z \times BU)$ as a finite sum

$$H_l(1) = \sum a_i x_i, \qquad x_i \in \mathscr{U}_*.$$

If d is an integer dividing x_i for all i, then d must be equal to ± 1, as proved just above.

Take any k and let $x \in \mathscr{U}_{2k}$ be an element not divisible by any integer $\neq \pm 1$. Since H_l is a $\mathscr{U}_*$-module map, we have

$$H_l(x) = H_l(1) \cdot x = \sum a_i x_i x.$$

Suppose that an integer d divides $x_i x$ for a fixed i. Then, since $\mathscr{U}_*$ is a polynomial ring over Z, d must divide x_i. Now if d is an integer dividing $H_l(x)$ then d divides $x_i x$ for all i. Thus d divides x_i for all i. Hence $d = \pm 1$. This proves (3.13) and completes the proof of Theorem (3.2).

§4. DEDUCTION OF THEOREM I

Let $\iota \in [BU, BU]$ denote the element represented by the identity map, where [,] means the set of base point preserving homotopy classes as before. Note that

$$[BU, BU] = [BU, Z \times BU] = \tilde{K}^0(BU).$$

Set $\theta = -\iota \in [BU, BU]$. We write θ also to denote a map $BU \to BU$ representing the homotopy class θ. Let X be a connected finite CW-complex with base point. Then

$$[X, BU] = [X, Z \times BU] = \tilde{K}^0(X).$$

Let $x \in [X, BU] = \tilde{K}^0(X)$. The it is clear that

$$\theta \circ x = -x \in \tilde{K}^0(X). \tag{4.1}$$

(4.1) says that, for a stable complex vector bundle x over X, $\theta \circ x$ is its inverse stable bundle.

Now, since the CW-complex BU is K^*-admissible, an argument similar to the proof of Proposition (2.18) using (4.1) proves the following equality:

$$\theta \circ \theta = \iota \in [BU, BU]. \tag{4.2}$$

From (4.2) it follows that

(4.3) $\theta^* : K^0(BU) \to K^0(BU)$ *is an involutive ring automorphism,*

$\theta^* : H^{**}(BU) \to H^{**}(BU)$ *is an involutive ring automorphism for any coefficient group.*

Let c_n denote the n-th Chern class of the universal U-bundle over BU. Then (4.1) and the product formula for Whitney sum yield

$$(1 + c_1 + c_2 + \cdots)(1 + \theta^*(c_1) + \theta^*(c_2) + \cdots) = 1. \tag{4.4}$$

Since the cohomology ring $H^{**}(BU; Z)$ is a ring of formal power series over Z with generators $c_1, c_2, c_3, \ldots$, the formula (4.4) completely determines the automorphisms $\theta^* : H^{**}(BU; Z) \to H^{**}(BU; Z)$ and $\theta^* : H^{**}(BU; Q) \to H^{**}(BU; Q)$. In particular, for the universal Todd class $\mathcal{T} \in H^{**}(BU; Q)$, we have

$$\theta^*(\mathcal{T}^{-1}) = \mathcal{T}. \tag{4.5}$$

If we write formally

$$1 + c_1 + c_2 + \cdots = \prod_i (1 + t_i),$$

then the Todd class $\mathcal{T}$ is given by

$$\mathcal{T} = \prod_i \frac{t_i}{1 - e^{-t_i}}.$$

Now let M be a connected closed weakly almost complex manifold of real dimension $2k$ imbedded in a sphere S^{2k+2N} with $k < N$. Let $\nu : M \to BU(N)$ be the normal map of the imbedding and let $\tau : M \to BU$ be the stable tangential map of M. The composition

$M \xrightarrow{\nu} BU(N) \to BU$ will be denoted by μ, where the second map is the obvious one. Then, by (4.1), we have

$$\theta \circ \mu = \tau.$$

Therefore, for any element $v \in H^{**}(BU; Q)$, we have

$$\tau^*(v) = \mu^*(\theta^*(v)). \tag{4.6}$$

Our task is to seek the subgroup I^{2k} of $H^{2k}(BU; Q)$ introduced in §1. It is equivalent to seek its transform $\theta^*(I^{2k})$ by the automorphism θ^*. From (4.6) it follows that

(4.7) *the subgroup* $\theta^*(I^{2k})$ *consists of those elements* v *such that* $\langle [M], \mu^*(v)\rangle \in Z$ *for any connected closed weakly almost complex manifolds* M.

Here $\langle\ ,\ \rangle$ denotes the usual Kronecker index and $[M]$ denotes the fundamental class of M. It is clear that we may restrict our attention to connected manifolds.

The natural homomorphism $H^{**}(BU; Q) \to H^{**}(BU(N); Q)$ is bijective in the dimensions $\leqq 2k$, so that in this range of dimensions we may identify both. Under that convention we may replace $\mu^*(v)$ by $\nu^*(v)$ in (4.7).

Let $x : S^{2k+2N} \to MU(N)$ be the map obtained by the Thom construction from the imbedding $M \subset S^{2k+2N}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^{2k}(BU(N)) & \xrightarrow{\nu^*} & H^{2k}(M) \\ \cong \downarrow \phi & & \cong \downarrow \phi \\ H^{2k+2N}(MU(N)) & \xrightarrow{x^*} & H^{2k+2N}(S^{2k+2N}), \end{array}$$

in which ϕ are the Gysin–Thom homomorphisms, both being bijective. This holds for the coefficient groups Z and Q.

In view of this and (4.7), we see that

(4.8) *the subgroup* $\phi\theta^*(I^{2k}) \subset H^{2k+2N}(MU(N); Q)$ *consists of the elements* $v \in H^{2k+2N}(MU(N); Q)$ *such that* $\langle s_{2k+2N}, x^*(v)\rangle \in Z$ *for any* $x \in \pi_{2k+2N}(MU(N))$.

Consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{K}^0(MU(N)) & \xrightarrow{\rho} & Hom(\pi_{2k+2N}(MU(N)), \tilde{K}^0(S^{2k+2N})) \\ \downarrow ch_{k+N} & & \downarrow ch_{k+N\#} = Hom(\ \ , ch_{k+N}) \\ H^{2k+2N}(MU(N); Q) & \underset{\cong}{\xrightarrow{\rho}} & Hom(\pi_{2k+2N}(MU(N)), H^{2k+2N}(S^{2k+2N}; Q)), \end{array}$$

where the homomorphism in the first row is defined in §1, and the second row is defined by the analogous formula

$$\rho(v)(x) = x^*(v).$$

The homomorphism ch_{k+N} is the $2(k+N)$-dimensional component of the character homomorphism ch. The second row is an isomorphism. This follows from the fact that the usual Hurewicz homomorphism $\pi_{2k+2N}(MU(N)) \otimes Q \to H_{2k+2N}(MU(N); Q)$ is an isomorphism.

Now (4.8) just says that *the subgroup* $\phi\theta^*(I^{2k})$ *corresponds to the subgroup* $Hom(\pi_{2k+2N}(MU(N)), H^{2k+2N}(S^{2k+2N}; Z))$ *of* $Hom(\pi_{2k+2N}(MU(N)), H^{2k+2N}(S^{2k+2N}; Q))$ *by the isomorphism* ρ. Also we know that $ch_{k+N}(\tilde{K}^0(S^{2k+2N})) = H^{2k+2N}(S^{2k+2N}; Z)$. Hence the above diagram induces the following commutative diagram in which the second row and the second column are isomorphisms:

$$\begin{array}{ccc} \tilde{K}^0(MU(N)) & \xrightarrow{\rho} & Hom(\pi_{2k+2N}(MU(N)), \tilde{K}^0(S^{2k+2N})) \\ \downarrow{\scriptstyle ch_{k+N}} & & \downarrow{\scriptstyle ch_{k+N\,*}} \\ \phi\theta^*(I^{2k}) & \underset{\cong}{\xrightarrow{\rho}} & Hom(\pi_{2k+2N}(MU(N)), H^{2k+2N}(S^{2k+2N}; Z)). \end{array}$$

Now the first row in this diagram is surjective by Theorem II. Hence ch_{k+N} is surjective. In other words,

$$ch_{k+N}(\tilde{K}^0(MU(N)) = \phi\theta^*(I^{2k}). \tag{4.9}$$

We now apply the differentiable Riemann–Roch theorem. $\tilde{K}^0(MU(N))$ is a free $\tilde{K}^0(BU(N))$-module on one generator u, and we have

$$ch(y\cdot u) = \phi(ch(y)\cdot\mathscr{T}^{-1}), \tag{4.10}$$

for $y \in K^0(BU(N))$. (See for example [6]. This type of theorem is usually proved for finite CW-complexes. In the present case, we may pass to the limit because of the K^*-admissisility of $BU(N)$ and $MU(N)$.)

Let $\chi_k(v)$ denote the $2k$-dimensional component of $v \in H^{**}(BU; Q)$. Then, comparing (4.9) and (4.10), we have

$$\theta^*(I^{2k}) = \chi_k(ch(BU(N))\cdot\mathscr{T}^{-1}),$$

where $ch(BU(N))$ is the image of $ch: K^0(BU(N)) \to H^{**}(BU(N); Q)$. We see easily that $K^0(BU) \to K^0(BU(N))$ is surjective. Hence, the homomorphism $ch(BU)\cdot\mathscr{T}^{-1} \to ch(BU(N))\cdot\mathscr{T}^{-1}$ is also surjective. Therefore we have

$$\theta^*(I^{2k}) = \chi_k(ch(BU)\cdot\mathscr{T}^{-1}). \tag{4.11}$$

Apply θ^* to both sides of (4.11). By (4.3), $\theta^{*2} = 1$, and $ch(BU)$ is invariant under θ^*. Taking account of (4.5) we finally get

$$I^{2k} = \chi_k(ch(BU)\cdot\mathscr{T}).$$

This is precisely the statement of Theorem I.

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Added in proof. After the preparation of the present paper, an article of R. E. Stong was published in which Theorem I was independently proved.

R. E. STONG: Relations among characteristic numbers—I, *Topology* **4** (1965), 267–281.