

# Universal Witt vectors and the “Japanese cocycle”

D. Kaledin

## Contents

<b>1</b>	<b>Recollection on the universal Witt vectors.</b>	<b>2</b>
<b>2</b>	<b>Recollection on the Japanese cocycle.</b>	<b>4</b>
<b>3</b>	<b>Multiplication.</b>	<b>6</b>
3.1	<i>K</i> -theoretic definition. . . . .	6
3.2	Group extensions. . . . .	7
3.3	Proof of Theorem 3.2. . . . .	10

## Introduction.

In contradiction to Stigler’s law of eponymy [S], Witt vectors were apparently invented by Ernst Witt [W]. The original definition depended on a rather non-trivial existence theorem: roughly speaking, one shows that for any prime  $p$ , the set  $A^{\mathbb{N}}$  of sequences  $\langle a_1, a_2, \dots \rangle$  of elements of a commutative ring  $A$  carries a unique commutative ring structure such that certain “ghost maps”  $w_n : A^{\mathbb{N}} \rightarrow A$  are ring maps. The resulting ring is denoted by  $W(A)$ . In the motivating example  $A = \mathbb{F}_p$ , the prime field, one has  $W(A) = \mathbb{Z}_p$ , the ring of  $p$ -adic numbers. One can also do the “universal” construction which combines the constructions for all primes; this again gives a certain ring structure on  $A^{\mathbb{N}}$ , with the resulting ring denoted  $\mathbb{W}(A)$ . The original  $W(A)$  is known as the “ $p$ -typical part” of  $\mathbb{W}(A)$  and corresponds, roughly speaking, to the subsequences  $\langle a_1, a_p, a_{p^2}, \dots \rangle$  in sequences  $\langle a_1, a_2, \dots \rangle$ .

Later on, it was realized that as a group,  $\mathbb{W}(A)$  admits a very simple description: all one has to do is to consider the multiplicative group  $A[[t]]^*$  of invertible power series in one variable  $t$ , and take the subgroup of power series with constant term 1. However, the ring structure on  $\mathbb{W}(A)$  is not visible in this description.

The goal of the present note is to remedy this situation and propose a direct construction of the product on  $\mathbb{W}(A) \subset A[[t]]^*$ .

The basic idea is very simple. We of course have  $A[[t]]^* = K_1(A[[t]])$ , the first algebraic  $K$ -group. Given two elements  $f, g \in K_1(A[[t]])$ , we can take their

external product and obtain an element in  $K_2(A[[t_1, t_2]])$ . Then one has to cut the number of variables down to one, and pass from  $K_2$  back to  $K_1$ . Both these tasks are easily accomplished at the same time by taking an appropriate tame symbol. One has a feeling that this simple observation ought to have been made back in the 1970ies, for example in [B], but apparently it was overlooked, possibly for lack of interesting applications (or at least, we could not find it in the literature).

In principle, we could use any of the descriptions of the tame symbol in algebraic  $K$ -theory. However, we found it more instructive to be slightly more conceptual and obtain the symbol by the same procedure as Tate used in his famous coordinate-free construction [T] of the residue map. Thus the tame symbol appears as a certain central extension of appropriate matrix group. For the proof, we also need a Lie algebra version of this extension which is actually quite basic in modern representation theory of infinite dimensional Lie algebras and related subjects. It was discovered by Date, Jimbo, Kashiwara and Miwa around 1980 (see e.g. [V], [FT1]), and it was informally known as “the Japanese cocycle” in mid-80ies Moscow; this is the Japanese cocycle of the title.

The paper is organized as follows. The first section contains a brief reminder on universal Witt vectors, and the second section contains a brief reminder on the Tate residue and the Japanese cocycle. Since the material of both sections is extremely well-known, we have allowed ourselves a slightly informal exposition, and omitted the proofs. There is absolutely nothing in Section 1 nor in Section 2 which has not been known for at least 25 years. Then in Section 3, we switch to the normal rigorous exposition: we give our invariant definition of the product of Witt vectors, and we prove that this is indeed the correct product. Morally, we use rather heavily the material in Section 2, but formally, nothing depends on it; in particular, a reader who wishes to check the proofs may completely skip Section 2.

**Acknowledgments.** I am grateful to A. Beilinson, A. Kuznetsov, N. Markarian and M. Verbitsky for useful discussions, to O. Shkabarnya for introducing me to Skype, and to KIAS for its hospitality and stimulating atmosphere.

## 1 Recollection on the universal Witt vectors.

A good general introduction to Witt vectors can be found for example in [D]; we only recall here the basic constructions.

Assume given a commutative ring  $A$ . Consider the algebra  $A[[t]]$  of formal power series in one variable  $t$  over  $A$ . Let  $A^*$ ,  $A[[t]]^*$  be the groups of invertible elements in  $A$ , resp.  $A[[t]]$  with the group structure given by multiplication. We have the natural map

$$(1.1) \quad A[[t]]^* \rightarrow A^*$$

**Definition 1.1.** The group  $\mathbb{W}(A)$  of *universal Witt vectors* of the ring  $A$  is the kernel of the map (1.1).

By definition,  $\mathbb{W}(A)$  is functorial in  $A$ , so that  $\mathbb{W}(-)$  is in fact a group scheme. Note that we have  $A^* = K_1(A)$ , the first algebraic  $K$ -group of the ring  $A$ , and similarly for  $A[[t]]$ . Thus by definition, we have a short exact sequence

$$(1.2) \quad 0 \longrightarrow \mathbb{W}(A) \longrightarrow K_1(A[[t]]) \longrightarrow K_1(A) \longrightarrow 0.$$

As a set,  $\mathbb{W}(A)$  is canonically identified with the set  $A^{\mathbb{N}}$  of sequences  $a_{\bullet} = \langle a_1, a_2, \dots \rangle$  of elements of  $A$  numbered by positive integers. Explicitly, the isomorphism

$$\alpha : A^{\mathbb{N}} \cong \mathbb{W}(A)$$

is given by

$$\alpha(a_{\bullet}) = \prod_{n \geq 1} (1 + a_n t^n) \in A[[t]]$$

(the infinite product obviously converges and gives an invertible element). To describe the group structure on  $\mathbb{W}(A)$  in terms of the parameterization  $\alpha$ , one introduces the *ghost map*

$$w : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$$

given by

$$(1.3) \quad \begin{aligned} w(a_{\bullet}) &= \langle w_1(a_{\bullet}), w_2(a_{\bullet}), \dots \rangle, \\ w_n(a_{\bullet}) &= \sum_{d|n} d a_d^{n/d}. \end{aligned}$$

Then one shows that there is a unique functorial commutative group structure on  $A^{\mathbb{N}}$  such that the map  $w$  is additive (with the group structure on its target given by termwise addition). This is the group structure of  $\mathbb{W}(A)$ . Indeed, by functoriality, it suffices to prove it for  $A = \mathbb{Z}$ ; since the natural map  $\mathbb{Z} \rightarrow \mathbb{Q}$  is injective, we can equally well prove it for  $A = \mathbb{Q}$ . But in this case, we have a natural isomorphism

$$\log : \mathbb{W}(\mathbb{Q}) \rightarrow \mathbb{Q}^{\mathbb{N}}$$

given by the logarithm series, and the composition  $\log \circ \alpha$  is given by

$$(1.4) \quad \log \alpha(a_{\bullet}) = \sum_{n \geq 1} \log(1 + a_n t^n) = \sum_{m \geq 1} \frac{1}{m} w_m(a_{\bullet}) t^m.$$

Thus on the level of groups, it makes no difference whether one defines  $\mathbb{W}(A)$  by the universal property with respect to the ghost map  $w$  or by the shorter Definition 1.1. However, the ghost map has the following advantage.

**Lemma 1.2.** *There exists a unique functorial commutative ring structure on  $\mathbb{W}(A) \cong A^{\mathbb{N}}$  such that the maps  $w_n : A^{\mathbb{N}} \rightarrow A$  are ring homomorphisms for any  $n \geq 1$ .*

*Proof.* Well-known. □

Thus  $\mathbb{W}(A)$  is not only a group scheme but actually a commutative ring scheme, and this ring structure is not at all obvious in the setting of Definition 1.1. The goal of this paper is to show how it can be described. To do this, we need another well-known tool from a seemingly unrelated subject, namely, the theory of infinite-dimensional Lie algebras and the Tate residue.

## 2 Recollection on the Japanese cocycle.

The shortest way to introduce the so-called *Tate residue* is the following. Fix a field  $k$ . Let  $k\text{-mod}^\infty$  be the exact category of countably generated  $k$ -vector spaces, and let  $k\text{-mod} \subset k\text{-mod}^\infty$  be the full subcategory of finitely generated vector spaces. This is a Serre subcategory, so that one can consider the quotient category  $\overline{k\text{-mod}} = k\text{-mod}^\infty / k\text{-mod}$ . By excision, we have long exact sequences

$$\begin{aligned} K_\bullet(k\text{-mod}) &\longrightarrow K_\bullet(k\text{-mod}^\infty) \longrightarrow K_\bullet(\overline{k\text{-mod}}) \longrightarrow , \\ K_\bullet^+(k\text{-mod}) &\longrightarrow K_\bullet^+(k\text{-mod}^\infty) \longrightarrow K_\bullet^+(\overline{k\text{-mod}}) \longrightarrow , \\ HH_\bullet(k\text{-mod}) &\longrightarrow HH_\bullet(k\text{-mod}^\infty) \longrightarrow HH_\bullet(\overline{k\text{-mod}}) \longrightarrow , \end{aligned}$$

where  $K_\bullet^+$  stands for the “additive  $K$ -theory” of [FT2], identified with the cyclic homology up to a shift of degree,  $HC_\bullet \cong K_{\bullet+1}^+$ , and  $HH_\bullet$  is Hochschild homology. By the standard argument, since  $k\text{-mod}$  has countable direct sums, the middle term of all the three triangles is identically 0, so that the connecting homomorphisms give natural isomorphisms

$$(2.1) \quad \begin{aligned} K_{\bullet+1}(\overline{k\text{-mod}}) &\cong K_\bullet(k) = K_\bullet(k\text{-mod}), \\ K_{\bullet+1}^+(\overline{k\text{-mod}}) &\cong K_\bullet^+(k) = K_\bullet^+(k\text{-mod}), \\ HH_{\bullet+1}(\overline{k\text{-mod}}) &\cong HH_\bullet(k) = HH_\bullet(k\text{-mod}). \end{aligned}$$

It is these isomorphisms that give rise to various canonical residue maps. For example, if  $k((z))$  is the ring of Laurent power series, then we have a natural functor

$$(2.2) \quad P_k : k((z))\text{-mod} \rightarrow \overline{k\text{-mod}}$$

sending a finitely generated module  $V$  over  $k((z))$  to its quotient by a  $k[[z]]$ -submodule  $V_- \subset V$  such that  $V \cong V_- \otimes_{k[[z]]} k((z))$ . The composition of the induced map  $HH_1(k((z))) \rightarrow HH_1(\overline{k\text{-mod}})$  with the isomorphism (2.1) is a map

$$\Omega^1(k((z))) \cong HH_1(k((z))) \rightarrow HH_0(k) \cong k$$

which precisely coincides with the usual residue map on differential forms (this is a modern reformulation of the original invariant construction of the residue

discovered by Tate in [T]). In  $K$ -theory, one obtains in the same way a canonical map

$$K_{\bullet+1}(k((z))) \rightarrow K_{\bullet}(k)$$

which turns out to be equal to the tame symbol.

In low degrees, the isomorphisms (2.1) can be made more explicit. Let  $\overline{\mathfrak{gl}}_{\infty}(k)$  be the Lie algebra of infinite matrices  $(r_{ij})$ ,  $i, j \in \mathbb{Z}$  over  $R$  such that  $r_{ij} = 0$  when  $|i - j| > C$  for some constant  $C$  depending on the matrix (in [FT1], these are called “generalized Jacobi matrices”). Equivalently, let  $V = k((z))$  be the ring  $k((z))$  considered as a  $k$ -vector space, and let  $V_+ = k[z^{-1}] \subset V$ ,  $V_- = k[[z]] \subset V$ ; then  $\overline{\mathfrak{gl}}_{\infty}(k) \subset \text{End}_k(V)$  consists of linear maps  $r : V \rightarrow V$  such that  $z^{l+C}V_- \subset z^l r(V_-) \subset z^{l-C}V_-$  for any  $l$  and some fixed  $C$  only depending on  $r$ . Let  $\mathfrak{gl}_{\infty}^+(k) = \text{End}_k(V_+) \cap \overline{\mathfrak{gl}}_{\infty}(k) \subset \overline{\mathfrak{gl}}_{\infty}(k)$  be the subalgebra of matrices such that  $r_{ij} = 0$  unless  $i, j \geq 0$  – that is,  $r(V_+) \subset V_+$  and  $r(V_-) = 0$  – and let  $\mathfrak{gl}_{\infty}(k) \subset \mathfrak{gl}_{\infty}^+(k)$  be the subalgebra of matrices with at most finite number of non-trivial entries  $r_{ij}$ . Then  $\mathfrak{gl}_{\infty}(k) \subset \mathfrak{gl}_{\infty}^+(k)$  is a Lie ideal, and we have a Lie algebra map

$$\overline{\mathfrak{gl}}_{\infty}(k) \rightarrow \mathfrak{gl}_{\infty}^+(k)/\mathfrak{gl}_{\infty}(k)$$

given by  $r \mapsto PrP$ , where  $p = (p_{ij})$  is the matrix given by  $p_{ij} = 1$  if  $i = j \geq 0$ , and 0 otherwise (this map is precisely the map induced by functor  $P_k$  of (2.2)). We then obtain the induced Lie algebra extension

$$0 \longrightarrow \mathfrak{gl}_{\infty}(R) \longrightarrow \widehat{\mathfrak{gl}}_{\infty}(R) \longrightarrow \overline{\mathfrak{gl}}_{\infty}(k) \longrightarrow 0.$$

One can further compose this extension with the trace map  $\mathfrak{gl}_{\infty}(k) \rightarrow k$ ; the results is the famous central extension  $\widehat{\mathfrak{gl}}_{\infty}(k)$  of  $\overline{\mathfrak{gl}}_{\infty}(k)$  by  $k$  discovered by Date, Jimbo, Kashiwara and Miwa that we have mentioned in the introduction. The corresponding cocycle (the “japanese cocycle”) is the transgression map

$$(2.3) \quad H_2(\overline{\mathfrak{gl}}_{\infty}(k)) \rightarrow H_1(\mathfrak{gl}_{\infty}(k)) \cong K_1^+(k) = HC_0(k) \cong HH_0(k).$$

The algebra  $k((z))$  acts on  $V$  by left multiplication, so that we have a Lie algebra map  $q : k((z)) \rightarrow \overline{\mathfrak{gl}}_{\infty}(k)$ . Composing it with (2.3), we obtain a map

$$(2.4) \quad H_2(k((z))) \rightarrow K_1^+(k),$$

where on the left-hand side,  $k((z))$  is considered as a Lie algebra with respect to the commutator bracket (which happens to be trivial). One can also do all the constructions above with  $k$  replaced by a matrix algebra  $\text{Mat}_n(k)$  and take the limit with respect to  $n$ ; the result is a map

$$(2.5) \quad H_2(\mathfrak{gl}_{\infty}(k((z)))) \rightarrow K_1^+(k).$$

If  $k$  is a field of characteristic 0, then the left-hand side by definition contains  $K_2^+(k((z))) \cong HC_1(k((z)))$ , so that altogether, we obtain the desired residue map

$$\text{res} : K_2^+(k((z))) \rightarrow K_1^+(k).$$

But in fact,  $HH_1(k((z))) \cong \Omega^1(k((z)))$  is spanned by differential forms  $fdg$ ,  $f, g \in k((z))$ , and  $K_2^+(k((z)))$  is spanned by images of these forms with respect to the natural map  $HH_1(k((z))) \rightarrow HC_1(k((z)))$ . In additive  $K$ -theory terms,  $fdg$  is just the product  $\{f, g\}$  of the elements  $f, g \in k((z)) \cong HH_0(k((z))) \cong K_1^+(k((z)))$ . To compute the residue of  $\{f, g\} \in K_2^+(k((z)))$ , it is not necessary to use the map (2.5), and it suffices to use the map (2.4). The result is as follows:

$$\text{res}(\{f, g\}) = \text{tr}(Pq(f)Pq(g)P - Pq(g)Pq(f)P).$$

This whole story can be generalized to an arbitrary  $k$ -algebra  $R$ . Moreover, it has a  $K$ -theoretic counterpart valid for any ring  $R$ . All one has to do is to replace the Lie algebras  $\mathfrak{gl}_\infty$ ,  $\mathfrak{gl}_\infty^+$ ,  $\mathfrak{gl}_\infty$  with their obvious group version  $GL_\infty$ ,  $GL_\infty^+$ ,  $GL_\infty$ , and use the determinant instead of the trace.

### 3 Multiplication.

We can now attain our goal: describe the product on Witt vectors defined as in Definition 1.1. The construction is obviously inspired by the material in Section 2 but formally, nothing depends on this material.

**3.1  $K$ -theoretic definition.** As in Section 1, fix a commutative ring  $A$ . For any  $n \geq 0$ , let  $A_n = A[t]/t^{n+1}$ . In particular, we have  $A_0 = A$ . Note that we have

$$K_1(A[[t]]) = A[[t]]^* \cong \lim_{\substack{\leftarrow \\ n}} K_1(A_n) = \lim_{\substack{\leftarrow \\ n}} A_n^*,$$

so that every element  $a \in \mathbb{W}(A) \subset K_1(A[[t]])$  can be represented by a projective system of elements  $a_{(n)} \in K_1(A_n)$ .

For any ring  $R$ , let  $R\text{-mod}^\infty$  be the exact category of countably generated projective  $R$ -modules, and let  $R\text{-mod} \subset R\text{-mod}^\infty$  be the subcategory of those modules which are finitely generated. Let  $\overline{R\text{-mod}} = R\text{-mod}^\infty / R\text{-mod}$  be the quotient category. Since  $R\text{-mod}^\infty$  has countable sums, we have  $K_\bullet(R\text{-mod}^\infty) = 0$ , so that the excision long exact sequence in algebraic  $K$ -theory provides a canonical isomorphism

$$\delta(R) : K_{\bullet+1}(\overline{R\text{-mod}}) \cong K_\bullet(R\text{-mod}) = K_\bullet(R).$$

Moreover, consider the category  $R((z))\text{-mod}$  of finitely generated projective modules over the Laurent power series ring  $R((z))$ . For any such module  $M$ , there exists an  $R[[z]]$ -lattice  $M_- \subset M$  – that is, an  $R[[z]]$ -submodule such that  $M \cong M_- \otimes_{R[[z]]} R((z))$ . Sending  $M$  to the quotient  $M/M_-$  gives a well-defined functor

$$(3.1) \quad P_R : R((z))\text{-mod} \rightarrow \overline{R\text{-mod}}.$$

Let

$$\pi(R) : K_\bullet(R((z))) \cong K_\bullet(R((z))) \rightarrow K_\bullet(\overline{R\text{-mod}})$$

be the map induced by the functor  $P(R)$ .

All of the above applies in particular to the rings  $A_n$ ,  $n \geq 1$ . To simplify notation, let  $\delta_n = \delta(A_n)$ ,  $\pi_n = \pi(A_n)$ . Moreover, let  $a_n : A[[t_1, t_2]] \rightarrow A_n((z))$  be the map sending  $t_2$  to  $z$  and  $t_1$  to  $tz^{-1}$  – since  $t^{n+1} = 0$  in  $A_n$ , the map is well-defined – and let  $\alpha_n$  be the induced map on algebraic  $K$ -groups.

**Definition 3.1.** The product  $fg$  of two elements  $f, g \in \mathbb{W}(A) \subset K_1(A[[t]])$  is given by

$$fg = \lim_{\substack{\rightarrow \\ n}} \delta_n \pi_n \alpha_n (f \otimes g) \in K_1(A[[t]]),$$

where  $f \otimes g \in K_2(A[[t_1, t_2]])$  is the exterior product.

**Theorem 3.2.** *The product  $fg$  of any two elements  $f, g \in \mathbb{W}(A)$  lies in  $\mathbb{W}(A) \subset K_1(A[[t]])$ , and for any  $n \geq 1$ , we have*

$$(3.2) \quad w_n(fg) = w_n(f)w_n(g),$$

where  $w_n$  is the ghost map (1.3).

**3.2 Group extensions.** In order to prove Theorem 3.2, we need to spell out the definition of algebraic  $K$ -groups in question. Since we only need  $K_1$  and  $K_2$ , it suffices to use [M].

For any ring  $R$ , let  $GL_\infty(R) = \lim_{\rightarrow} GL_n(R)$ , and let  $E(R) \subset GL_\infty(R)$  be its commutant, so that we have a group extension

$$1 \longrightarrow E(R) \longrightarrow GL_\infty(R) \longrightarrow K_1(R) \longrightarrow 1.$$

Recall that we have a canonical group extension

$$(3.3) \quad 1 \longrightarrow K_2(R) \longrightarrow \text{St}(R) \longrightarrow E(R) \longrightarrow 1,$$

where  $\text{St}(R)$  is the Steinberg group. Consider the ring  $R((z))$  of Laurent power series as a right  $R$ -module, and let

$$V_n = (R((z))/R[[z]])^{\oplus n}$$

and

$$GL_\infty^+(R, n) = GL_R(V_n)$$

for any  $n \geq 1$ . Inside  $GL_\infty^+(R)$ , we have the normal subgroup of automorphisms  $f : V_n \rightarrow V_n$  such that  $\text{Im}(\text{id} - f) \subset V_n$  is finitely generated over  $R$  (equivalently, annihilated by  $z^l$  for  $l \gg 0$ ). This subgroup is naturally identified with  $GL_\infty(R)$  (irrespective of  $n$ ). Let  $\overline{GL}_\infty(R, n) = GL_\infty^+(R, n)/GL_\infty(R)$  be the quotient group, so that we have a diagram of group extensions

$$(3.4) \quad \begin{array}{ccccccc} 1 & \longrightarrow & GL_\infty(R) & \longrightarrow & GL_\infty^+(R, n) & \longrightarrow & \overline{GL}_\infty(R, n) \longrightarrow 1 \\ & & \det \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & K_1(R) & \longrightarrow & \widetilde{GL}_\infty(R, n) & \longrightarrow & \overline{GL}_\infty(R, n) \longrightarrow 1, \end{array}$$

where we denote  $\widetilde{GL}_\infty(R, n) = GL_\infty^+(R, n)/E(R)$  (if  $R$  is commutative, then the map  $\det$  on the left-hand side is the determinant map  $GL_\infty(R) \rightarrow R^* \cong K_1(R)$ ). For any  $n \geq 1$ , we have  $P_R(R((z))^{\oplus n}) = V_n$ , where  $P_R$  is the functor (3.1), and  $V_n$  is considered as an object of the category  $\overline{R}\text{-mod}$ . Then  $\overline{GL}_\infty(R, n) = \text{Aut}(V_n)$ , and the functor  $P_R$  induces a map

$$p^{(n)}(R) : GL_n(R((z))) \rightarrow \overline{GL}_\infty(R, n).$$

Explicitly, this map can be described as follows. Let  $P : R((z)) \rightarrow R((z))$  be the projector onto  $z^{-1}R[z^{-1}] \subset R((z))$ , and let  $P^{(n)} = P^{\oplus n}$ . Then we have

$$(3.5) \quad p^{(n)}(R)(f) = P^{(n)}fP^{(n)}$$

(and it is easy to see directly that the right-hand side is a well-defined group map modulo  $GL_\infty(R)$ ).

The extensions (3.4) and the maps  $p^{(n)}(R)$  are compatible for various  $n$ , so that we can pass to the limit and obtain a group extension

$$1 \longrightarrow K_1(R) \longrightarrow \widetilde{GL}_\infty(R, \infty) \longrightarrow \overline{GL}_\infty(R, \infty) \longrightarrow 1$$

and a map

$$p(R) : GL_\infty(R((z))) \rightarrow \overline{GL}_\infty(R, \infty).$$

But the extension (3.3) is universal. Therefore the restriction of the map  $p(R)$  to  $E(R((z))) \subset GL_\infty(R((z)))$  uniquely extends to a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & K_2(R((z))) & \longrightarrow & \text{St}(R((z))) & \longrightarrow & E(R((z))) \longrightarrow 1 \\ & & \text{res}(R) \downarrow & & \downarrow & & \downarrow p(R) \\ 1 & \longrightarrow & K_1(R) & \longrightarrow & \widetilde{GL}_\infty(R, \infty) & \longrightarrow & \overline{GL}_\infty(R, \infty) \longrightarrow 1. \end{array}$$

The map  $\text{res}(R)$  is equal to the composition  $\delta(R) \circ \pi(R)$  of the maps  $\pi(R)$ ,  $\delta(R)$  of Subsection 3.1.

Now, as in Subsection 3.1, let  $R = A_n$ ,  $n \geq 1$ , and denote  $p_n = p(A_n)$ ,  $p_n^{(l)} = p^{(l)}(A_n)$ ,  $l \geq 1$ ,  $\text{res}_n = \text{res}(A_n) = \delta \circ \pi_n$ . Consider the map  $a_n : A[[t_1, t_2]] \rightarrow A_n((z))$ , and assume given two elements  $f, g \in \mathbb{W}(A) \subset K_1(A[[t]])$ . To compute their product  $fg$ , we first need to evaluate the exterior product  $\{f, g\} \in K_2(A[[t_1, t_2]])$ . Let  $\iota_1, \iota_2 : A[[t]] \rightarrow A[[t_1, t_2]]$  be embeddings sending  $t$  to  $t_1$  resp.  $t_2$ ; then by [M, §8], to compute  $\{f, g\}$ , one has to represent  $f_1 = \iota_1(f)$  and  $g_2 = \iota_2(g)$  by commuting diagonal matrices in  $E(A[[t_1, t_2]])$ , for example

$$(3.6) \quad \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_1^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} g_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g_2^{-1} \end{pmatrix},$$

lift these matrices to elements in the Steinberg group, and take their commutator. Since we are only interested in the image  $\text{res}_n(a_n(\{f, g\}))$  we may just as well apply the maps  $a_n$  and  $p_n$ , and then compute the commutator in the

group  $\widetilde{GL}_\infty(A_n, 3)$ . Moreover, since the map  $p_n$  is actually defined on the whole group  $GL_\infty(A_n((z)))$ , there is no need to use matrices (3.6) or any other diagonal matrices in  $E(A_n((z)))$  – we may replace them with the matrices of size 1. We conclude that  $\text{res}_n(a_n(\{f, g\})) \in K_1(A_n)$  is obtained as the commutator of arbitrary liftings  $\tilde{f}, \tilde{g} \in \widetilde{GL}_\infty(A_n, 1)$  of elements

$$p_n^{(1)}(a_n(f_1)), p_n^{(1)}(a_n(g_2)) \in \overline{GL}_\infty(A_n, 1).$$

While we do not have any natural liftings to the group  $\widetilde{GL}_\infty(A_n, 1)$ , convenient liftings to the group  $GL_\infty^+(A_n)$  are provided by (3.5), where we interpret the right-hand side as an automorphism of  $V_1 = zA_n[z^{-1}] \subset A_n((z))$ . Indeed, let  $b_1, b_2 : A[[t] \rightarrow A((z))[[t]]$  be the maps that send  $t$  to  $tz^{-1}$  resp.  $z$ , and let

$$(3.7) \quad \bar{f} = Pb_1(f)P, \quad \bar{g} = Pb_2(g)P.$$

Then modulo  $t^n$  for any fixed  $n \geq 1$ , both  $\bar{f}$  and  $\bar{g}$  are locally unipotent, thus invertible. Using (3.4) and passing to the limit with respect to  $n$ , we obtain the following conclusion.

**Lemma 3.3.** *For any  $f, g \in \mathbb{W}(A) \subset K_1(A[[t]])$ , their product  $fg$  in the sense of Definition 3.1 is given by*

$$fg = \det(\bar{f}\bar{g}(\bar{f})^{-1}(\bar{g})^{-1}),$$

where  $\bar{f}$  and  $\bar{g}$  are as in (3.7).

**Corollary 3.4.** *In the assumptions of Theorem 3.2, the product  $fg$  lies in  $\mathbb{W}(A) \subset K_1(A[[t]])$ .*

*Proof.* By definition, we have to show that the image of  $fg$  in  $K_1(A) = K_1(A_0)$  with respect to the natural projection  $K_1(A[[t]]) \rightarrow K_1(A_0)$  is equal to 1. But by definition  $a_0(t_1) = z^{-1}t = 0$ , so that we have  $a_0(f) = 1 \in A_0((z))$ , and  $Pb_1(f)P$  becomes the identity map after projection to the group  $GL_\infty^+(A)$ .  $\square$

The same game with the group extensions can be played in a different way. Let  $\mathbb{W}^{[2]}(A) \subset A[[t_1, t_2]]^*$  be the kernel of the natural map  $A[[t_1, t_2]]^* \rightarrow A^*$ , and let  $GL_\infty^+(A_n, 1)_o \subset GL_\infty^+(A_n, 1)$  be the subgroup of automorphisms  $f$  of  $V_n = A_n((z))/A_n[[z]]$  such that

- for any  $l \geq 1$ , the map  $\text{id} - f$  sends the submodule  $z^{-l}A_n[[z]]/A_n[[z]] \subset V_n$  into the submodule  $z^{1-l}A_n[[z]]/A_n[[z]] \subset V_n$ .

Moreover, let

$$\begin{aligned} GL_\infty(A_n)_o &= GL_\infty(A_n) \cap GL_\infty^+(A_n, 1)_o \subset GL_\infty(A_n), \\ \overline{GL}_\infty(A_n, 1)_o &= GL_\infty^+(A_n, 1)_o / GL_\infty(A_n)_o \subset \overline{GL}_\infty(A_n). \end{aligned}$$

Then the determinant map  $\det$  obviously send  $GL_\infty(A_N)_o \subset GL_\infty(A_n)$  into the kernel  $\mathbb{W}(A)_n \subset K_1(A_n)$  of the map  $K_1(A_n) \rightarrow K_1(A)$ , so that (3.4) induces a commutative diagram of group extensions of the form

$$(3.8) \quad \begin{array}{ccccccc} 1 & \longrightarrow & GL_\infty(A_n)_o & \longrightarrow & GL_\infty^+(A_n, 1)_o & \longrightarrow & \overline{GL}_\infty(A_n, 1)_o \longrightarrow 1 \\ & & \det \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \mathbb{W}(A)_n & \longrightarrow & \widetilde{GL}_\infty(A_n, 1)_o & \longrightarrow & \overline{GL}_\infty(A_n, 1)_o \longrightarrow 1, \end{array}$$

for a certain group  $\widetilde{GL}_\infty(A_n, 1)_o$ .

Denote  $q_n = p_n^{(1)} \circ a_n : A[[t_1, t_2]]^* \rightarrow \overline{GL}_\infty(A_n, 1)$ . Then  $q_n$  sends  $\mathbb{W}^{[2]}(A)$  into  $\overline{GL}_\infty(A_n, 1)_o$ , so that the second extension of (3.4) induces an extension of  $\mathbb{W}^{[2]}(A)$  by  $\mathbb{W}(A)_n$ . Passing to the limit with respect to  $n$ , we obtain an extension  $\widetilde{\mathbb{W}}^{[2]}(A)$  of  $\mathbb{W}^{[2]}(A)$  by  $\mathbb{W}(A)$  such that for any  $n$ , we have a commutative diagram

$$(3.9) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{W}(A) & \longrightarrow & \widetilde{\mathbb{W}}^{[2]}(A) & \longrightarrow & \mathbb{W}^{[2]}(A) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow q_n \\ 1 & \longrightarrow & \mathbb{W}(A)_n & \longrightarrow & \widetilde{GL}_\infty(A_n, 1)_o & \longrightarrow & \overline{GL}_\infty(A_n, 1)_o \longrightarrow 1. \end{array}$$

It is this extension  $\widetilde{\mathbb{W}}^{[2]}(A)$  which encodes the product  $fg$  of two elements  $f, g \in \mathbb{W}(A)$ . Namely, choose arbitrary liftings  $\tilde{f}, \tilde{g} \in \widetilde{\mathbb{W}}^{[2]}(A)$  of the elements  $\iota_1(f), \iota_2(g) \in \mathbb{W}^{[2]}(A)$ , and take their commutator in  $\mathbb{W}(A) \subset \widetilde{\mathbb{W}}^{[2]}(A)$ . Then by the commutativity of the diagrams (3.9) and (3.8), one immediately sees that we have

$$(3.10) \quad fg = \tilde{f}\tilde{g}\tilde{f}^{-1}\tilde{g}^{-1}.$$

**3.3 Proof of Theorem 3.2.** We can now prove Theorem 3.2. We start with the following reduction.

Fix  $f, g \in \mathbb{W}(A)$  and an integer  $n \geq 1$ . Let  $f' = \text{id} - b_1(f)$ ,  $g' = \text{id} - b_2(g)$ , considered as  $A_n$ -linear endomorphisms of  $A_n((z))$ . Note that  $f'$  always preserves  $z^{-1}A_n[[z^{-1}]] \subset A_n((z))$ , thus commutes with the projector  $P$ . Moreover, we have  $(z^{-1}t)^n = 0$  in  $A_n((z))$ . Therefore for any  $l$ ,  $f'(z^l A_n[[z]]) \subset A_n((z))$  lies inside  $z^{l-n}A_n[[z]] \subset A_n((z))$ , and if  $f = 1 \pmod{t^n}$ , we have  $f' = 0$ , hence  $w_n(fg) = 0$ . Moreover, if  $g = 1 \pmod{t^n}$ , then  $g'$  sends  $z^l A_n[[z]] \subset A_n((z))$  into  $z^{l+n+1}A_n[[z]] \subset A_n((z))$ , so that  $Pg'P$  sends  $z^l A[[z]]/A[[z]] \subset A((z))/A[[z]]$  into  $z^{l+n+1}A[[z]]/A[[z]] \subset A((z))/A[[z]]$ . Therefore the commutator bracket

$$[\bar{f}, \bar{g}] = [f', Pg'P] = f'Pg'P - Pg'Pf$$

is locally nilpotent, the commutator  $\bar{f}\bar{g}(\bar{f})^{-1}(\bar{g})^{-1}$  is locally unipotent, and its determinant is equal to 1, so that we again have  $w_n(fg) = 0$ . Since the product

$fg$  is bilinear by construction, in studying  $w_n(fg)$  we may therefore assume that  $f$  and  $g$  are of the form

$$f = \prod_{0 \leq i < n} (1 + f_i t^{i+1}), \quad g = \prod_{0 \leq i < n} (1 + g_i t^{i+1}).$$

But then  $g'$  sends  $z^{-n-1}A_n[z^{-1}] \subset A_n((z))$  into  $z^{-1}A_n[z^{-1}]$ , so that on the submodule  $z^{-n-1}A_n[z^{-1}]$ ,  $g' = Pg'P$  and  $\bar{f}$  commutes with  $\bar{g}$ . Thus the determinant in Lemma 3.3 is effectively computed on the  $A_n$ -module  $z^{-n}A_n[[z]]/A_n[[z]]$  of finite rank independent on  $f$  and  $g$ .

We conclude that  $w_n(fg)$  – that is, the left-hand side of (3.2) – is given by a polynomial in  $f_1, \dots, f_n, g_1, \dots, g_n$  of bounded degree. The same is obviously true for the right-hand side. Moreover, since all the constructions are functorial, these polynomials are universal and defined over  $\mathbb{Z}$ . Thus to prove the identity, it suffices to consider the case  $A = \mathbb{Z}$ . Moreover, since the natural map  $\mathbb{W}(\mathbb{Z}) \rightarrow \mathbb{W}(\mathbb{Q})$  is injective, it suffices to consider the case  $A = \mathbb{Q}$ . Thus from now on, we assume  $A = \mathbb{Q}$ .

Now it is convenient to switch the viewpoint – let us use the extensions (3.8) and (3.9) to compute the product  $fg$  by (3.10) rather than Lemma 3.3. Indeed, all the groups that appear in these diagrams are actually pronipotent groups. Since  $A = \mathbb{Q}$ , every pronipotent group is identified with its Lie algebra by means of the logarithm map.

In particular, we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{W}(A) & \longrightarrow & \widetilde{\mathbb{W}^{[2]}(A)} & \longrightarrow & \mathbb{W}^{[2]}(A) \longrightarrow 1 \\ & & \downarrow \log & & \downarrow \log & & \downarrow \log \\ 0 & \longrightarrow & \mathfrak{w}(A) & \longrightarrow & \widetilde{\mathfrak{w}^{[2]}(A)} & \longrightarrow & \mathfrak{w}^{[2]}(A) \longrightarrow 0, \end{array}$$

where the bottom line is a Lie algebra extension; the Lie algebra  $\mathfrak{w}^{[2]}(A)$  of the group  $\mathbb{W}^{[2]}(A)$  is the kernel of the map  $A[[t_1, t_2]] \rightarrow A$  with the trivial Lie bracket, and the Lie algebra  $\mathfrak{w}(A)$  of the group  $\mathbb{W}(A)$  is  $tA[[t]] \subset A[[t]]$ , again with the trivial Lie bracket. Thus the Lie algebra  $\widetilde{\mathfrak{w}^{[2]}(A)}$  of the group  $\widetilde{\mathbb{W}^{[2]}(A)}$  is a Heisenberg Lie algebra defined by a canonical skew-symmetric cocycle  $c(-, -) : \widetilde{\mathfrak{w}^{[2]}(A)} \times \mathfrak{w}^{[2]}(A) \rightarrow \mathfrak{w}(A)$ . In particular,  $[\xi_1, [\xi_2, \xi_3]] = 0$  for any  $\xi_1, \xi_2, \xi_3 \in \widetilde{\mathfrak{w}^{[2]}(A)}$ , so that by the Campbell-Hausdorff formula, (3.10) implies

$$(3.11) \quad \log(fg) = \log(\widetilde{f}\widetilde{g}\widetilde{f}^{-1}\widetilde{g}^{-1}) = c(\log f, \log g)$$

for any  $f, g \in \mathbb{W}(A)$ .

To compute the cocycle  $c(-, -)$ , we again turn to matrix algebras. The

diagrams (3.8), (3.9) induce commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{w}(A) & \longrightarrow & \widetilde{\mathfrak{w}^{[2]}}(A) & \longrightarrow & \mathfrak{w}^{[2]}(A) \longrightarrow 0 \\ & & \downarrow & & \downarrow q_n & & \parallel \\ 0 & \longrightarrow & \mathfrak{w}(A)_n & \longrightarrow & \widetilde{\mathfrak{gl}_\infty(A_n, 1)_o} & \longrightarrow & \overline{\mathfrak{gl}_\infty(A_n, 1)_o} \longrightarrow 0. \end{array}$$

and

$$(3.12) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{gl}_\infty(A_n)_o & \longrightarrow & \mathfrak{gl}_\infty^+(A_n, 1)_o & \longrightarrow & \overline{\mathfrak{gl}_\infty(A_n, 1)_o} \longrightarrow 0 \\ & & \text{tr} \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{w}(A)_n & \longrightarrow & \widetilde{\mathfrak{gl}_\infty(A_n, 1)_o} & \longrightarrow & \overline{\mathfrak{gl}_\infty(A_n, 1)_o} \longrightarrow 0, \end{array}$$

of Lie algebra extensions; to compute  $a_n(c(-, -))$ , we may use the extension  $\widetilde{\mathfrak{gl}_\infty(A_n, 1)_o}$ , and by (3.12), this extension is compatible with the extension  $\mathfrak{gl}_\infty^+(A_n, 1)_o$ , so that

$$a_n(c(\xi_1, \xi_2)) = \text{tr}[Pa_n(\xi_1)P, Pa_n(\xi_2)P]$$

for any  $\xi_1, \xi_2$ . This is exactly the Japanese cocycle [V], and it is equal to

$$a_n(c(\xi_1, \xi_2)) = \text{res}_z(a_n(\xi_1)d_z a_n(\xi_2)),$$

where  $d_z$  is the de Rham differential with respect to  $z$ . Substituting (3.11) and (1.4), we obtain

$$\begin{aligned} \sum_n \frac{1}{n} w_n(fg)t^n &= \text{res}_z \left( \sum_i \frac{1}{i} w_i(f)t^i z^{-i} \right) \left( \sum_j \frac{1}{j} w_j(g) \cdots j z^{j-1} dz \right) \\ &= \sum_n \frac{1}{n} w_n(f)w_n(g)t^n, \end{aligned}$$

and the  $t^n$ -term of this equality is exactly (3.2).  $\square$

## References

- [B] S. Bloch, *Algebraic K-theory and crystalline cohomology*, Publ. Math IHES, **47** (1977), 187-268.
- [D] M. Demazure, *Lectures on p-divisible groups*, Lecture Notes in Math. **302**, Springer-Verlag, Berlin-New York, 1972.
- [FT1] B. Feigin and B. Tsygan, *Cohomology of Lie algebras of generalized Jacobi matrices*, (in Russian), Funktsional. Anal. i Prilozhen. **17** (1983), 86-87.
- [FT2] B. Feigin and B. Tsygan, *Additive K-theory*, in *K-theory, arithmetic and geometry (Moscow, 1984-1986)*, Lecture Notes in Math. **1289**, Springer, Berlin, 1987; 67-209.

- [M] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies **72**, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971.
- [S] [http://en.wikipedia.org/wiki/Stigler's\\_law\\_of\\_eponymy](http://en.wikipedia.org/wiki/Stigler's_law_of_eponymy).
- [T] J. Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. (4) **1** (1968), 149-159.
- [V] J.-L. Verdier, *Les représentations des algèbres de Lie affines: applications à quelques problèmes de physique (d'après E. Date, M. Jimbo, M. Kashiwara, T. Miwa)*, Bourbaki Seminar, Vol. 1981/1982, Astérisque, **92-93**, Soc. Math. France, Paris, 1982; 365-377.
- [W] E. Witt, *Zyklische Körper und Algebren der Charakteristik  $p$  vom Grad  $p^n$ . Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik  $p^n$* , Journal für Reine und Angewandte Mathematik **176** (1936), 126-140.

STEKLOV MATH INSTITUTE  
MOSCOW, USSR

AND

KOREAN INSTITUTE FOR ADVANCED STUDIES  
SEOUL, REP. OF KOREA

*E-mail address:* `kaledin@mccme.ru`