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Hochschild and Cyclic Homology via Functor Homology

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Abstract. A description of Hochschild and cyclic homology of commutative algebras via homological algebra in functor categories was achieved in [4]. In this paper we extend this approach to associative algebras and provide an interpretation of Hochschild and cyclic homology as derived functors of tensor products in appropriate categories of functors.

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1. Preliminaries and the Main Result

1.1. INTRODUCTION

The aim of the present paper is to show that Hochschild homology and cyclic homology of any associative algebra in any characteristic can be described via homological algebra of functor categories over the category of non-commutative sets constructed by Fiedorowicz and Loday [2]. Our results should be considered as a different version of the theorem of Connes [1], where cyclic (co)homology is also described via functor (co)homology, but in the category of cyclic modules. Our results and ideas are non-commutative versions of the recent development of commutative algebra homology via functor homology given in [4–8].

1.2. CATEGORY OF NON-COMMUTATIVE SETS

We introduce the category $\mathcal{F}(\mathsf{as})$, which we call the category of non-commutative sets. Lemma 1.1 shows that this category is isomorphic to the category ΔS introduced by Fiedorowicz and Loday in [2] (see also [3]). Objects of the category $\mathcal{F}(\mathsf{as})$ are finite sets

$$[n] = \{0, \dots, n\}, \quad n \geqslant 0.$$

A morphism $[n] \to [m]$ in $\mathcal{F}(\mathsf{as})$ is a map $f : [n] \to [m]$ together with a total ordering of the preimages $f^{-1}(j)$ for all $j \in [m]$. If $f : [n] \to [m]$ and $g : [m] \to [k]$ are morphisms in $\mathcal{F}(\mathsf{as})$, then the composite of g and f as a map is gf and the total ordering in $(gf)^{-1}(i)$, $i \in [n]$ is given via the ordered union of ordered sets:

$$(gf)^{-1}(i) = \coprod_{j \in g^{-1}(i)} f^{-1}(j).$$

Clearly there is a forgetful functor $\mathcal{F}(\mathsf{as}) \to \mathcal{F}$. Here $\mathcal{F}$ denotes the category of finite sets. The objects of the category $\mathcal{F}$ are still the sets $[n]$, $n \geqslant 0$ but morphisms in $\mathcal{F}$ are just set maps.

We let $\Gamma(\mathsf{as})$ (resp. Γ) be the subcategory of $\mathcal{F}(\mathsf{as})$ (resp. $\mathcal{F}$) whose morphisms $f : [n] \to [m]$ preserve the zero element, that is $f(0) = 0$. Again, there is a forgetful functor $\Gamma(\mathsf{as}) \to \Gamma$.

If $g : [n] \to [m]$ is an order preserving map (in the usual ordering of $[n]$ and $[m]$) then the restriction of the total ordering of $[n]$ to $g^{-1}(i)$, $i \in [m]$ allows us to consider g as a morphism in $\mathcal{F}(as)$. In this way one obtains a functor $\Delta \to \mathcal{F}(\mathsf{as})$, which is the identity on objects and an inclusion on morphisms. Here Δ is the standard category of simplicial topology: objects of Δ are the sets $[n]$, $n \geqslant 0$, and morphisms are non-decreasing maps. Thus one can identify Δ with a subcategory of $\mathcal{F}(\mathsf{as})$.

A morphism of the category $\mathcal{F}(\mathsf{as})$ is called *injective* (resp. *bijective, surjective*) if it so as a set map. Clearly the forgetful functor $\mathcal{F}(\mathsf{as}) \to \mathcal{F}$ is bijective on injective morphisms. In particular any bijection $[n] \to [n]$ is a morphism in $\mathcal{F}(\mathsf{as})$.

LEMMA 1.1. *Any morphism $f : [n] \to [m]$ in $\mathcal{F}(\mathsf{as})$ has a unique decomposition $g \circ h$, where h is a bijection and g is a morphism in Δ.*

Proof. It is enough to observe that there exists a unique order preserving map $g : [n] \to [m]$ such that

$$\mathsf{Card}(g^{-1}(i)) = \mathsf{Card}(f^{-1}(i)), \quad i \in [m]$$

and for this g there exists a unique bijection h with $f = g \circ h$. Conversely, if $f = g \circ h$ with bijective h, then for each $i \in [n]$ the number of elements in $g^{-1}(i)$ and $f^{-1}(i)$ are the same and the lemma is proved. □

If $f : [n] \to [m]$ is a morphism in $\mathcal{F}(\mathsf{as})$ and $f = g \circ h$ as in Lemma 1.1 then we write $g = \mu(f)$ and $h = \omega(f)$.

COROLLARY 1.2. *The symmetric group Σ_{n+1} acts freely on $\mathsf{Hom}_{\mathcal{F}(\mathsf{as})}([n], [m])$ and the set of orbits can be identified with $\mathsf{Hom}_{\Delta}([n], [m])$ via the map $f \mapsto \mu(f)$.*

Another consequence of Lemma 1.1 is that the category $\mathcal{F}(\mathsf{as})$ is isomorphic to the category ΔS considered in [3] and [2].

1.3. ASSOCIATIVE ALGEBRAS AS FUNCTORS ON NON-COMMUTATIVE SETS

Let A be an associative and unital algebra over a commutative ring K with unit and let M be an A-bimodule. We let

$$\mathcal{L}(A, M)\colon \Gamma(\mathsf{as}) \to \mathsf{mod} \quad \text{and} \quad \mathcal{L}(A, A)\colon \mathcal{F}(\mathsf{as}) \to \mathsf{mod}$$

be the functors given on objects by $[n] \mapsto M \otimes A^{\otimes n}$ and $[n] \mapsto A^{\otimes(n+1)}$, respectively. Here mod denotes the category of K-modules. In order to describe the action of morphisms on $\mathcal{L}(A, M)$ and $\mathcal{L}(A, A)$, we need some additional notation. Let I be an arbitrary subset $[n]$ but the elements of I may be ordered differently and let $a_i \in A$ for $i = 1, \dots, k$ and $a_0 \in M$. Then we denote by $\prod_{i\in I}^{<} a_i$ the product of the elements a_i according to the ordering in I. For a morphism $f : [n] \to [m]$ in $\Gamma(\mathsf{as})$, the action of f on $\mathcal{L}(A, M)$ is given by

$$f_*(a_0 \otimes \cdots \otimes a_n) := b_0 \otimes \cdots \otimes b_m,$$

where $b_j = \prod_{f(i)=j}^{<} a_i$, $j = 0, \dots, n$. Moreover, for $M = A$ the same formula shows that $\mathcal{L}(A, A)$ factors through $\mathcal{F}(\mathsf{as})$. One observes that if A is commutative and M is a symmetric A-bimodule the functor $\mathcal{L}(A, M)$ factors through the category Γ and the functor $\mathcal{L}(A, A)$ factors through the category $\mathcal{F}$.

1.4. THE NON-COMMUTATIVE CIRCLE

Before we define the non-commutative circle, let us recall the construction of the smallest simplicial model of the circle. Consider the finite pointed simplicial set $C\colon \Delta^{op} \to \Gamma$ which assigns $[n]$ to $[n]$. The face and degeneracy maps in the simplicial pointed set C are given as follows. The map $s_i : [n] \to [n+1]$ is the unique monotone injection, whose image does not contain $i + 1$, while $d_i : [n] \to [n-1]$ is given by

$$d_i(j) = \begin{cases} j & \text{if } j < i, \\ i & \text{if } j = i < n, \ \text{resp.} \quad 0 \quad \text{if } j = i = n \\ j-1 & \text{if } j > i. \end{cases}$$

We claim that C considered as a pointed simplicial set is a simplicial model of the circle. Indeed, it is clear that $0 \in [0]$ and $1 \in [1]$ are non-degenerate simplices. The first one is zero-dimensional and the second one is one-dimensional. On the other hand, if $j \in [n]$ and $n \geqslant 2$, then $j = s_0(j-1)$ provided $j > 1$. Similarly $0 = s_0(0)$ and $1 = s_1(1)$, which shows that all other simplices are degenerate and hence, the geometric realization of C is a 1-sphere.

The functor C fits in the commutative diagram (see p. 221 of [3], with slightly different notation):

$$\begin{array}{ccc} \Delta^{op} & \longrightarrow & \Delta C^{op} \\ \downarrow{\scriptstyle C} & & \downarrow \\ \Gamma & \longrightarrow & \mathcal{F}, \end{array}$$

where $\Gamma \to \mathcal{F}$ and $\Delta^{op} \to \Delta C^{op}$ are inclusions. Here ΔC is the category constructed by A. Connes, with the property that contravariant functors from ΔC to mod are cyclic objects in mod (see § 6.1 of [3]).

The following important observation was made by Loday (see Exercise 6.4.1 on p. 222 of [3]). The functor $C : \Delta^{op} \to \Gamma$ has a canonical lifting $\hat{C} : \Delta^{op} \to \Gamma(\mathsf{as})$. So the non-commutative circle is the simplicial object in $\Gamma(\mathsf{as})$, which is $[n]$ in dimension n. Since any injective map has a unique lift in $\Gamma(\mathsf{as})$, we have only to define the face maps. If $i < n$ then $d_i^{-1}(j)$ is a singleton for all j except $j = i$. We define the total ordering on $d_i^{-1}(i) = \{i, i+1\}$ by declaring that $i < i+1$. Since $d_n^{-1}(j)$ is a singleton for all j except $j = 0$ we need only to define the total ordering on $d_n^{-1}(0) = \{0, n\}$, which is now given by $n < 0$. It is tedious, but straightforward to check that in this way we get, in fact, a simplicial object in $\Gamma(\mathsf{as})$. Moreover, this simplicial object is compatible with the unique lift of $t_n : [n] \to [n]$, which is given by $t_n(i) = i + 1$ for $i < n$ and $t_n(n) = 0$. Hence, $\hat{C}$ is indeed a cyclic object in $\mathcal{F}(\mathsf{as})$.

1.5. DEFINITION OF HOCHSCHILD AND CYCLIC HOMOLOGY OF FUNCTORS

Now we define the Hochschild homology $\mathsf{H}_*(F)$ of a functor $F : \Gamma(\mathsf{as}) \to \mathsf{mod}$ as the homotopy of the simplicial module $F \circ \hat{C}$. Similarly, the cyclic homology $\mathsf{HC}_*(T)$ of a functor $T : \mathcal{F}(\mathsf{as}) \to \mathsf{mod}$ is defined as the cyclic homology of the cyclic module $T \circ \hat{C}$. Of course, for such T we can also define the Hochschild homology of T as the Hochschild homology of the composite functor $\Gamma(\mathsf{as}) \subset \mathcal{F}(\mathsf{as}) \to \mathsf{mod}$.

These definitions generalize the classical definition of Hochschild and cyclic homology of associative algebras as follows. Let A be a unital associative algebra and let M be an A-bimodule. One observes that the simplicial module $\mathcal{L}(A, M) \circ \hat{C}$ is exactly the standard Hochschild complex of A with coefficients in M. Hence,

$$\mathsf{H}_*(\mathcal{L}(A, M)) \cong \mathsf{H}_*(A, M) \quad \text{and} \quad \mathsf{HC}_*(\mathcal{L}(A, A)) \cong \mathsf{HC}_*(A).$$

1.6. FUNCTOR HOMOLOGY

For any small category $\mathcal{C}$ we denote by $\mathcal{C}$-*mod* the category of all covariant functors from $\mathcal{C}$ to mod. Similarly, *mod*-$\mathcal{C}$ denotes the category of contravariant functors from $\mathcal{C}$ to the category of K-modules. The categories $\mathcal{C}$-*mod* and *mod*-$\mathcal{C}$ are Abelian categories with sufficiently many projective and injective objects. Projective generators of the category $\mathcal{C}$-*mod* (resp. *mod*-$\mathcal{C}$) are the functors $\mathcal{C}^c$ (resp. $\mathcal{C}_c$), $c \in \mathcal{C}$, where $c \in Ob(\mathcal{C})$ and

$$\mathcal{C}^c := K[\mathsf{Hom}_{\mathcal{C}}(c, -)] \quad \text{and} \quad \mathcal{C}_c := K[\mathsf{Hom}_{\mathcal{C}}(-, c)],\ c \in \mathcal{C}.$$

Here $K[S]$ denotes the free K-module generated by a set S.

If $F \in \mathcal{C}$-*mod* and $T \in$ *mod*-$\mathcal{C}$ one defines a module $T \otimes_{\mathcal{C}} F$ as a quotient of $\bigoplus_{c \in \mathcal{C}} T(c) \otimes F(c)$ modulo the relations $\alpha^*(x) \otimes y = x \otimes \alpha_*(y)$.

Here $\alpha: c \to c'$ is a morphism in $\mathcal{C}$, $x \in T(c')$ and $y \in F(c)$. It is well known (see §16.7 of [9]) that the bifunctor

$$- \otimes_{\mathcal{C}} - : (mod - \mathcal{C}) \times (\mathcal{C} - mod) \to \mathsf{mod}$$

is right exact with respect to both variables and preserves sums. It is also important to note that

$$T \otimes_{\mathcal{C}} \mathcal{C}^c \cong T(c) \quad \text{and} \quad \mathcal{C}_c \otimes_{\mathcal{C}} F \cong F(c).$$

Moreover, the derived functors of $- \otimes_{\mathcal{C}} -$ with respect to each variable are isomorphic and we will denote the common value by $\mathsf{Tor}_*^{\mathcal{C}}(-, -)$.

1.7. THE FUNCTOR b

Let us return to the category of non-commutative sets $\mathcal{F}(\mathsf{as})$. For simplicity we will write P_n and P^n instead of $\mathcal{F}(\mathsf{as})_n$ and $\mathcal{F}(\mathsf{as})^n$, i.e.,

$$P_n := K[\mathsf{Hom}_{\mathcal{F}(as)}(-, [n])].$$

Similarly, we write $\bar{P}^n$ and $\bar{P}_n$ instead of $\Gamma(\mathsf{as})^n$ and $\Gamma(\mathsf{as})_n$. The morphism $d_i : [n] \to [n-1]$ of $\mathcal{F}(\mathsf{as})$ yields a natural transformation $P_n \to P_{n-1}$, which is still denoted by d_i. We define the contravariant functor b as the cokernel of the morphism $d = d_0 - d_1 : P_1 \to P_0$. We claim that the evaluation of b on the set $[m]$ can be identified with the free K-module spanned on all total orderings on $\{1, \dots, m\}$. Indeed, the generators of $P_0([m])$ are morphisms of non-commutative sets $[m] \to [0]$ and this is the same as total orderings of $[m]$. Similarly, generators of $P_1([m])$ are morphisms of non-commutative sets $[m] \to [1]$ and can, therefore, be identified with partitions of $[m]$ into two disjoint subsets (ξ_0, ξ_1) together with a total ordering on each of them. The map d maps (ξ_0, ξ_1) to $\xi_0 \coprod \xi_1 - \xi_1 \coprod \xi_0$, where $\coprod$ means the ordered union of ordered sets. Therefore, the cokernel of d consists of equivalence classes of total orderings of the set $[m]$ and each equivalence class contains exactly one total ordering of $[m]$ with minimal element 0, and hence the claim is proved.

We let $\bar{\mathsf{b}}$ be the restriction of b on $\Gamma(\mathsf{as})$. Since the morphisms $d_i : [n] \to [n-1]$ respect 0 they yield natural transformations: $d_i : \bar{P}_n \to \bar{P}_{n-1}$. Thus we can form the cokernel of the map $d = d_0 - d_1 : \bar{P}_1 \to \bar{P}_0$. We claim that this is isomorphic to the functor $\bar{\mathsf{b}}$. Indeed, we have $\bar{P}_0([m]) = P_0([m])$, thus the evaluation of the cokernel on $[m]$ can be identified with the free K-module spanned by the equivalence classes of total orderings of $[m]$. The equivalence relation is similar to the one above. The only difference is that now partitions (ξ_0, ξ_1) of $[m]$ satisfy the property $0 \in \xi_0$. But this has no effect on the quotient.

1.8. THE MAIN THEOREM

The main results of the paper are the identifications of Hochschild and cyclic homology of functors as derived functors of the tensor product with $\bar{\mathsf{b}}$ resp. b.

THEOREM 1.3. *For functors* $F : \Gamma(\mathsf{as}) \to \mathsf{mod}$ *and* $T : \mathcal{F}(\mathsf{as}) \to \mathsf{mod}$ *one has natural isomorphisms*

$$\mathsf{H}_* F \cong \mathsf{Tor}_*^{\Gamma(\mathsf{as})}(\bar{\mathsf{b}}, F) \quad \text{and} \quad \mathsf{Tor}_*^{\mathcal{F}(\mathsf{as})}(\mathsf{b}, T) \cong \mathsf{HC}_*(T).$$

Proof. We will use the well-known axiomatic characterization of Tor functors. Thanks to Section 1.7, one has an exact sequence $\bar{P}_1 \to \bar{P}_0 \to \bar{\mathsf{b}} \to 0$ in *mod*-$\Gamma(\mathsf{as})$. Tensoring with F and using the isomorphism $\bar{P}_i \otimes_{\Gamma(\mathsf{as})} F \cong F([i])$ one obtains that the first isomorphism in question holds in dimension zero. Clearly, the functors $F \mapsto \mathsf{H}_* F$ form an exact connected sequence of functors and it is enough to show that $\mathsf{H}_* F$ vanishes in positive dimensions for any projective F. Thus one only needs to consider functors like $F = \bar{P}^k$. Proposition 2.2 below gives the result. Similarly, Proposition 2.5 below shows that the same argument proves the second isomorphism as well, provided we start with the exact sequence $P_1 \to P_0 \to \mathsf{b} \to 0$ constructed in Section 1.7. □

2. Propositions 2.2 and 2.5

In order to prove the statements needed for the proof of Theorem 1.3 we introduce three families of auxiliary simplicial sets $X(k)$, $Z(k)$ and $Y(k)$ for $k \geqslant 0$.

2.1. THE SIMPLICIAL SET $X(k)$

Let $X(k)$ be the composite of $\hat{C} : \Delta^{op} \to \Gamma(\mathsf{as})$ and $\mathsf{Hom}_{\Gamma(\mathsf{as})}([k]\,, -) : \Gamma(\mathsf{as}) \to \mathsf{Sets}$: $X(k) : \Delta^{op} \to \Gamma(\mathsf{as}) \to \mathsf{Sets}$.

PROPOSITION 2.1. *The simplicial set* $X(k)$ *is isomorphic to the disjoint union of the* $k!$ *copies of the standard* k*-dimensional standard simplex* Δ^k.

Proof. We will extensively use the fact that the symmetric group Σ_k can be identified with $\mathsf{Aut}_{\Gamma(\mathsf{as})}([k])$. Thus it acts on $\mathsf{Hom}_{\Gamma(\mathsf{as})}([k], [m])$ via precomposition. Thanks to Corollary 1.2 this action is free and hence Σ_k acts freely on $X(k)$.

The zero simplices of $X(k)$ are the maps in $\Gamma(\mathsf{as})$ from $[k]$ to $[0]$; hence they correspond to total orderings of $[k]$. The one-dimensional simplices are the maps from $[k]$ to $[1]$, thus they consist of partitions of $[k]$ say (ξ_0, ξ_1) with a given ordering in every part ξ_i of the partition. The boundary maps take such an element to the fusion of the partition but with different orderings (see Section 1.7). Therefore, given a zero simplex $i_0 < \cdots < i_k$ every zero simplex with a cyclic variation of this ordering is in the same component and all other elements are not connected to $i_0 < \cdots < i_k$. Therefore, any connected component contains exactly one total ordering with minimal element 0. Thus we can identify the set of connected components with Σ_k or equivalently with the set of total orderings of the set $\{1, \ldots, k\}$.

Furthermore, the induced action of Σ_k on the set of connected components of $X(k)$ is free and transitive. Thus all connected components are isomorphic to each other. Moreover, the set of vertices of each component has exactly $(k+1)!/k! = (k+1)$ elements. On the other hand, an n-simplex of $X(k)$ is non-degenerate if the corresponding morphism $[k] \to [n]$ is surjective. Therefore, $X(k)$ is k-dimensional and the highest-dimensional non-degenerate simplices correspond to isomorphisms $[k] \to [k]$, i.e., to elements in Σ_k. Any other non-degenerate simplex is a (maybe iterated) face of a such a permutation. Furthermore, the action of Σ_k via precomposition on the set $\Sigma_k \subset X(k)_k$ is free and transitive and we can conclude that in every component there is exactly one highest-dimensional non-degenerate simplex.

We have only to prove that the connected component X corresponding to the standard ordering of $[k]$ is isomorphic to the standard k-simplex. To this end, we let x be the identity map $[k] \to [k]$, which is the unique highest-dimensional non-degenerate simplex of X. Then there is a unique morphism of simplicial sets $\Delta^k \to X$ which takes the unique highest-dimensional non-degenerate simplex of Δ^k to x. Obviously this map is surjective. It is also injective, because the induced map on vertices is bijective. □

PROPOSITION 2.2. *For any $n \geqslant 0$ one has*

$$\mathsf{H}_i(\bar{P}^n) = 0,\ i > 0, \qquad \mathsf{H}_0(\bar{P}^n) \cong \bar{\mathsf{b}}([n]).$$

Proof. Since $\bar{P}_n$ is the free K-module on $\mathsf{Hom}_{\Gamma(\mathsf{as})}([n], -)$, we see that $\mathsf{H}_*(\bar{P}^n)$ is nothing but the homology of the simplicial set $X(n)$ with coefficients in K and the statement follows from Proposition 2.1.

2.2. THE SIMPLICIAL SET $Z(k)$

We introduce a simplicial set $Z(k)$, which can be described as follows. The set of m-simplices of $Z(k)$ is $\mathsf{Hom}_\Delta([k], [m])$. So, in some sense $Z(k) = \mathsf{Hom}_\Delta([k], -)$ is the dual of $\Delta^k = \mathsf{Hom}_\Delta(-, [k])$. The degeneracy maps $s_i : Z(k)_m \to Z(k)_{m+1}$, $i = 0, \dots, k$, are induced from the degeneracy maps $s_i : [m] \to [m+1]$ in the simplicial set C. Since the degeneracy maps are non-decreasing, they indeed induce well-defined maps $Z(k)_m \to Z(k)_{m+1}$. The same is true for the face maps $d_0, \dots, d_{m-1}$. Only the last face map $d_m : [m] \to [m-1]$ is not monotonic. In order to define the map $d_m : Z(k)_m \to Z(k)_{m-1}$ we need a different description of the set $Z(k)_m$. Let $f : [k] \to [m]$ be a non-decreasing map and let a_i be the number of elements in $f^{-1}(i)$, $i = 0, \dots, m$. Then $k+1 = a_0 + \cdots + a_m$ and $a_i \geqslant 0$. It is clear that in this way one obtains a one-to-one correspondence between the elements of $Z(k)_m$ and $(m+1)$-tuples $(a_0, \dots, a_m)$ with the properties $a_i \geqslant 0$ and $a_0 + \cdots + a_m = k+1$. Having this identification in mind, one sees that for the maps s_i and d_j, $i = 0, \dots, m$ and $0 \leqslant j < m$ one has

$$\begin{aligned} s_i(a_0, \dots, a_m) &= (a_0, \dots, a_i, 0, \dots, a_m), && 0 \leqslant i \leqslant m \\ d_j(a_0, \dots, a_m) &= (a_0, \dots, a_j + a_{j+1}, \dots, a_m), && 0 \leqslant j < m. \end{aligned}$$

We now define

$$d_m(a_0, \dots, a_m) := (a_m + a_0, \dots, a_m).$$

In this way we get a well-defined simplicial set $Z(k)$. Clearly $Z(0)$ is isomorphic to the circle C. Actually the simplicial set $Z(k)$ is a cyclic set, where the cyclic structure is compatible with the corresponding cyclic structure on $\hat{C}$ and it is given by

$$t(a_0, \dots, a_m) = (a_m, a_0, \dots, a_{m-1}).$$

We let $|Z(k)|^{cy}$ be the cyclic geometric realization of $Z(k)$ (see p. 235 of [3]), which is nothing but the Borel construction of the geometric realization of $Z(k)$ with respect to the natural S^1-action induced by the cyclic structure $|Z(k)|^{cy} := ES^1 \times_{S^1} |Z(k)|$. Let us also recall (see loc. cit.) that the cyclic homology of the cyclic module $K[Z(k)]$ is isomorphic to the S^1-equivariant homology $\mathsf{H}_*^{S^1}(|Z(k)|, K)$ of $|Z(k)|$. In other words

$$\mathsf{HC}_*(K[Z(k)]) \cong \mathsf{H}_*(|Z(k)|^{cy}, K).$$

LEMMA 2.3. (i) *The space $|Z(k)|$ is weakly homotopy equivalent to the circle.*
(ii) *The cyclic realization $|Z(k)|^{cy}$ has the same homotopy type as*

$$K(\mathbb{Z}/(k+1)\mathbb{Z}, 1).$$

Proof. (i) Since $Z(k)_0$ is a singleton, we see that $Z(k)$ is connected. The 1-simplices of $Z(k)$ are just pairs (i, j) of non-negative integers with $i + j = k + 1$, while the 2-simplices are triples (a, b, c) of non-negative integers with $a + b + c = k + 1$. Hence, the fundamental group is generated by such pairs (i, j) modulo the relations $(a + b, c)(c + a, b) = (a, b + c)$. It is obvious that this group is an infinite cyclic group generated by $(k, 1)$. Let $f : C \to Z(k)$ be the unique simplicial map, which sends the unique non-degenerate 1-simplex to $(k, 1)$. By our description of the fundamental group it is clear that f induces an isomorphism on π_1. We have to show that f induces an isomorphism on homology with local coefficients. Let M be a module over the ring $\mathbb{Z}[t, t^{-1}]$. The homology of $Z(k)$ with coefficients in M is defined as the homology of the reduced chain complex $C_*(Z(k), M)$, where $C_n(Z(k), M) = \bigoplus_{(a_0,\dots,a_n)} M$. Here $(a_0, \dots, a_n)$ corresponds to a non-degenerate simplex of $Z(k)$, that is $a_0 \geqslant 0$, $a_i > 0$, $i \geqslant 1$ and $a_0 + \dots + a_n = k + 1$. A typical element of $C_n(Z(k), M)$ is denoted by $(a_0, \dots, a_n; x)$, where $x \in M$. The boundary map $d : C_n(Z(k), M) \to C_{n-1}(Z(k), M)$ is given by

$$\begin{aligned} d(a_0, \dots, a_n; x) &= (a_0 + a_1, \dots, a_n; x) + \\ &+ \sum_{i=1}^{n-1} (-1)^i (a_0, \dots, a_i + a_{i+1}, \dots, a_n; x) + \\ &+ (-1)^n (a_n + a_0, \dots, a_{n-1}; t^{a_n} x). \end{aligned}$$

We let $F_i C_n$ be the subgroup of $C_n(Z(k), M)$ corresponding to the summands $(a_0, \dots, a_n)$ with $a_0 \geqslant k + 1 - i$. Then the boundary formula shows that $F_i C_*$ is a subcomplex of $C_*(Z(k), M)$. In this way we obtain a filtered complex

$$F_0 C_* \subset F_1 C_* \subset \dots \subset F_k C_* = C_*(Z(k), M).$$

Clearly

$$F_1C_* = (M \leftarrow M \leftarrow 0 \leftarrow \cdots)$$

with only one nontrivial boundary map given by the multiplication on $(1-t)$. Thus it suffices to prove that the homology of $F_iC_*/F_{i-1}C_*$ is zero for $i \geqslant 2$. We have

$$F_iC_n/F_{i-1}C_n = \bigoplus_{(k+1-i,a_1,\dots,a_n)} M$$

where $a_1 + \cdots + a_n = i$. The boundary map

$$\delta\colon F_iC_n/F_{i-1}C_n \to F_iC_{n-1}/F_{i-1}C_{n-1}$$

is induced by d. One observes that the first and last summand of d lie in $F_{i-1}C_{n-1}$, hence they disappear in δ. In particular δ does not depend on the $\mathbb{Z}[t, t^{-1}]$-module structure on M. We let $h : F_iC_n/F_{i-1}C_n \to F_iC_{n+1}/F_{i-1}C_{n+1}$ be the map given by

$$h(a_0, a_1, \dots, a_n; x) = 0, \quad \text{if } a_1 = 1$$

and

$$h(a_0, a_1, \dots, a_n; x) = -(a_0, 1, a_1 - 1, \dots, a_n; x) \quad \text{if } a_1 > 1.$$

Here $a_0 = k + 1 - i$ and $i \geqslant 2$. Then $h\delta + \delta h = 1$ and part (i) is proved.

(ii) By part (i) we know that the integral homology of $Z(k)$ is $\mathbb{Z}$ in dimensions 0 and 1 and is zero in dimensions >1. This fact can be seen also by noting that the simplicial Abelian group $\mathbb{Z}[Z(k)]$ is isomorphic to the degree $(k + 1)$-part of $C_*(\mathbb{Z}[x], \mathbb{Z}[x])$. Here $C_*(R, R)$ denotes the Hochschild complex of a ring R and the grading of $C_*(\mathbb{Z}[x], \mathbb{Z}[x])$ corresponds to the grading of the polynomial ring $Z[x]$ with $deg(x) = 1$. It is a classical fact that the Hochschild homology of $\mathbb{Z}[x]$ is zero in dimensions > 1, while

$$\mathsf{H}_0(\mathbb{Z}[x], \mathbb{Z}[x]) = \mathbb{Z}[x], \qquad \mathsf{H}_1(\mathbb{Z}[x], \mathbb{Z}[x]) \cong \mathbb{Z}[x]\, dx.$$

Hence, the degree $(k+1)$-part of it is zero in dimensions >1 and is $\mathbb{Z}$ in dimensions 0 and 1, spanned respectively by x^{k+1} and $x^k dx$. Therefore, the same is true for the integral homology of $Z(k)$. Moreover, this shows also that Connes' homomorphism $B\colon \mathsf{H}_0(Z(k), \mathbb{Z}) \to \mathsf{H}_1(Z(k), \mathbb{Z})$ corresponding to the cyclic space $Z(k)$ is the multiplication by $(k + 1)$. Therefore, it follows from Connes' exact sequence that the integral homology of $|Z(k)|^{cy}$ is $\mathbb{Z}$ in dimension 0 and is $\mathbb{Z}/(k + 1)\mathbb{Z}$ in odd dimensions. All other homology groups vanish. Furthermore, the fibration

$$S^1 \to |Z(k)| \to |Z(k)|^{cy}$$

corresponding to the Borel construction shows that

$$\pi_i(|Z(k)|^{cy}) = 0, \quad \text{if } i \neq 1, 2$$

and one has an exact sequence

$$0 \to \pi_2(|Z(k)|^{cy}) \to \mathbb{Z} \to \mathbb{Z} \to \pi_1(|Z(k)|^{cy}) \to 0.$$

As a consequence we see that $\pi_1(|Z(k)|^{cy})$ is Abelian and, therefore, it is the same as the first homology of $|Z(k)|^{cy}$, i.e., $\mathbb{Z}/(k+1)\mathbb{Z}$. Hence, the map $\mathbb{Z} \to \mathbb{Z}$ is injective and we obtain $\pi_2(|Z(k)|^{cy}) = 0$. □

2.3. A SIMPLICIAL SET $Y(k)$

Let $Y(k)$ be the composite of $\hat{C}$ and $\mathsf{Hom}_{\mathcal{F}(\mathsf{as})}([k], -)$.

$$\Delta C^{op} \to \mathcal{F}(\mathsf{as}) \to \mathsf{Sets}.$$

Clearly the cyclic structure on $\hat{C}$ yields a cyclic structure on $Y(k)$.

LEMMA 2.4. *The underlying simplicial set of the cyclic set $Y(k)$ is weakly homotopy equivalent to the disjoint union of $k!$ copies of the circle.*

Proof. A similar argument as in Proposition 2.1 shows that the connected components of the simplicial set $Y(k)$ are in one-to-one correspondence with Σ_k. Thanks to Corollary 1.2 Σ_{n+1} acts freely on $\mathsf{Hom}_{\mathcal{F}(\mathsf{as})}([n],[m])$ and the set of orbits can be identified with $\mathsf{Hom}_{\Delta}([n],[m])$. Thus the action of the group $\Sigma_{k+1} \subset \mathsf{Hom}_{\mathcal{F}(\mathsf{as})}([k],[k])$ on $Y(k)$ is free and the orbits form a simplicial set, which is isomorphic to $Z(k)$. Then the first part of Lemma 2.3 implies the result.

PROPOSITION 2.5. *The space $|Y(k)|^{cy}$ is homotopy equivalent to the discrete space with $k!$ points. Thus $\mathsf{HC}_i(P_k) = 0$, $i > 0$ and $\mathsf{HC}_0(P_k) \cong \mathsf{b}([k])$.*

Proof. The fibration $S^1 \to |Y(k)| \to |Y(k)|^{cy}$ together with Lemma 2.4 shows that $\pi_0(|Y(k)|^{cy})$ is a set with $k!$ elements. Since the group Σ_{k+1} acts freely on $Y(k)$ and the orbits are $Z(k)$ it follows that Σ_{k+1} acts also freely on $|Y(k)|^{cy}$ with orbits $|Z(k)|^{cy}$. Thus the result follows from the second part of Lemma 2.3. □

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