Geometry and topology of symplectic resolutions

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Introduction.

By Hironaka, every singular algebraic variety Y over $\mathbb{C}$ admits a resolution of singularities – that is, a smooth algebraic variety X equipped with a projective birational map $X \to Y$. In many problems of algebraic geometry the mere existence of X is enough, but sometimes it is not. Especially when algebraic geometry is being used as a tool in some other area of mathematics, more control over the resolution X is needed.

This is very prominently the case, for instance, in Geometric Representation Theory (see e.g. [CG]). Ideally, given a singular variety Y encoding some representation theory problem, one wants a resolution X which is semismall (that is, $\operatorname{dim} X \times_Y X = \operatorname{dim} X$), and with some restrictions on the topology of the fibers. If Y carries some group action, one also wants X to be equivariant with respect to this action.

In many cases it is actually possible to achieve the ideal. For instance, if Y is the nilpotent cone in the adjoint representation of a semisimple Lie group, then it admits a semismall resolution X known as *Springer resolution*. The resolution is equivariant with respect to all possible group actions on Y. Its fibers, although singular, cohomologically behave in the same way as smooth homogeneous spaces: all the cohomology groups are pure with respect to the weight filtration, and are in fact

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spanned by classes of algebraic cycles. A completely parallel picture holds for the so-called *quiver varieties* of H. Nakajima, and for Hilbert schemes of n points on $\mathbb{C}^2$.

The standard proofs of these facts (e.g. [CLP]) work by explicit constructions and rely heavily on the specific geometry of the variety Y in question. For some of the statements, it also helps to consider the homolomorphic symplectic form on the resolution X, which is present in all the cases mentioned above.

The goal of the present paper is to report on a recent series of results that somewhat changes this conventional perspective. Namely, upon closer inspection, it turns out that this auxiliary and almost accidental piece of structure, the homolorphic symplectic form, actually insures all of the other good properties of the resolution X – the semismall property, the cohomological purity of the fibers, and so on and so forth. Moreover, the theory can pushed through so far as to give a complete algebraic description of the derived category of coherent sheaves on the resolution X. This gives new information even in the well-studied cases such as the Springer resolution or the Hilbert scheme.

Since the only thing needed from X is the holomorphic symplectic form, the results we are going to describe lie entirely within algebraic geometry (or even algebraic symplectic geometry, if such a thing exists at present). Thus no knowledge of geometric representation theory is needed, nor assumed. Moreover, while most known applications come from geometric representation theory, all the results can also be used in the local study of contractions of compact holomorphic symplectic and hyperkähler manifolds – or, more generally, in that part of the Minimal Model Programm which deals with varieties with trivial canonical bundle. In particular, some of the results on the derived category are actually difficult conjectures which should hold in larger generality, and at least for general Calabi-Yau varieties (see e.g. the general program sketched in [BO1], [BO2]). In the holomorphic symplectic case, these conjectures can actually be proved.

1 Definitions and general results.

Fix a base field K of characteristic 0. A convenient starting point is the following definition introduced by A. Beauville [B].

Definition 1.1. A *symplectic singularity* is a normal irreducible algebraic variety Y over k equipped with a non-degenerate symplectic form $\Omega \in H^0(Y^{sm}, \Omega^2)$ on the smooth locus $Y^{sm} \subset Y$ which extends to a possibly degenerate symplectic form on a smooth projective resolution $X \to Y$.

Here and elsewhere in the paper *symplectic* is understood in the algebraic sense – it should not be confused with C^∞ symplectic froms which appear in Kähler geometry. By a *resolution* we understand a smooth variety X equipped with a projective birational map $X \to Y$. Originally, Beauville only required the existence of the form Ω; we prefer to include it into the definition as a part of the data.

An easy observation ([B]) is that if Ω extends to one smooth resolution X, it also extends to any other resolution X' – thus in Definition 1.1 we can replace "a resolution" with "any resolution" without any loss of generality.

In the present paper, we are mostly concerned with local study of symplectic singularities – in particular, Y will be usually assumed to be affine. Since we also assume Y normal, we must have $Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$, so that, once a resolution X is given, it is no longer necessary to specify Y.

Let us give some examples of symplectic singularities.

Example 1.1. $Y = W/G$, where W is the 2-dimensional vector space considered as an affine variety, and $G \subset SL(W)$ is a finite subgroup. In this classic case (see e.g. [L]), there exists a unique smooth resolution X with trivial canonical bundle – since we are in $\dim 2$, this is equivalent to having a symplectic form.

Example 1.2. $Y = \mathbb{A}^{2n}/S_n$, the quotient of the affine space of dimension $2n$ by the symmetric group on n letters – equivalently, Y is the n-th symmetric power of the affine plane $\mathbb{A}^2$. X is the Hilbert scheme of 0-dimensional subschemes of length n in $\mathbb{A}^2$ (abbreviated to "Hilbert scheme of n points").

Example 1.3. A combination of the previous two examples: Y is the n-th symmetric power of a quotient $Y_0 = W/G$, $\mathsf{dim}\, W = 2$, $G \subset SL(W)$, X is the Hilbert scheme of n points on the canonical symplectic resolution X_0 of Y_0.

Example 1.4. $Y = V/G$ is the quotient of a sympletic vector space V by a finite subgroup $G \subset Sp(V)$, X is any resolution.

Example 1.5. $Y \subset \mathfrak{g}$ is the nilpotent cone in a Lie algebra $\mathfrak{g}$ of a semisimple Lie group G, $X = T^*(G/B)$ is the cotangent bundle of the flag variety G/B associated to G (the *Springer resolution*).

Example 1.6. More generally, $X = T^*(G/P)$ is the cotangent bundle to a homogeneous variety G/P associated to a parabolic subgroup $P \subset G$ in a semisimple Lie group. $Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$ is in this case a closure of a certain nilpotent orbit in $\mathfrak{g}$.

Example 1.7. Even more generally, Y is the normalization of the closure of a nilpotent orbit in a semisimple Lie algebra $\mathfrak{g}$, X is any resolution.

Example 1.8. Y and X are *quiver varieties* constructed by H. Nakajima [N1] starting from certain combinatorail data.

We note that Examples 1.1-1.3 are particular cases of Example 1.4, and Examples 1.5-1.6 are particular cases of Example 1.7. The reason we have separated them from the rest is that they actually satisfy a stronger assumption: there exists a resolution X to which the symplectic form Ω extends *as a non-degenerate* 2*-form*. This property, unlike the general defintion of the symplectic singularity, depends on the resolution. To emphasize this, we introduce the following definition.

Definition 1.2. A *symplectic resolution* is a smooth algebraic variety X over K equipped with a closed non-degenerate 2-form Ω such that the canonical map $X \to Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$ is a birational projective map.

Not all symplectic singularities have symplectic resolutions. In the quotient singularity case (Example 1.4), the only known examples of symplectic resolutions are those in Example 1.3. Moreover, it has been proved by M. Verbitsky [Ve] that the existence of a symplectic resolution yields a strong necessary condition on the subgroup $G \subset Sp(V)$, and even this condition is not sufficient ([GiKa]). In the nilpotent orbit case (Example 1.7), the existence question has been studied exhaustively by Baohua Fu [F]; all the symplectic resolutions that do exist are covered by Example 1.6. Finally, in the quiver variety case a resolution always exists, but this is in fact a corollary of a certain technical assumption on the combinatorial data imposed in [N1]. If one drops this assumption, one obtains some quiver varieties which do not admit a resolution (for instance, those considered in [KLS]). Even more surprisingly, there are quiver varieties which do admit a symplectic resolution, but *not* of the quiver type – such is the O'Grady singularity studied in [KL].

An obvious source of smooth symplectic varieties is the cotangent bundles, $X = T^*M$ for a smooth algebraic variety M. However, this is not very promising from the point of view of symplectic singularities. Indeed, there is the following conjecture variously attributed to J.-P. Demailly, F. Campana, Th. Peternell, which is very difficult, but kind of old and well-established.

Conjecture 1.3. *Assume given a smooth algebraic variety M, let $X = T^*M$, and assume that the natural map $X \to Y = H^0(X, \mathcal{O}_X)$ is a projective birational map. Then $M = G/P$, the quotient of a semisimple algebraic group G by a parabolic subgroup $P \subset G$.*

Thus all the symplectic resolutions of the form T^*M are conjecturally covered by Example 1.6. We note that if in assumtions of Conjecture 1.3 one in addition requires that Y has an isolated singularity, then one can prove that M must be a projective space – this is S. Mori's famous theorem on smooth varieties with ample tangent bundle.

Another natural source of symplectic resolution comes from global homolorphic symplectic geometry. Given a projective homolorphic symplectic manifold $\overline{X}$, one can sometimes construct a projective birational contraction $\overline{X} \to \overline{Y}$; the preimage $X \subset \overline{X}$ of any open affine $Y \subset \overline{Y}$ is then an example of a symplectic resolution according to Definition 1.2. However, so far, all the singularities obtained in this way are also covered by Example 1.6 (prominent examples are the Mukai Contraction, where $X = T^*\mathbb{P}^n$, and the O'Grady singularity, where $X = T^*L$, L the Grassmanian of Lagrangian 2-planes in the 4-dimensional symplectic vector space).

Let us now list some of the general properties of symplectic singularities and symplectic resolutions. First of all, assume given a symplectic singularity Y. The first observation is the following.

Lemma 1.4 ([B]). *A symplectic singularity Y is necessarily canonical and rational.* □

Corollary 1.5. *For any fiber F of a smooth resolution $\pi : X \to Y$ of a symplectic singularity Y, the cohomology group $H^1(F_{an}, \mathbb{Z})$ is trivial.*

Proof. By proper base change, it suffices to prove that $R^1\pi_*\mathbb{Z}_X = 0$; this immediately follows from $R^1\pi_*\mathcal{O}_X = 0$ by considering the exponential exact sequence. □

To proceed, it turns out to be very productive to notice that Y carries a natural structure of a Poisson scheme, [K2, Definition 1.2]. Namely, for every two local functions $f, g \in \mathcal{O}_Y$ on Y, we define

$$\{f, g\} = \Theta \lrcorner\, df \wedge dy$$

on the smooth part $Y^{sm} \subset Y$, where $\Theta \in H^0(Y^{sm}, \Lambda^2\mathcal{T})$ is the bivector on Y^{sm} dual to the symplectic form, and we note that since Y is normal, any function extends uniquely from Y^{sm} to the whole Y. The advantage of the Poisson bracket over the symplectic form is that the bracket is perfectly well-defined on the singular locus of Y. This allows to prove the following.

Theorem 1.6 ([K2, Theorem 2.3]). *Every symplectic singularity Y admits a finite stratification by locally closed Poisson subschemes $Y_i \subset Y$ such that every Y_i is in fact smooth, and the induced Poisson structure on Y_i comes from a symplectic form. All the closures $\overline{Y_i}$ are also symplectic singularities. Moreover, for any closed point $y \in Y$, the formal completion $\widehat{Y}_y$ admits a decomposition*

$$\widehat{Y}_y = Y_{y,0} \widehat{\times} \widehat{Y}_{iy} \tag{1.1}$$

into the Poisson scheme product of the formal germ of the stratum Y_i containing y and a certain symplectic singularity $Y_{y,0}$.

The decomposition (1.1) into the product of a symplectic stratum and a transversal slice is an algebraic version of the *Weinstein decomposition* known in the C^∞ Poisson geometry, see [We]. Unfortunately, in algebraic geometry it only exists after passing to the formal completion. To remedy the situation, it would be very convenient to have one additional piece of structure on a symplectic singularity – namely, a $\mathbb{G}_m$-action which dilates the sympletic form.

Definition 1.7. An action of the algebraic group $\mathbb{G}_m$ on a symplectic singularity Y is called *dilating* if it preserves the line $K \cdot \Omega \in H^0(Y^{sm}, \Omega^2_Y)$ and acts on this line via a representation of weight $l > 0$ (in other words, $\lambda \cdot \Omega = \lambda^l \Omega$ for some fixed integer $l > 0$ any $\lambda \in \mathbb{G}_m(K) = K^*$). A $\mathbb{G}_m$-action on Y is said to be *positive-weight* if every finite-dimensional $\mathbb{G}_m$-equivariant subquotient of $H^0(Y, \mathcal{O}_Y)$ decomposes into a sum of representations of non-negative weights.

Conjecture 1.8. *Every transversal slice $Y_{y,0}$ in* (1.1) *admits a dilating positive-weight action of $\mathbb{G}_m$ such that $y \in Y_{y,0}$ is its only fixed point.*

This conjecture is not as wild as one might suppose, since it is actually possible to prove that $Y_{y,0}$ does admit a dilating $\mathbb{G}_m$-action whose only fixed point is y ([K2, Theorem 2.4]). It is *not* true, however, that such an action always has positive weights. It often happens in examples that Y admits a commuting $\mathbb{G}_m$ action which is Hamiltonian, and even if one starts with a positive-weight dilating action, composing it with a commuting Hamiltonian action may produce a dilating $\mathbb{G}_m$ action whose weights are no longer positive. Effectively, Conjecture 1.8 says that this is the only source of problems: whenever a dilating action provided by [K2, Theorem 2.4] is not positive-weight, there is a commuting Hamiltonian action which can be used to correct the weights.

In examples, a good $\mathbb{G}_m$-action is always present. In the quotient singularity case $Y = V/G$, the strata Y_i and the transversal slices $Y_{y,0}$ are also symplectic quotient singularities, and the $\mathbb{G}_m$-action on V with weight 1 induces a good dilating action. In the nilpotent orbit case, the strata are smaller nilpotent orbits, and it is possible to obtain a dilating positive-weight $\mathbb{G}_m$-action on transversal slice by an explicit construction. The same is true in the quiver variety case.

The existence of a positive-weight dilating $\mathbb{G}_m$-action is in fact necessary for some of the results that we are going to describe; since we cannot at present prove Conjecture 1.8, it has to be introduced as an additonal hypothesis in the statements.

Assume now given a symplectic resolution X in the sense of Definition 1.2. Then the affine scheme $Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$ is a symplectic singularity, so that Theorem 1.6 applies. However, one can also prove some stronger results concerning the topology of X.

Theorem 1.9 ([K2, Corollary 2.8, Lemma 2.11, Theorem 2.12]). *Let X be a symplectic resolution of $Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$. Then X is semismall over Y – that is, $\mathsf{dim}\, X \times_Y X = \mathsf{dim}\, X$. More exactly, for any stratum $Y_i \subset Y$ of codimension $\mathsf{codim}\, Y_i = 2l$, its preimage $X_i \subset X$ has codimension $\geq l$. Moreover, we have*

$$H^p(X, \Omega^q_X) = 0$$

whenever $p > q$, and for any fiber $E \in X$ of the map $X \to Y$, the odd cohomology groups $H^{2p+1}(E_{an}, \mathbb{C})$ of the corresponding complex-analytic space E_{an} are trivial, while the even cohomology groups $H^{2p}(E_{am}, \mathbb{C})$ carry a pure $\mathbb{R}$-Hodge structure of weight $2p$ and type (p,p). Finally, the symplectic form Ω is exact in the formal neighborhood of any fiber $E \in X$.

Sketch of the proof. For any integer $i \geq 0$, denote by $F^i H^{\bullet}_{DR}(X)$ the image of the cohomology $H^{\bullet}(X, F^i\Omega^{\bullet}_X)$ with coefficients in the i-th term of the stupid filtration on the de Rham complex $\Omega^{\bullet}_X$. This is not quite the Hodge filtration, since X is not compact, but it is functorial, and restricts to Hodge filtration on compact fibers. By definition, the de Rham cohomology class $[\Omega] \in H^2_{DR}(X)$ of the symplectic form Ω lies in $F^2 H^2_{DR}(X)$. Since Y has rational singularities, we have $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$, so that $H^{\bullet}_{DR}(X) = F^1 H^{\bullet}_{DR}(X)$, and the same is then true for the complex-conjugate to the filtration $F^{\bullet}$. In particular, $[\Omega] \in \overline{F^1 H^2_{DR}(X)}$. By Hodge theory, this implies that $[\Omega]$, hence also Ω itself restricts to 0 on any fiber E.

Since the de Rham cohomology $H^{\bullet}_{DR}(\widetilde{E})$ of the formal neighborhood $\widetilde{E}$ of the fiber $E \in X$ is isomorphic to the cohomology of E itself, $[\Omega]$ is also tridial in $H^2_{DR}(\widetilde{E})$. Noting that $H^1(\widetilde{E}, \mathcal{O}_{\widetilde{E}}) = 0$ by Lemma 1.4, we deduce that Ω is exact on $\widetilde{E}$.

Let now $Y_i \subset Y$ be some smooth stratum, let $y \in Y_i$ be a closed point, and let $E \subset X$ be its preimage. Since $\Omega = 0$ on E, for any tangent vector $\xi \in T_y Y_i$ we have a well-defined 1-form $\alpha = \xi \lrcorner \Omega$ on the smooth part of E. By careful analysis of the construction of the Hodge structure on the cohomology of E – this is a delicate point, for details we refer the reader to [K2, Lemma 2.9] – one checks that α actually has a well-defined class in $H^1(E_{an}, \mathbb{C})$. But this group vanishes by Corollary 1.5. Again by Hodge theory, this implies that $\alpha = 0$ on the smooth part of E.

We conclude that the restriction of Ω to the smooth part of the preimage X_i is obtained by pullback from a symplectic form on Y_i (more careful analysis actually shows that this is the symplectic from induced by the Poisson structure on Y). Since Ω is non-degenerate on X, this gives the dimension estimates $2\operatorname{codim} X_i \geq \operatorname{codim} Y_i$, $\dim X \times_Y X = \dim X$.

Now one can apply one of the vanishing theorems of [EV], see [K2, Lemma 2.10], to conclude that $H^p(X, \Omega^q_X) = 0$ whenever $p + q \geq \dim X$, or, equivalently, whenever $p \geq q$ (since X is symplectic, we have $\Omega^q_X \cong \Omega^{\dim X - q}_X$). Again analysing the Hodge structure on the fiber E, we deduce the cohomological purity claim. $\square$

2 Deformations and quantization.

As before, let us fix a symplectic resolution X over a field K of characteristic 0. Next we would like to discuss deformation theory of X.

Since X is not compact, it usually does not have a reasonable deformation theory as an algebraic variety (in particular, the deformation space is infinite-dimensional). What one has to do is to consider deformations of the pair $\langle X, \Omega\rangle$ – that is, deformations of X together with the symplectic form. For this deformation problem, the order-1 deformations are controlled by the second de Rham cohomology group $H^2_{DR}(X)$ which is finite-dimensional. Moreover, in higher orders we have a complete analog of the Bogomolov-Tian-Todorov Unobstructedness Theorem.

Theorem 2.1 ([KV, Theorem 1.1]). *The pair $\langle X, \Omega\rangle$ admits a universal formal deformation $\langle \mathfrak{X}, \Omega_{\mathfrak{X}}\rangle/S$ whose base is the competion of the vector space $H^2_{DR}(X)$ at the point $[\Omega] \in H^2_{DR}(X)$.* $\square$

In fact, for any deformation $\langle \mathfrak{X}', \Omega_{\mathfrak{X}'}\rangle/S'$, the classifying map $S' \to S$ is the period map for the symplectic form Ω: it sends a point $s \in S'$ to $[\Omega_s] \in H^2_{DR}(X)$, where Ω_s is the symplectic form on the fiber $\mathfrak{X}'_s$ over $s \in S'$, and we identify $H^2_{DR}(\mathfrak{X}'_s) \cong H^2_{DR}(X)$ by using the Gauss-Manin connection. Of course, the deformation $\mathfrak{X}/S$ is only formal, so that $s \in S'$ should be understood her as an A-valued point for some Artin local K-algebra A. The proof imitates the usual prove of Bogomolov-Tian-Todorov Lemma using the T_1-lifting principle of Z. Ran ([R], [K]).

However, it turns out that it also makes sense to consider the *non-commutative* deformations of X. Namely, we introduce the following.

Definition 2.2. A *quantization* of a Poisson variety X is a sheaf $\mathcal{O}_h$ of flat $K[[h]]$-algebras on X, complete in h-adic topology and equipped with an isomorphism $\mathcal{O}_h/h \cong \mathcal{O}_X$, $f \mapsto \overline{a}$, so that for any two local sections $f, g \in \mathcal{O}_h$ we have

$$fg - gf = h\{\overline{f}, \overline{g}\} \mod h^2,$$

where $\{-,-\}$ is the Poisson bracket on X.

This can be applied to our symplectic resolution X, or to some deformation $\mathfrak{X}'/S'$ with the Poisson structure induced by the relative symplectic form (functions in $\mathcal{O}_{S'}$ are in the Poisson center). This allows one to give a complete classification of quantization.

Theorem 2.3 ([BK1, Theorem 1.8, Lemma 6.4]). *Assume given a symplectic resolution X, and let $\mathfrak{X}/S$ be its universal symplectic deformation provided by Theorem* 2.1. *Then there exists a canonical qunatization $\widetilde{\mathcal{O}}_h$ of the Poisson variety $\mathfrak{X}$ which is universal in the following sense: for any quantization $\mathcal{O}_h$ of the symplectic resolution X, there exists a unique section*

$$s : \operatorname{Spec} K[[h]] \to \operatorname{Spec} K[[h]] \widehat{\times} S$$

of the projection $\operatorname{Spec} K[[h]] \widehat{\times} S \to \operatorname{Spec} k[[h]]$ *such that $s^*(\widetilde{\mathcal{O}}_h)$ is isomorphic to $\mathcal{O}_h$.*

Algebraically speaking, quantizations of X exist, and they are classified up to an isomorphism by power series $H^2_{DR}(X)[[h]]$ with coefficients in $H^2_{DR}(X)$ and leading term $[\Omega]$.

We would like to explain briefly why Theorem 2.3 holds. For this, we need to start with the local theory.

2.1 Local theory. Let $A = K[[x_1, \dots, x_n, y_1, \dots, y_n]]$ be the algebra of functions on the formal neighborhood of 0 in the $2n$-dimensional vector space. Equip $\operatorname{Spec} A$ with the standrd symplectic form $\Omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$ and consider the associated Poisson structure. Then A has a standard quantization D given by

$$D = K[[x_1, \dots, x_n, y_1, \dots, y_n, h]]/\{x_i x_j - x_j x_i, y_i y_j - y_j y_i, x_i y_j - y_j x_i - \delta_{ij} h\}, \tag{2.1}$$

where δ_{ij} is the Kronecker delta-symbol. Denote by $\mathsf{Aut}(D)$ the algebraic group of $K[[h]]$-linear automorphisms of the algebra D. For any $l \geq 1$, denote by $\mathsf{Aut}_{\geq l}(D) \subset \mathsf{Aut}(D)$ the subgroup of automorphisms which are identical on $D/h^l D$.

Proposition 2.4. (i) *For any other symplectic form Ω' on* $\operatorname{Spec} A$*, the pair $\langle A, \Omega\rangle$ is isomorphic to $\langle A, \Omega'\rangle$.*

(ii) *Any quantization of $\langle A, \Omega\rangle$ is isomorphic to D.*

(iii) *The subgroup $\mathsf{Aut}_{\geq 1}(D) \subset \mathsf{Aut}(D)$ coincides with the subgroup on inner autmorphisms of the algebra D, and we have a central group extension*

$$1 \longrightarrow K[[h]]^* \longrightarrow D^* \longrightarrow \mathsf{Aut}_{\geq 1}(D) \longrightarrow 1. \tag{2.2}$$

Sketch of a proof. (i) is the Darboux Theorem; all the standard proofs work in our formal algebraic situation. For (ii) and (iii), we first prove the following.

Lemma 2.5. *For any quantization D' of the Poisson algebra A, the relative Hochschild cohomology groups $HH^l_{K[[h]]}(D')$ of the algebra D' over $K[[h]]$ are annihilated by h for $l \geq 1$, while $HH^0_{K[[h]]}(D') \cong K[[h]]$.*

Sketch of a proof. Consider the spectral sequence computing $HH^{\bullet}_{K[[h]]}(D')$ associated to the h-adic filtration on D'. Its first term $E_1^{\bullet,\bullet}$ is $HH^{\bullet}_{K[[h]]}(\mathsf{gr}^{\bullet} D')$, and the associated graded quotient $\mathsf{gr}^{\bullet} D'$ with respect tot he h-adic filtration by definition coincides with $A[[h]]$. Therefore we have

$$E_1^{\bullet,\bullet} \cong \Lambda^{\bullet}(\mathcal{T}(A))[[h]],$$

the algebra of polyvector fields on $\operatorname{Spec} A$. One checks easily that the differential in the spectral sequence is given by

$$d_1(\alpha) = h[\Theta, \alpha],$$

where $\Theta \in \Lambda^2(\mathcal{T}(A))$ is the Poisson bivector, and $[-,-]$ is the Schouten bracket of polyvector fields. If we use the symplectic form Ω on A to identify $\mathcal{T}(A) \cong \Omega^1(A)$, then $\Lambda^{\bullet}\mathcal{T}(A)$ becomes identified with the algebra $\Omega^{\bullet}(A)$ of differential forms, and the differential d_1 becomes identified with hd, where $d : \Omega^{\bullet}(A) \to \Omega^{\bullet+1}(A)$ is the de Rham differential. But by the Poincarè Lemma, $\operatorname{Spec} A$ has no higher de Rham cohomology, and $H^0_{DR}(A) \cong K$. Therefore for dimension reasons, the spectral sequence degenerates already at $E_2^{\bullet,\bullet}$, and the term $E_2^{\bullet,\bullet} \cong E_\infty^{\bullet,\bullet}$ is as required by the statement. □

Now, since all the groups in Proposition 2.4 (iii) are unipotent, it suffices to prove the corresponding statement for Lie algebras, where it immediately follows from Lemma 2.5: since $HH^1(D)$ is annihilated by h, every derivation of the algebra D which is divisible by h is inner, and the center of the algebra D is $Hh^0(D) = K[[h]]$. To prove (ii), we take a different quantization D', and we prove by induction that $D/h^lD \cong D'/h^lD'$ for any l. Indeed, D/h^lD is an extension of the algebra $D/h^{l-1}D$ by A/K, and as as such, it is given by an extension class β in

$$HH^2_{K[h]/h^{l-1}}(D/h^{l-1}D, A/K) \cong \Omega^2(A).$$

If $l = 2$, then this class β is exactly the Poisson bivector for A, and therefore it is the same for D and for D'. If $l \geq 2$, then β comes from a class in $HH^2(D/h^{l-1}D, D/h^2D)$, so that it survives in the E_2-term of the h-adic spectral sequence. By the proof of Lemma 2.5, this means that the corresponding form $\beta \in \Omega^2(A)$ must be closed. The same is true for the class $\beta' \in \Omega^2(A)$ associated to D'/h^lD'. But by the Poincaré Lemma, all closed forms on A are exact, $\beta = \beta' + d\alpha$; therefore one can change an isomorphism $D/h^{l-1}D \cong D'/h^{l-1}D'$ by composing it with $\exp(1+h^{l-1}\alpha)$ so that we have $\beta = \beta'$ and $D'/h^lD' \cong D/h^lD$. □

For any $l \geq 0$, denote the quotient $\mathsf{Aut}(D)/\mathsf{Aut}_{\geq l+1}(D)$ by $\mathsf{Aut}^l(D)$. By Proposition 2.4 (ii), every symplectic automorphis of A lifts to an automorphism of D, so that $\mathsf{Aut}^0(D)$ is the group of symplectic automorphisms of A. For any $l \geq 1$, we have a group extension

$$1 \longrightarrow A/K \longrightarrow \mathsf{Aut}^l(D) \longrightarrow \mathsf{Aut}^{l-1}(D) \longrightarrow 1, \tag{2.3}$$

where A/K on the left-hand side is considered as the additive group. By definition, $\mathsf{Aut}^l(D)$ acts on $D/h^{l+1}D$, and we have an injective map

$$\mathsf{Aut}^l(D) \hookrightarrow \mathsf{Aut}(D/h^{l+1}D).$$

However, this map is not surjective – not every $K[[h]]$-linear automorphism of $D/h^{l+1}D$ lifts to an automorphism of D. To describe the groups $\mathsf{Aut}^l(D)$ exactly, we introduce the following.

Definition 2.6. A *quantized algebra* over the field K is an associative algebra B over $K[[h]]$ equipped with a Lie bracket $\{-,-\}$ such that $\{a,-\}$ is a derivation of B for any $b \in B$, and for any $a, b \in B$ we have

$$ab - ba = h\{a,b\}.$$

If a quantized algebra B is flat over $K[[h]]$, then the bracket can be recovered uniquely from the product, and Definition 2.6 simply requires that the commutator $ab - ba$ is divisible by h (in other words, B/hB is commutative). Conversely, a quantized algebra annihilated by h is the same thing as a Poisson algebra. In general, Definition 2.6 interpolates between the two notions.

Lemma 2.7. *For any $l \geq 0$, the group $\mathsf{Aut}^l(D)$ is the group of $K[[h]]$-linear automorphisms of the quantized algebra $D/h^{l+1}D$.*

Proof. Left to the reader. □

2.2 Globalization by formal geometry. Assume now given a smooth symplectic variety X of dimension $2n$ over K. To obtain global results about quantization of X, we use the technique called *formal geometry* ([GeKa]). Namely, we note that X defines a completely canonical *variety of formal coordinate systems* $\mathcal{M}_{coord}$ whose points are pairs $\langle x, \varphi\rangle$ of a point $x \in X$ and an isomorphism $\varphi : \widehat{\mathcal{O}_{X,x}} \cong A$ between the standard algebra A and the algebra of functions on the formal neighborhood of x in X. Forgetting φ gives a map

$$\mathcal{M}_{coord} \to X,$$

and one check easily that $\mathcal{M}_{coord}$ is a torsor over X with respect to the group $\mathsf{Aut}(A)$ of all continuous automorphisms of the algebra A. The symplectic structure on X defines a subtorsor

$$\mathcal{M}_0 \subset \mathcal{M}_{coord}$$

of *symplectic* coordinate systems – that is, such pairs $\langle x, \varphi \rangle$ that φ is compatible with the symplectic form. By Proposition 2.4 (i), the forgetfull map $\mathcal{M}_0 \to X$ sirjective, and $\mathcal{M}_0$ is a torsor with respect to the group $\mathsf{Aut}^0(D) \subset \mathsf{Aut}(A)$ of all symplectic automorhpisms of A – in other words, we have a restriction of the torsor $\mathcal{M}_{coord}$ to the subgroup $\mathsf{Aut}^0(D) \subset \mathsf{Aut}(A)$.

Analogously, every quantization $\mathcal{O}_h$ of the symplectic manifold X defines the space $\widetilde{\mathcal{M}}$ of pairs $\langle x, \widetilde{\varphi} \rangle$, $x \in X$, $\widetilde{\varphi}$ is an isomorphism between tha natual completion $\widehat{\mathcal{O}_h}$ at $x \in X$ and the standard quantization D. By Proposition 2.4 (ii), $\widetilde{\mathcal{M}}$ is restriction – or maybe, a lifting would be a better term – of the $\mathsf{Aut}^0(D)$-torsor $\mathcal{M}_0$ to the group $\mathsf{Aut}(D)$ with respect to the natural map $\mathsf{Aut}(D) \to \mathsf{Aut}^0(D)$.

Thus to any quantization $\mathcal{O}_h$, we associate a restriction of the $\mathsf{Aut}^0(D)$-torsor $\mathcal{M}_0$ to the group $\mathsf{Aut}(D)$.

This is not a one-to-one correspondence, but it can be made into one if we enlarge the automorphism groups. Namely, every automorphism of the algebra D must necessarily preserve the maximal ideal $\mathfrak{m} \subset D$ generated by $x_1, \dots, x_n, y_1, \dots, y_n, h \in D$. Thus the Lie algebra of the group $\mathsf{Aut}(D)$ is the algebra $\mathsf{Der}_0(D)$ of all $K[[h]]$-linear derivations $D \to D$ which preserve $\mathfrak{m} \subset D$. It lies naturally within a larger algebra $\mathsf{Der}(D)$ of *all* $K[[h]]$-linear derivations, but those derivations which do not preserve $\mathfrak{m}$ cannot be integrated to actual automorphisms. We have a similar picture for $\mathsf{Aut}(A)$. To take account of the non-integrable derivations, one has to either consider groups which are partially formal, or, which is simpler, to work with the so-called *Harish-Chandra pairs* $\langle G, \mathfrak{h} \rangle$ of an algebraic group G, a Lie algebra $\mathfrak{h}$ equipped with an action of G, and a G-equivariant Lie algebra embedding $\mathfrak{g} \hookrightarrow \mathfrak{h}$, where $\mathfrak{g}$ is the Lie algebra of the group G. When the appropriate notion of a $\langle G, \mathfrak{h} \rangle$-torsor is introduced (see e.g. [BK1, Definition 2.3], or [BD, Section 2.6]), $\mathcal{M}_{coord}$ becomes a $\langle \mathsf{Aut}\, A, \mathsf{Der}\, A \rangle$-torsor over X, $\mathcal{M}_0$ becomes a $\langle \mathsf{Aut}^0(D), \mathsf{Der}^0(D) \rangle$-torsor, and we have the following.

Proposition 2.8 ([BK1, Lemma 3.4]). *Let X be a smooth symplectic variety over K of dimension $2n$. Then there exists a natural one-to-one correspondence between the isomorphism classes of quantizations $\mathcal{O}_h$ of the variety X and the isomorphism classes of liftings $\widetilde{\mathcal{M}}$ of the symplectic coordinate torsor $\mathcal{M}_0$ from the Harish-Chandra pair $\langle \mathsf{Aut}^0(D), \mathsf{Der}^0(D) \rangle$ to the Harish-Chandra pair $\langle \mathsf{Aut}(D), \mathsf{Der}(D) \rangle$.* □

An analogous statement holds in the relative situation – for a variety $\mathfrak{X}$ smooth and sympletic over a base S and of relative dimension $2n$.

With the use of Proposition 2.8, the problem of finding and classifying quantizations redices to the study of torsors. This can be done step-by-step using the extensions (2.3). By the standard obstruction theory, at each step, the obstruction to lifting an $\mathsf{Aut}^l(D)$-torsor to $\mathsf{Aut}^{l+1}(D)$ lies in the group

$$H^2(X, \mathcal{E}),$$

where $\mathcal{E}$ is the vector bundle on X associated to the given $\mathsf{Aut}^l(D)$-torsor via the action of $\mathsf{Aut}^l(D)$ on A/K. This action factors through $\mathsf{Aut}^0(D)$; therefore $\mathcal{E}$ does not really depend on the $\mathsf{Aut}^l(D)$-torsor. An easy computation shows that $\mathcal{E} \cong J^\infty\mathcal{O}_X/\mathcal{O}_X$, the quotient of the jet bundle $J^\infty\mathcal{O}_X$ of the structure sheaf by the structure sheaf itself. When we pass to Harish-Chandra pairs, the cohomology gets replaced with de Rham cohomology (see [BK1]; note that both $\mathcal{O}_X$ and $J^\infty\mathcal{O}_X$ carry natural flat connections). Thus the group that contains the obstruction fits as the middle term into an exact sequence

$$H^2_{DR}(X, J^\infty\mathcal{O}_X) \longrightarrow H^2_{DR}(X, J^\infty\mathcal{O}_X/\mathcal{O}_X) \longrightarrow H^3_{DR}(X)$$

Moreover, since the central extension (2.2) is obviously compatible with the filtration $\mathsf{Aut}_{\geq\bullet}(D)$, one shows that the obstruction actually comes from an element in $H^2_{DR}(X, J^\infty\mathcal{O}_X)$. But by the universal property of the jet bundle, this group coincides with $H^2(X, \mathcal{O}_X)$, which is trivial for a symplectic resolution X. Thus there are no obstructions. Analogously, we see that possible liftings at each step are classified by elements of the group

$$H^1_{DR}(X, J^\infty\mathcal{O}_X/\mathcal{O}_X),$$

and this is isomorphic to $H^2_{DR}(X)$, as required in Theorem 2.3.

Remark 2.9. In fact, one can use a parallel argument to study commutative symplectic deformations and prove Theorem 2.1. To do this, one replaces the standard quantization D by the standard (trivial) symplectic deformation $A[[h]]$, and considers its automorphism group. An exact sequence completely analogous to (2.3) exists, where A/K on the left-hand side consists of exponentials of Hamiltonian vector fields on $\operatorname{Spec} A$.

3 Positive characteristic case.

Interestingly, the quantization theory for symplectic resolutions can be developed still further and in a slightly unexpect direction. Namely, assume now that everything is defined over a perfect base field k of *positive* characteristic $\mathsf{char}\, k = p > 0$.

3.1 New phenomena. Our definition of a quantization, Definition 2.2, makes sense in positive characteristic without any changes, and so does the standard example of a quantization, the algebra D of (2.1). However, in $\mathsf{char}\, p$ we immediately see a new feature: the algebra D has a large center. The p-th powers x_i^p, y_j^p of the generators become central in D. This observation motivates the following.

Definition 3.1 ([BK3, Definition 1.4]). A *Frobenius-constant* quantization of a Poisson scheme X over a field k of characteristic p is a pair of a quantization $\mathcal{O}_h$ of X in the sense of Definition 2.2 and a map $s : \mathcal{O}_X \to \mathcal{O}_h$ such that for any $f \in \mathcal{O}_X$, $s(f)$ is central in $\mathcal{O}_h$ and satisfies

$$s(f) = f^p \mod h^{p-1}.$$

In other words, the natural surjection $\mathcal{O}_h \to \mathcal{O}_X$ should be split by the map s on the subalgebra $\mathcal{O}_X^p \subset \mathcal{O}_X$ of p-th powers of functions on X.

Remark 3.2. Definition 3.1 first appeared in [BK2] with a weaker condition, $s(f) = f^p \mod h$, not $\mod h^{p-1}$. The need for a stronger condition was realized in [BK3].

One would like to prove a version of Theorem 2.3 for Frobenius-constant quantizations in positive characteristic. We can start in the same way, and we immediately notice that the situation is somewhat simpler: instead of considering the whole algebra A, it is enough to consider the truncated algebra

$$\overline{A} = A/\{x_i^p, y_i^p\} \tag{3.1}$$

and its standard quantization

$$\overline{D} = D/\{x_i^p, y_i^p\}.$$

Again, we consider the algebraic group $\mathsf{Aut}(\overline{D})$, its subgroups $\mathsf{Aut}_{\geq l}(\overline{D})$, and the quotient groups $\mathsf{Aut}^l(\overline{D})$. But since D is a finitely-generated $k[[h]]$-module, these groups are now finite-dimensional. For the same reason, there is no need to consider Harish-Chandra pairs: the non-integrable automorphisms can be included into the group $\mathsf{Aut}(\overline{D})$ itself as a certain non-reduced part (formally, $\mathsf{Aut}(\overline{D})$ is the group scheme over k which represents the functor $T \mapsto \mathsf{Aut}_{T[[h]]}(D \otimes \mathcal{O}_T)$).

However, there is a price to pay for this simplification, and it is Proposition 2.4. Out of its three claims, only (iii) survives in positive characteristic essentially in the same form.

Lemma 3.3. *For any Frobenius-constant quantization $\overline{D}'$ of the Poisson algebra $\overline{A}$, the tensor product $\overline{D}' \otimes_{k[[h]]} k((h))$ is a matrix algebra over the Laurent power series field $k((h))$, and we have a group extension*

$$1 \longrightarrow k[[h]]^* \longrightarrow \overline{D}'^* \longrightarrow \mathsf{Aut}_{\geq 1}(\overline{D}') \longrightarrow 1, \tag{3.2}$$

where $\mathsf{Aut}_{\geq 1}(\overline{D}')$ *is the algebraic group of all* $k[[h]]$*-linear automorphisms of the algebra* $\overline{D}'$ *which are identical on* $A = \overline{D}'/h$.

Sketch of the proof The first claim is [BK3, Lemma 3.2]; the second claim immediately follows from the Scolem-Noether Theorem. □

Remark 3.4. Lemma 3.3 is true even for a weaker notion of a Frobenius-constant quantization used in [BK2].

Proposition 2.4 (ii) also has a positive-characteristic counterpart – if one uses the strong notion of a Frobenius-constant quantization, and imposes an additional technical assumtion (Definition 3.12). However, the proof is much more delicate, since Lemma 2.5 no longer holds.

Indeed, our of Lemma 2.5 goes through in positive characteristic without any changes up to the point where one needs to invoke the Poincaré Lemma. But the Poincaré Lemma now is false: the higher de Rham cohomology groups $H^{\bullet}_{DR}(A)$ are not trivial. In a sense, they are as large as possible – there exists a so-called *Cartier isomorphism* $C : H^{\bullet}_{DR}(A) \cong \Omega^{\bullet}(A^p)$ which identifies the de Rham cohomology groups of $\operatorname{Spec} A$ with the spaces of differential forms on $\operatorname{Spec} A^p$ (recall that the subring $A^p \subset A$ of p-th powers of elements of A is isomorphic to A as an abstract ring). For the reduced algebra $\overline{A}$, the Cartier map identifies de Rham cohomology algebra $H^{\bullet}_{DR}(\operatorname{Spec} \overline{A})$ with the exterior algebra $\Lambda^{\bullet} W$, where $W = T^*_o \operatorname{Spec} A$ is the cotangent vector space at the closed point $o \in \operatorname{Spec} A$.

For the same reason, Proposition 2.4 (i) – in other words, the Darboux Theorem – completely breaks down in positive characteristic. For example, while the standard symplectic form Ω on A satisfies $C(\Omega) = 0$, it is very easy to construct a symplectic form Ω' with a $C(\Omega')$ not equal to 0, or in fact equal to an arbitrary prescribed non-zero 2-form on A^p. Since the Cartier map is canonical, A equipped with such a form Ω' cannot be isomorphic to $\langle A, \Omega \rangle$.

Remark 3.5. Since Hochschild cohomology is Morita-invariant, a matrix algebra has the same Hochschild cohomology as its center; therefore Lemma 3.3 implies that $HH^{\bullet}_{k[[h]]}(\overline{D}')$ is annihilated by some power of h, just as in Lemma 2.5. What happens is that there is a second non-trivial differential in the spectral sequence, namely, the differential at the term E^p. After that term, the spectral sequence does degenerate, and we still get the statement of Lemma 2.5, but the higher Hochschild cohomology groups are only annihilated by h^{p-1}, not by h. This, in particular, explains why, if we want Proposition 2.4 to hold, we have to introduce a stronger assumption on $s(f)$ in Definition 3.1.

3.2 Restricted structures. To recover the full statement of Proposition 2.4 in positive characteristic, and then prove a version of Proposition 2.8, we need to equip our quantizations with an additional structure.

Denote by $Q(x, y)$ the free quantized algebra in the sense of Definition 2.6 generated by two elements x, y. Explicitly, we have

$$Q(x, y) = \bigoplus W^{\bullet}_{PBW} T^{\bullet}(x, y),$$

where $T^{\bullet}(x, y)$ is the free associative algebra generated by x and y, and $W^{\bullet}_{PBW}$ is the Poincaré-Birkhoff-Witt increasing filtration (see [BK3, Subsection 1.2] for details). The quantization paramter h acts on $Q(x, y)$ by the natural embedding $W^{\bullet}_{PBW} T^{\bullet}(x, y) \to W^{\bullet+1}_{PBW} T^{\bullet}(x, y)$. We call elements of the algebra $Q(x, y)$ *quantized polynomials* in variables x, y.

Lemma 3.6. *Assume that the base field* k *is of characteristic* $\mathsf{char}\, k = p > 0$. *Then there exists quantized polynomials* $F(x, y)$, $P(x, y)$ *in* x, y *such that*

$$(x + y)^p - x^p - y^p = h^{p-1} F(x, y) \qquad (xy)^p - x^p y^p = h^{p-1} P(x, y).$$

Proof. Immediately follows from [BK3, Lemma 1.3]. □

Definition 3.7. A *restricted quantized algebra* A is a quantized algebra A over a field k of characteristic $\operatorname{\mathsf{char}} k = p > 0$ equipped with an additional operation $x \mapsto x^{[p]}$ such that $h^{[p]} = k$ and

$$
\begin{aligned}
\{x^{[p]}, y\} &= (\operatorname{\mathsf{ad}} x)^p(y),\\
(x+y)^{[p]} &= x^{[p]} + y^{[p]} + F(x,y),\\
(xy)^{[p]} &= x^p y^{[p]} + x^{[p]} y^p - h^{p-1} x^{[p]} y^{[p]} + P(x,y),
\end{aligned}
\tag{3.3}
$$

where we denote by $\operatorname{\mathsf{ad}} x : A \to A$ the endomorphism given by $y \mapsto \{x, y\}$. A restricted quantized algebra A is called *regular* is whenever $x \in A$ satisfies $x^p = 0 \mod h$, it also satisfies $x^p = 0 \mod h^{p-1}$.

Remark 3.8. The polynomial $F(x,y)$ is a well-known Lie polynomial in x, y which can also be given by an explicit formula (Jacobson formula, [DP, II, §7.3, Définition 3.1]). The first two equations of (3.3) define the standard notion of a *restricted Lie algebra.* The standard example of a restricted Lie algebra is the algebra of vector fields on a scheme or, more generally, the algebra of all derivations of an associative algebra B/k (the restricted power operation sends a derivation $D : B \to B$ to its p-th power D^p, which, as one checks easily, is also a derivation).

Recall that a quantized algebra A which is h-adically complete and has no h-torsion is the same as a quantization in the sense of Definition 2.2 (of the quotient A/hA). The notion of a restricted quantized algebra plays the same role in positive characteristic.

Lemma 3.9. *A restricted quantized algebra A which is h-adically complete and has no h-torsion is the same as a Frobenius-constant quantization of the quotient A/hA.*

Sketch of a proof. Given a restricted quantized algebra A, we define the splitting map $s : A/hA \to A$ by

$$
s(a) = a^p - h^{p-1} a^{[p]}.
\tag{3.4}
$$

The first equation of (3.3) guarantees that this is a central map from A to itself, and the last two equations insure that it is an algebra map; since $h^{[p]} = h$, this map vanishes on $hA \subset A$. Conversely, by our assumption, for any Frobenius-constant quantization the difference $s(a) - a$ is divisible by h^{p-1} for any $a \in A$; since A has no h-torsion, (3.4) uniquely defines a restricted power operation $a \mapsto a^{[p]}$, and the conditions on the map s insure that (3.3) is satisfied. □

On the other hand, if $h = 0$ on a restricted algebra A, then A is a Poisson algebra. In this case Definition 3.7 gives a notion of a *restricted Poisson algebra.* The first two equations of (3.3) insure that A equipped with the Poisson bracket is a restricted Lie algebra in the usual sense, and the last equation gives a compatibility condition between the restricted power operation and the multiplication which, it seems, was first introduced in [BK3]. The main source of restricted Poisson algebras is the following.

Proposition 3.10 ([BK3, Theorems 1.11,1.12]). *Let $X = \operatorname{Spec} A$ be a smooth affine scheme over a field k of characteristic $\operatorname{\mathsf{char}} k = p > 0$, and assume that X is equipped with a symplectic form Ω. Then the following conditions are equivalent.*

(i) *We have $C([\Omega]) = 0$.*

(ii) *The form Ω is exact, $\Omega = d\alpha$ for some $\alpha \in \Omega^1_X$.*

(iii) *The Lie subalgerba $H \subset \mathcal{T}(A)$ of* Hamiltonian *vector fields on X is closed with respect to the natural rstricted power operation on $\mathcal{T}(A)$.*

(iv) *The Poisson algebra A admits a restricted Poisson structure.*

Moreover, restricted Poisson structures in (iv) *are in one-to-one correspondence with* 1*-forms* α *in* (ii) *considered modulo exact* 1*-forms,* $\alpha \sim \alpha + df$ *for any* $f \in A$. $\square$

3.3 Quantization. We can now give the main results about quantization in positive characteristic. First of all, we need the following technical notion introduced in [BK3]

Definition 3.11. A *good quantization base* B is a complete local k-algebra B with residue field k equipped with an element h in its maximal ideal $\mathfrak{m} \subset B$ and an algebra map $s : B/h \to B$ such that for any $b \in B$ we have $s(b) = b^p \mod h^{p-1}$.

In other words, a good quantization base is a *commutative* restricted quantized algebra, with a completeness condition. The quotient B/hB is then a complete local restricted Poisson algebra whose Poisson bracket is trivial. The restricted power operation on B/hB need *not* be trivial (and in applications, it is not). However, since $\{-,-\} = 0$ tautologically, we must have $b^{[p]} = K(b)$ for some additive map $K : B/h \to B/h$ which satisfies $K(ab) = a^pK(b) + K(a)b^p$. In [BK3], such maps are called *Frobenius-derivations.*

Given a quantization base B, by a *restricted quantized algebra* A over B we will understand a quantized algebra over B equipped with a restricted structure in such a way that the natural central embedding $B \to A$ is compatible with the restricted structures.

We will also need the following notions.

Definition 3.12. A Frobenius-constant quantization $\langle \mathcal{O}_h, s\rangle$ of a scheme X/k is called *regular* if for any local section $f \in \mathcal{O}_X$ such that $f^p = 0$, we have $s(f) = 0 \mod h^{p-1}$. A restricted quantized algebra A is called regular if for any $a \in A$ with $a^p = 0 \mod h$, we have $a^{[p]} = 0 \mod h$.

Definition 3.13. A *small Dieudonné module* I over k is a k-vector space equipped with an additive operation $I \to I$, $a \mapsto a^{[p]}$ which is Frobenius-semilinear, that is, $(\lambda a)^{[p]} = \lambda^p a^{[p]}$ for any $\lambda \in k$, $a \in I$.

Small Dieudonné modules obviously form an abelian category. For any good quantization base B with an ideal $I \subset B$ such that $\mathfrak{m}_B \cdot I = 0$, the restricted power operation on B induces a structure of a small Dieudonné module on I.

Proposition 3.14 ([BK3, Proposition 3.8, Lemmas 3.10,3.11]). *Assume given a good quantization base B with maximal ideal $\mathfrak{m} \subset B$. Then there exists a unique regular restricted quantized algebra A^B over B whose quotient $A^B/\mathfrak{m}A^B$ is isomorphic as a Poisson algebra to the standard Poisson algebra $\overline{A}$ defined in* (3.1). *Moreover, for any restricted ideal $I \subset B$ such that $\mathfrak{m} \cdot I = 0$ we have a natural extension of algebraic groups*

$$1 \longrightarrow \mathcal{H}(\overline{A}, I) \longrightarrow \mathsf{Aut}(A^B) \longrightarrow \mathsf{Aut}(A^{B/I}) \longrightarrow 1, \tag{3.5}$$

where $\mathsf{Aut}(A^B)$ is the group of B-linear automorphisms of the restricted quantized algebra A^B, $\mathsf{Aut}(A^{B/I})$ is the group of B/I-linear automorphisms of the restricted quantized algebra $A^{B/I}$, and $\mathcal{H}(\overline{A}, I)$ is a certain commutative algebraic group which only depends on the small Dieudonné module $I \subset B$. Finally, if $I = \mathfrak{m}$, then the group extension (3.5) *is a semi-direct product.* $\square$

This Proposition is the positive characteristic counterpart of Proposition 2.4, together with Lemma 2.7. We will not need the precise form of the group $\mathcal{H}(\overline{A}, I)$, see [BK3, Definition 1.16,Subsection 3.3]. Let us just say that $\mathcal{H}(\overline{A}, -)$ is a exact functor from small Dieudonné modules to commutative algebraic groups, and that the algebraic group $\mathsf{Aut}(\overline{A})$ of all automorphisms of the algebra $\overline{A}$ acts naturally on $\mathcal{H}(\overline{A}, I)$ for any I (the action of $\mathsf{Aut}(A^{B/I})$ on $\mathcal{H}(\overline{A}, I)$ coming from the extension (3.5) is induced by this action via the natural map $\mathsf{Aut}(A^{B/I}) \to \mathsf{Aut}(\overline{A})$). Moreover, there are two particular cases that we will need.

(i) If the restricted structure on I is trivial, $a^{[p]} = 0$ for any $a \in I$, then $\mathcal{H}(\overline{A}, I)$ is the k-vector space $(\overline{A}/k) \otimes_k I$ considered as an (additive) algebraic group.

(ii) If $I = k \cdot h$ for some element $h \in I$ with $h^{[p]} = h$, then $\mathcal{H}(\overline{A}, I) = \overline{A}^*/k^*$ (with the natural structure of an algebraic group).

To state the global quantization result, let us from now on, for any scheme X/k, denote $X^{(1)} = \langle X, \mathcal{O}_X^p \rangle$ – that is, X with the subsheaf $\mathcal{O}_X^p \subset \mathcal{O}_X$ of p-th powers as the structure sheaf. Let us denote by $\mathsf{Fr} : X \to X^{(1)}$ the natural map (if X is reduced, $X^{(1)}$ is isomorphic to X, and Fr becomes the Frobenius map). For any restricted Poisson scheme X/k and a good quantization base B with maximal ideal $\mathfrak{m} \subset B$, by a *B-quantization* $\mathcal{O}_B$ of X we will understand a sheaf of flat and complete restricted quantized B-algebras $\mathcal{O}_B$ on X equipped with a restricted Poisson isomorphism $\mathcal{O}_B/\mathfrak{m}_B \cong \mathcal{O}_X$.

Theorem 3.15 ([BK3, Proposition 1.22]). *Assume given a good quantization base B in the sense of Definition* 3.11 *and a smooth symplectic variety X/k. Assume also that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2, 3$. Then the isomorphism classes of B-quantizations of X are in one-to-one correspondence with elements of the étale cohomology group*

$$H^1_{et}(X^{(1)}, \mathsf{Loc}(\mathcal{H}(\overline{A}, \mathfrak{m}/\mathfrak{m}^2))), \tag{3.6}$$

where $\mathsf{Loc}(\mathcal{H}(\overline{A}, I))$ is an étale sheaf on $X^{(1)}$ which only depends on the $\mathsf{Aut}(\overline{A})$-action on the algebraic group $\mathcal{H}(\overline{A}, I)$. In particular, Frobenius-constant quantizations – or, equivalently, $k[[h]]$-quantizations – of X/k are classified up to an automorphism by elements of the group

$$H^1_{et}(X^{(1)}, (\mathsf{Fr}_*\mathcal{O}_X)^*/(\mathcal{O}_{X^{(1)}})^*).$$

Finally, every B/h-quantization of the scheme X extends to a B-quantization.

Sketch of the proof. As in the proof of Theorem 2.3, we use the technique of formal geometry. To any smooth variety X/k of dimension $2n$ one associates the torsor

$$\mathcal{M}_{coord}(X) = \mathsf{Maps}(\operatorname{Spec} \overline{A}, X)$$

of étale maps $\operatorname{Spec} \overline{A} \to X$ – or, informally speaking, of formal coordinate systems in the Frobenius neighborhoods of points of X. By definition, $\mathcal{M}_{coord}(X)$ is a torsor over X with respect to the subgroup group $\mathsf{Aut}^0(\overline{A}) \subset \mathsf{Aut}(\overline{A})$ of automorphisms of the algebra $\overline{A}$ which preserve the maximal ideal $\mathfrak{m} \subset \overline{A}$. The larger group $\mathsf{Aut}(\overline{A})$ also acts on $\mathcal{M}_{coord}(X)$, and the quotient is isomorphic to $X^{(1)}$ – we have a sequence of maps

$$\mathcal{M}_{coord}(X) \longrightarrow X \xrightarrow{\mathsf{Fr}} X^{(1)},$$

where the map on the right-hand side is the Frobenius map. Both torsors $\mathcal{M}_{coord}(X)/X$ and $\mathcal{M}_{coord}(X)/X^{(1)}$ are locally trivial in étale topology.

For any algebraic group V equipped with an action of the group $\mathsf{Aut}(\overline{A})$, we denote by $\mathsf{Loc}(V)$ the associated étale sheaf on $X^{(1)}$. One checks easily that $\mathsf{Loc}(\overline{A}) \cong \mathsf{Fr}_*\mathcal{O}_X$, $\mathsf{Loc}(k) \cong \mathcal{O}_{X^{(1)}}$, $\mathsf{Loc}(\overline{A}^*) \cong (\mathsf{Fr}_*\mathcal{O}_X)^*$, and $\mathsf{Loc}(k^*) = \mathcal{O}^*_{X^{(1)}}$.

Just as in Proposition 2.8, one deduces from Proposition 3.14 that giving a B-quantization of X is equivalent to giving a lifting of the torsor $\mathcal{M}_{coord}(X)/X^{(1)}$ to the group $\mathsf{Aut}(A^B)$ with respect to the natural group map

$$\mathsf{Aut}(A^B) \to \mathsf{Aut}(\overline{A}).$$

We filter B by the powers of the maximal ideal $\mathfrak{m} \subset B$, and we lift the torsor $\mathcal{M}_{coord}(X)/X^{(1)}$ to $\mathsf{Aut}(A^B)$ step-by-step, by going through the groups $\mathsf{Aut}(A^{B/\mathfrak{m}^l})$. At the first step, we have

$$\mathsf{Aut}(A^{B/\mathfrak{m}^2}) = \mathsf{Aut}(\overline{A}) \rtimes \mathcal{H}(\overline{A}, \mathfrak{m}/\mathfrak{m}^2);$$

therefore a lifting exists, and all liftings are classified by elements of the cohomology group (3.6). At each subsequent step, we apply the standard obstruction theory to (3.5) and deduce that the obstruction to the lifting lies in the group

$$H^2_{et}(X^{(1)}, \mathsf{Loc}(\mathcal{H}(\overline{A}, \mathfrak{m}^l/\mathfrak{m}^{l+1}))),$$

and if this obstruction vanished, the liftings are classified by elements of the group

$$H^1_{et}(X^{(1)}, \mathsf{Loc}(\mathcal{H}(\overline{A}, \mathfrak{m}^l/\mathfrak{m}^{l+1}))).$$

However, when $l \geq 1$, the restricted structure on $\mathfrak{m}^l/\mathfrak{m}^{l+1}$ is obviously trivial. Therefore we have $\mathcal{H}(\overline{A}, \mathfrak{m}^l/\mathfrak{m}^{l+1}) = (\overline{A}/k)^N$ for some integer N, and

$$\mathsf{Loc}(\mathcal{H}(\overline{A}, \mathfrak{m}^l/\mathfrak{m}^{l+1})) = (\mathsf{Fr}_*\mathcal{O}_X/\mathcal{O}_{X^{(1)}})^N.$$

By assumption on X, this étale sheaf has no cohomology. Therefore the lifting exists and is unique. This proves the first claim. To prove the second claim, it suffices to notice that $\mathsf{Loc}(A^*/k^*) = (\mathsf{Fr}_*\mathcal{O}_X)^*/\mathcal{O}^*_{X^{(1)}}$. Finally for the last claim, one checks that the small Dieudonné module $k \cdot h$ is injective in the abelian category of small Dieudonné modules, so that, whatever is the restricted operation on $\mathfrak{m}/\mathfrak{m}^2$, the exact sequence

$$0 \longrightarrow k \cdot h \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow (\mathfrak{m}/\mathfrak{m}^2)/h \longrightarrow 0$$

splits. $\square$

When studying Frobenius-constant quantizations by Theorem 3.15, one can further consider the Kummer spectral sequence, and deduce the short exact sequence

$$0 \to \mathsf{Pic}(X)/p\,\mathsf{Pic}(X) \to H^1_{et}(X^{(1)}, (\mathsf{Fr}_*\mathcal{O}_X)^*/\mathcal{O}_{X^{(1)}}*) \to \mathsf{Br}_p(X) \to 0, \tag{3.7}$$

where $\mathsf{Pic}(X^{(1)}) \cong \mathsf{Pic}(X)$ is the Picard group of X and $X^{(1)}$, and $\mathsf{Br}_p(X)$ is the p-torsion part of the (cohomological) Brauer group $\mathsf{Br}(X^{(1)}) = \mathsf{Br}(X) = H^2_{et}(X, \mathcal{O}^*_X)$. An additional result clarifies the appearance of the Brauer group.

Proposition 3.16 ([BK3, Proposition 1.24]). *In the assumption of Theorem* 3.15*, assume that a Frobenius-constant quantization $\mathcal{O}_h$ is classified by $a \in H^1_{et}(X^{(1)}, (\mathsf{Fr}_*\mathcal{O}_X)^*/\mathcal{O}^{*p}_X)$, and denote by $b \in \mathsf{Br}_p(X)$ the image of the class a under the canonical projection given in* (3.7)*. Denote*

$$X^{(1)}[[h]] = \mathcal{S}pec(X^{(1)}, \mathcal{O}_{X^{(1)}}[[h]]), \qquad X^{(1)}((h)) = \mathcal{S}pec(X^{(1)}, \mathcal{O}_{X^{(1)}}((h))),$$

and let $\pi : X^{(1)}((h)) \to X^{(1)}$ be the natural projection. Using the splitting map $s : \mathcal{O}^p_X \to \mathcal{O}_h$, consider $\mathcal{O}_h$ as a sheaf of algebras over $X^{(1)}[[h]]$, and consider the localization $\mathcal{O}_h(h^{-1})$ as a sheaf of algebras over $X^{(1)}((h))$. Then $\mathcal{O}_h(h^{-1})$ is an Azumaya algebra over $X^{(1)}((h))$, and its class in the Brauer group of $X^{(1)}((h))$ is equal to $\pi^*(b)$. $\square$

4 Derived equivalence.

Quantization theory in positive characteristic summarized in Theorem 3.15 may have some independent interest; however, the reason for its development in [BK3] was a somewhat unexpected application purely in characteristic 0. This is what we are going to describe now.

4.1 Tilting generators. We start with some generalities. Assume given an affine normal algebraic variety Y over a field K and a smooth projective resolution $\pi : X \to Y$. Consider the derived category $D^b_c(X)$ of bounded complexes of coherent sheaves on X, and assume given a vector bundle $\mathcal{E}$ on X. Then, if we denote $R = \operatorname{End}(\mathcal{E})$, we have a natural functor

$$\mathbf{R}\mathcal{H}om(\mathcal{E}, -) : D^b_c(X) \to D^b(R\text{-mod}^{\mathfrak{fg}}),$$

where $D^b(R\text{-mod}^{\mathfrak{fg}})$ is the bounded derived category of fin itely generated left R-modules. Denote this functor by $R^{\bullet}\pi^{\mathcal{E}}_*$. If we pass to the derived categories bounded from above, we have an adjoint functor

$$L^{\bullet}\pi^*_{\mathcal{E}} : D^-(R\text{-mod}^{\mathfrak{fg}}) \to D^-_c(X).$$

Definition 4.1. (i) A vector bundle $\mathcal{E}$ on X is said to be *tilting* if we have $\operatorname{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ for all $i \geq 1$.

(ii) A vector bundle $\mathcal{E}$ is called a *tilting generator* for X if in addition $\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$ for any $\mathcal{F} \in D^-_c(X)$.

A vector bundle $\mathcal{E}$ is tilting if and only if the composition $R^{\bullet}\pi^{\mathcal{E}}_* \circ L^{\bullet}\pi^*_{\mathcal{E}}$ is the identity endofunctor of the category $D^-(R\text{-mod}^{\mathfrak{fg}})$. This in turn happens if and only if the functor $L^{\bullet}\pi^*_{\mathcal{E}} : D^-(R\text{-mod}^{\mathfrak{fg}}) \to D^-_c(X)$ is fully faithful. If $\mathcal{E}$ is a tilting generator, then $L^{\bullet}\pi^*_{\mathcal{E}} \circ R^{\bullet}\pi^{\mathcal{E}}_*$ is also the identity, and both $L^{\bullet}\pi^*_{\mathcal{E}}$ and $R^{\bullet}\pi^{\mathcal{E}}_*$ are equivalence of categories which induce equivalences between $D^b_c(X)$ and $D^b(R\text{-mod}^{\mathfrak{fg}})$ (for details, see [K3, Lemma 1.2]).

Thus if X admits a tilting generator $\mathcal{E}$, the "geometric" category $D^b_c(X)$ is equivalent to the purely algebraic category $D^b(R\text{-mod}^{\mathfrak{fg}})$. This is interesting in its own right, and also has rather strong implications concerning the topology of X which we will describe in the next Section.

However, tilting generators seem to be quite rare. Presently there are only two situations where the existence of a tilting generator is known.

(i) $\mathsf{dim}\, X = 3$, X is a crepant resolution of a quotient singularity $Y = V/G$, where V is a 3-dimensional vector space, and $G \subset SL(V)$ is a finite subgroup. This is the situation of a so-called *McKay equivalence* established in [BKR].

(ii) Again $\mathsf{dim}\, X = 3$, X has a trivial canonical bundle, and $\pi : X \to Y$ is *small*, that is, X has relative dimension at most 1 over Y. This has been described by M. Van den Bergh [V1], following the work of T. Bridgeland [Br].

It turns out that the quantization in positive characteristic allows one to construct a tilting generator in a third rather general situation – namely, for a symplectic resolution X.

Theorem 4.2 ([K3, Theorem 1.4]). *Let $X \to Y$ be a symplectic resolution over a filed K of characteristic* 0. *Then for any closed $y \in Y$, there exists an étale neighborhood $Y_0 \to Y$ of the point $y \in Y$ such that the fibered product $X_0 = Y_0 \times_Y X$ admits a tilting generator.*

Very sketchily, the reason Theorem 3.15 is useful for this result is the following. Note that if X and Y were defined over a perfect field k of positive characteristic $\mathsf{char}\, k = p > 0$, and $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$, then for any $a \in \mathsf{Pic}(X)/p\,\mathsf{Pic}(X)$ Theorem 3.15 gives a quantization $\mathcal{O}_a$ of X associated to the image of the class a in the group $H^1_{et}(X, \mathcal{O}^*_X/\mathcal{O}^{p*}_X)$. Moreover, by Proposition 3.3, the sheaf $\mathcal{O}_a(h^{-1})$ is a *split* Azumaya algebra over $X^{(1)}((h))$ – in other words, $\mathcal{O}_a(h^{-1}) = \mathcal{E}nd(\mathcal{E}_a)$ for some vector bundle $\mathcal{E}_a$ on $X^{(1)}((h))$. On the other hand, $H^i(X, \mathcal{O}_a) = 0$ for $i \geq 1$ by semicontinuity, and therefore $H^i(X^{(1)}((h)), \mathcal{E}nd(\mathcal{E}_a)) = 0$ for $i \geq 1$. Thus the vector $\mathcal{E}_a$ is automatically tilting.

Elementary obstruction theory shows that tilting vector bundles are rigid, that is, they extend uniquely to any formal deformation of X. Thus by standard reduction to positive characteristic, we obtain, in the assumption of Theorem 4.2, a series of tilting vector bundles $\mathcal{E}$ on $X((h)) \cong X^{(1)}((h))$ depending on a prime number p and a class $a \in \mathsf{Pic}(X)$ (since Y is not proper, we may have to replace it by a étale neighborhood Y_0). At the cost of shrinking Y_0 even further, we can get of rid of the quantization parameter h, and obtain a series of tilting vector bundles on the symplectic resolution X_0.

Now, a careful analysis shows that for almost all values of the parameter $a \in \mathsf{Pic}(X)$ the corresponding vector bundle $\mathcal{E}_a$ will actually be a tilting generator (in fact, it suffices to consider values proportional to a class $[L] \in \mathsf{Pic}(X)$ of an ample line bundle L). More precisely, there is constant M independent of p such that $\mathcal{E}_a$ is *not* a generator for at most M values of a. Thus if we take p large and a generic enough, we get a tilting generator required by Theorem 4.2.

We will now try to fill the gaps in this sketch to some extent, so that the reader would get an idea about how the actual proof of Theorem 4.2 works.

4.2 Twistor deformations. The first thing to do is to collect various quantization $\mathcal{E}_a$ in a single multi-paramater family. To do this, we will apply the last claim of Theorem 3.15 to a certain special one-parameter deformation of the symplectic resolution X.

Recall that if a smooth symplectic variety Z over a field k is equipped with a symplectic action of the multiplicative group $\mathbb{G}_m$, then a map $\mu : Z \to \mathbb{A}^1 = \operatorname{Spec} k[t]$ is called a *moment map* if $\Omega_Z \lrcorner \xi_0 = \mu^* dt$, where ξ_0 is the infinitesemal generator of the $\mathbb{G}_m$-action. If a moment map is given, and the quotient $Z/\mathbb{G}_m$ exists, this quotient becomes naturally a Poisson scheme over $\mathbb{A}^1$, and its fiber X over the origin $o \subset \mathbb{A}^1$ is again symplectic. This is known as *Hamiltonian reduction.*

It turns out that sometimes the Hamiltonian reduction procedure can be inversed to some extent. Namely, let X/Y be a symplectic resolution over a field k, and let L be a line bundle on X. Denote $S = \operatorname{Spec} k[[t]]$, the formal disc over k, and let $o \in S$ be the special point (given by the maximal ideal $tk[[t]] \subset k[[t]]$).

Definition 4.3. By a *twistor deformation* Z associated to the pair $\langle X, L\rangle$ we will understand a smooth symplectic deformation $\langle \mathfrak{X}, \mathcal{L}\rangle$ of the pair $\langle X, L\rangle$ over S and a symplectic form Ω_Z on the total space Z of the $\mathbb{G}_m$-torsor associated to $\mathcal{L}$ such that Ω_Z is $\mathbb{G}_m$-invariant, and the projection $\rho : Z \to \mathfrak{X} \to S$ is the moment map for the $\mathbb{G}_m$-action on Z. A twistor deformation is called *exact* if the symplectic form Ω_Z is exact.

Lemma 4.4 ([K1, Lemma 2.2]). *Assume that* $\mathsf{char}\, k = 0$. *Then for any line bundle L on X, there exists a twistor deformation $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z\rangle$ associated to the pair $\langle X, L\rangle$. Moreover, $\mathfrak{X}$ is projective over $\mathcal{Y} = \operatorname{Spec} H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, while $\mathcal{Y}$ is normal and flat over S.*

In terms of the period map as in Theorem 2.1, a twistor deformation is parametrized by a straight line $[\Omega] + t[L] \in H^2_{DR}(X)$ – it is easy to see that this is equivalent to the moment map condition in Definition 4.3. Thus Lemma 4.4 can be deduced from Theorem 2.1. Interestingly, twistor deformations exist in a much wider generality – namely, for a arbitraty Poisson scheme X with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ (see [K1]).

Remark 4.5. The name *twistor deformation* comes from hyperkähler geometry. Namely, if the Chern class $[L] \in H^2_{DR}(X)$ can be represented by the Kähler form of a hyperkähler metric on X, then the corresponding twistor deformation $\mathfrak{X}/S$ can be extended over the whole affine line $\mathbb{A}^1$ and in fact even over the projective line $\mathbb{P}^1$. The total space of this extended deformation is known as the *twistor space* associated to the hyperkähler metric.

Lemma 4.6 ([K1, Lemma 2.5]). *Assume that the line bundle L on X is ample, and consider the twistor deformation $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z\rangle$ associated to the pair $\langle X, L\rangle$ by Lemma 4.4. Let $\widehat{A} = H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, $\mathcal{Y} = \operatorname{Spec} \widehat{A}$, and let $\pi : \mathfrak{X} \to \mathcal{Y}$ be the natural map. Then the map π is one-to-one over the complement $S \setminus \{o\}$. Moreover, if Y is the spectrum of a Henselian local k-algebra, so that $\widehat{A}$ is a local k-algebra with maximal ideal $\mathfrak{m} \subset \widehat{A}$, then there exists a finitely generated k-subalgebra $\widetilde{A} \subset \widehat{A}$ such that*

(i) *the t-adic completion of the Henselization of the algebra $\widetilde{A}$ in $\mathfrak{m} \cap \widetilde{A} \subset \widetilde{A}$ coincides with $\widehat{A}$, and*

(ii) *all the data $\langle \mathfrak{X}, \mathcal{L}, \Omega_Z \rangle$ are defined over $\widetilde{A}$.*

Sketch of the proof. Since the map $\pi : \mathfrak{X} \to \mathcal{Y}$ is projective and $\mathcal{Y}$ is normal, for the first claim it suffices to show that the generic fiber $\mathfrak{X}_\eta$ over the generic point $\eta \in S$ has no compact curves. Indeed, if $\rho : C_\eta \to \mathfrak{X}_\eta$ is a map from a proper curve C_η, the class $[\Omega_\eta]$ of the symplectic from Ω_η on X_η satisfies

$$\rho^*[\Omega_\eta] = \rho^*[\Omega] + t\rho^*[L],$$

where $[\Omega]$ is the class of the form Ω on X, and $[L]$ is the first Chern class of the line bundle $[L]$. Since C_η is a curve, $\rho^*\Omega_\eta = 0$, so that in particular, $\rho^*[L] = 0$. Since L is ample, this implies that $\rho(C_\eta) \subset X_\eta$ is a point, which proves the first claim.

As a corollary, we deduce that the generic fiber $\mathcal{Y}_\eta \cong X_\eta$ is smooth. Then the second claim is a particular case of Artin's Theorem [A, Theorem 3.9]. Namely, since the formal scheme $\mathcal{Y}$ contains a closed subscheme $Y \subset \mathcal{Y}$ if finite type, and the completement $\mathcal{Y} \setminus Y \cong \mathcal{Y}_\eta$ is regular, the formal scheme $\mathcal{Y}$ is a completion of a scheme $\widetilde{Y} = \operatorname{Spec} \widetilde{A}$ of finite type. □

Assume now that we are in the situation of Theorem 4.2 – we are given a symplectic resolution X/Y, $\pi : X \to Y$ over a field K of characteristic 0 and a closed point $y \in Y$. By Theorem 1.9 the form Ω is exact in the formal neighborhood of the fiber $\pi^{-1}(y) \subset X$. Changing Ω is necessary and possibly replacing Y with an open neighborhood of y, we can assume that Ω is exact on the whole of X. Fix an ample line bundle L on X, and consider the twistor deformation $\langle \mathfrak{X}, Z, \Omega_Z \rangle$ associated to $\langle X, L \rangle$ by Lemma 4.4.

One shows – this is [K3, Lemma 2.4] – that since Ω is exact on X, the symplectic form Ω_Z on Z is also exact (that is, the twistor deformation is exact in the sense of Definition 4.3). Moreover, by Lemma 4.6 $\mathfrak{X}$ and all the other data are actually defined over a scheme $\widetilde{Y} = \operatorname{Spec} \widetilde{A}$ of finite type over K whose completion near $Y \subset \widetilde{Y}$ is $\mathcal{Y}$. Therefore we can find a subring $O \subset K$ of finite type over $\mathbb{Z}$ so that everything is defined and smooth over O. Since O is of finite type over $\mathbb{Z}$, the residue field $O/\mathfrak{m}$ for any maximal ideal $\mathfrak{m} \subset O$ is a finite field.

To sum up: starting from a symplectic resolution X/Y over K and a point $y \in Y$, we can construct a symplectic resolution X_O/Y_O, its exact twistor deformation $\mathfrak{X}_O/\mathcal{Y}_O$, and an O-valued point $y_O \in Y_O$. Localizing O is necessary, we can also assume that $\mathfrak{X}_O$ satisfies all the topological properties of X given in Theorem 1.9 – in particular, X_O/Y_O is semismall, and $H^i(X_O, \mathcal{O}_{X_O}) = 0$ for $i \geq 1$.

4.3 Estimates for critical lines. Let us now fix $O \subset K$, the symplectic resolution X_O/Y_O and its exact twistor deformation described above. For any maximal ideal $\mathfrak{m} \subset O$, we obtain a symplectic resolution $X_\mathfrak{m}/Y_\mathfrak{m}$ and its exact twistor deformation $\mathfrak{X}_\mathfrak{m}$ over the field $k = O/\mathfrak{m}$ of some positive characteristic $\operatorname{\mathsf{char}} k = p > 0$. We also obtain the closed point $y_\mathfrak{m} \in Y_\mathfrak{m}$.

Since the twistor deformation $\mathfrak{X}_\mathfrak{m}$ is exact, the corresponding variety $Z_\mathfrak{m}$ carries a restricted Poisson structure in the sense of Definition 3.7, see Proposition 3.10. A more careful analysis ([K3, Proposition 2.6]) shows that this restricted structure is $\mathbb{G}_m$-invariant, so that it descends to $\mathfrak{X}_\mathfrak{m} = Z_\mathfrak{m}/\mathbb{G}_m$, and that the deformation parameter t satisfies $t^{[p]} = t$. Setting $t^{[p]} = t$ obviously defines uniquely a Frobenius-derivation of $k[[t]] = \mathcal{O}_S$, in other words, a restricted structure on $k[[t]]$. Thus $\mathfrak{X}_\mathfrak{m}/S$ is a $k[[t]]$-quantization of the restricted Poisson variety $X_\mathfrak{m}$.

In order to apply Theorem 3.15 to $\mathfrak{X}_\mathfrak{m}/S$, we need to choose a good quantization base in the sense of Definition 3.11 which is compatible with this restricted structure on $k[[t]]$. We let $B = k[[h, t]]$, the algebra of power series in two variables, and define the splitting map $s : k[[t]] \to k[[h, t]]$ by $s(t) = t(t^{p-1} - h^{p-1})$. Then $\langle B, s \rangle$ obviously satisfies the conditions of Definition 3.11, and

$$t^{[p]} = \frac{1}{h^{p-1}}(t^p - s(t)) = t,$$

as required. Applying the last claim of Theorem 3.15, we obtain a B-quantization $\mathcal{O}_h$ of the Poisson scheme $X_{\mathfrak{m}}$ extending the given B/h-quantization $\mathfrak{X}_{\mathfrak{m}}$ – in other words, a Frobenius-constant quantization $\mathcal{O}_h$ of the restricted Poisson scheme $\mathfrak{X}_{\mathfrak{m}}$ which is a sheaf of algebras over B.

Geometrically, the map $s : S_h = \operatorname{Spec} B \to S$ is a flat map ramified over the origin $o \in S$. The preimage $S_o = s^{-1}(o) \subset S_h$ is the union of the lines $S_a \subset S_h$ defined by equations $h - at$ for all $a \in \mathbb{Z}/p\mathbb{Z}$. We let $S_t = \operatorname{Spec} k[t]/t^p = S_o \times_{k[[h]]} k$. By definition, $\mathcal{O}_h$ is a sheaf of algebras on

$$\mathfrak{X}_h = \mathfrak{X}_{\mathfrak{m}}^{(1)} \widehat{\times}_{S^{(1)}} S_h,$$

where $\widehat{\times}$ means the tensor product completed with respect to the natural adic topology. For every line $S_a \subset S_h$, the subscheme $X_a = \mathfrak{X}^{(1)} \widehat{\times}_{S^{(1)}} S_a \subset \mathfrak{X}_h$ is canonically identified with $X_h = X_{\mathfrak{m}}^{(1)}[[h]]$. If we restrict $\mathcal{O}_h$ to $X_a \subset \mathfrak{X}_h$, we obtain a Frobenius-constant quantization $\mathcal{O}_a$ of the scheme $X_{\mathfrak{m}}$ associated by Theorem 3.15 to the parameter $a[L] \in \mathsf{Pic}(X_{\mathfrak{m}})/p\,\mathsf{Pic}(X_{\mathfrak{m}}) \subset H^1_{et}(X_{\mathfrak{m}}, \mathcal{O}^*_{X_{\mathfrak{m}}}/\mathcal{O}^{*p}_{X_{\mathfrak{m}}})$.

By Proposition 3.3, the algebra $\mathcal{O}_a(h^{-1})$ is a matrix algebra for any $a \in \mathbb{Z}/p\mathbb{Z}$, that is, $\mathcal{O}_a(h^{-1}) = \mathcal{E}nd(\mathcal{E}_a)$ for some vector bundle $\mathcal{E}_a$ on $\overline{X}_a = X_a \setminus X_{\mathfrak{m}}^{(1)} \cong X_{\mathfrak{m}}((h))$. If we consider the union

$$X_o = \bigcup_a X_a = X_{\mathfrak{m}}^{(1)} \widehat{\times} S_o \subset \mathfrak{X}_h,$$

then the complement $\overline{X}_o = X_o \setminus X_{\mathfrak{m}}^{(1)} \times S_t$ is the disjoint union of the complements $\overline{X}_a$, and $\mathcal{O}_h$ restricts to a matrix algebra over the whole $\overline{X}_o$.

Definition 4.7. A line $S_a \subset S_h$ is said to be *regular* for $X_{\mathfrak{m}}$ if the vector bundle $\mathcal{E}_a$ is a tilting generator on the completion of $X_{\mathfrak{m}}^{(1)}((h))$ near the preimage $\pi^{-1}(y_{\mathfrak{m}}((h)))$. A line which is not regular is called *critical*.

The main technical result needed for Theorem 4.2 is the following.

Proposition 4.8. *There exists a constant C, independent of the choice of the ideal $\mathfrak{m} \subset O$, such that there are at most C lines $S_a \subset S_h$ which are critical for $X_{\mathfrak{m}}$.*

Thus if we take the ideal $\mathfrak{m} \subset O$ so that $p = \operatorname{\mathsf{char}} O/\mathfrak{m}$ is high enough, there exists at least one regular line S_a, $a \in \mathbb{Z}/p\mathbb{Z}$.

The proof of Proposition 4.8 takes up Section 3 and most of Section 4 of [K3]; we refer a reader interested in technical details to that paper. Here we only list the main steps of the proof.

Step 1. We need a way to measure whether the direct image functor $R^{\bullet}\pi^{\mathcal{E}}_*$ is an equivalence. We work in a general situation: consider a scheme X equipped with a coherent sheaf of algebras $\mathcal{A}$ such that $H^i(X, \mathcal{A}) = 0$ for $i \geq 1$ and the global sections functor $R^{\bullet}\pi^{\mathcal{A}}_* : D^b_c(X, \mathcal{A}) \to D^b(A\text{-mod}^{\mathfrak{fg}})$ from the derived category of coherent sheaves of $\mathcal{A}$-modules to the category of finitely generated modules over $A = H^0(X, \mathcal{A})$. The functor has an adjoint $L^{\bullet}\pi^*_{\mathcal{A}} : D^-(A\text{-mod}^{\mathfrak{fg}}) \to D^-_c(X, \mathcal{A})$, and since $A = R^{\bullet}\pi^{\mathcal{A}}_*(\mathcal{A})$, we have $R^{\bullet}\pi^{\mathcal{A}}_* \circ L^{\bullet}\pi^*_{\mathcal{A}} \cong \mathsf{Id}$. We show (see [K3, Subsection 3.1]) that the composition in the other direction, – that is, the functor

$$L^{\bullet}\pi^*_{\mathcal{A}} \circ R^{\bullet}\pi^{\mathcal{A}}_* : D^-_c(X, \mathcal{A}) \to D^-_c(X, \mathcal{A}) \tag{4.1}$$

is a kernel functor associated to a kernel $\mathcal{M}^{\bullet}(X, \mathcal{A}) \in D^-_c(X \times X, \mathcal{A}^{opp} \boxtimes \mathcal{A})$. The kernel $\mathcal{M}^{\bullet}(X, \mathcal{A})$ maps naturally to the diagonal sheaf $\mathcal{A}_{\Delta}$ on $X \times X$, and the cone $\mathcal{K}^{\bullet}(X, \mathcal{A})$ of this map is trivial if and only if the functor (4.1) is an equivalence. The kernel $\mathcal{K}^{\bullet}(X, \mathcal{A})$ essentially depends only on the topology of X, not on its scheme structure; in particular, for any finite radical map $\rho : X \to X'$ we have

$$\mathcal{K}^{\bullet}(X', \rho_*\mathcal{A}) \cong \rho_*(X, \mathcal{A}). \tag{4.2}$$

If the scheme X is flat over a scheme S, then a similar construction exists relatively over X, and the kernel $\mathcal{K}^{\bullet}(X, \mathcal{A})$ is actually supported on $X \times_S X \subset X \times X$. Moreover, this construction is compatible with the base change: for any map $\rho : S' \to S$, we have $\rho^*\mathcal{K}^\bullet(X,\mathcal{A}) \cong \mathcal{K}^\bullet(X \times_S S', \rho^*\mathcal{A})$. Finally, we note that if $D^b_c(X,\mathcal{A})$ has finite homological dimension, say d, then it suffices to check that $\mathcal{K}^i(X,\mathcal{A})$ is trivial for $0 \geq i \geq -2d$ – if this is satisfied, then $\mathcal{K}^i(X,\mathcal{A}) = 0$ for all i.

Step 2. Applying Step 1 to Proposition 4.8, we see that a line $S_a \subset S_h$ is regular if and only if the kernel

$$\overline{\mathcal{K}}^\bullet_a = \mathcal{K}^\bullet(\overline{X}_a, \mathcal{O}_a) = \mathcal{K}^\bullet(X^{(1)}_{\mathfrak{m}}((h)), \mathcal{E}nd(\mathcal{E}_a))$$

is trivial on the fiber $F_y = \pi^{-1}(y_h \times y_h) \subset (X^{(1)}_{\mathfrak{m}} \times X^{(1)}_{\mathfrak{m}})((h))$. Moreover, since $D^b_c(X^{(1)}((h)))$ has homological dimension $\mathsf{dim}\, X$, we can claim that if the line S_a is critical, then there exists a non-positive integer $i \geq -2\,\mathsf{dim}\, X$ and a point $x \in F_y$ such that $\mathcal{K}^j_a = 0$ on F_y for all $j > i$, while the fiber $\left(\mathcal{K}^i_a\right)_x$ is non-trivial. However, $\mathcal{K}^i_a$ is by definition a sheaf of modules over $\mathcal{E}nd(\mathcal{E}_a)^{opp} \boxtimes \mathcal{E}nd(\mathcal{E}_a)$, and this is a matrix algebra of rank $p^{2\,\mathsf{dim}\, X}$. Therefore we can claim that

$$\mathsf{dim}_{k((h))} \left(\overline{\mathcal{K}}^i_a\right)_x \geq p^{\mathsf{dim}\, X}. \tag{4.3}$$

Step 3. Choose a projective embedding $X^{(1)}_{\mathfrak{m}} \times X^{(1)}_{\mathfrak{m}} \to \mathbf{P}_Y = \mathbb{P}^N \times Y_{\mathfrak{m}} \times Y_{\mathfrak{m}}$, and denote by $\mathbf{P}_y \subset \mathbf{P}_Y$ the fiber of $\mathbf{P}_Y/Y$ over the point $y_{\mathfrak{m}} \times y_{\mathfrak{m}} \in Y^{(1)}_{\mathfrak{m}} \times Y^{(1)}_{\mathfrak{m}}$, with the embedding map $i_y : \mathbf{P}_y \to \mathbf{P}_Y$. Let $\mathbf{P}_h = \mathbf{P}_y((h)) \subset \mathbf{P}_Y((h))$. Applying pushforward, we can treat the complex $\overline{\mathcal{K}}^\bullet_a$ as a complex of sheaves on $\mathbf{P}_Y$; by restriction we obtain a complex of sheaves $\overline{\mathcal{K}}^\bullet_{y,a} = L^\bullet i^*_y \overline{\mathcal{K}}^\bullet_a$ on $\mathbf{P}_h$, and (4.3) implies

$$\mathsf{dim}_{k((h))} \mathbb{H}^i\left(\mathbf{P}_h, \overline{\mathcal{K}}^\bullet_{y,a} \overset{L}{\otimes} k_x\right) = \mathsf{dim}_{k((h))} \left(\overline{\mathcal{K}}^i_{y,a}\right)_x \geq p^{\mathsf{dim}\, X}.$$

But by definition, $\mathbf{P}_h$ is the projective space $\mathbb{P}^N_{k((h))}$ of dimension N over $k((h))$. Therefore the skyscraper sheaf k_x at the point $x \in \mathbf{P}_h$ has a Koszul resolution by sheaves of the form $\mathcal{O}(-n)^{\oplus\binom{N}{n}}$, $0 \geq n \geq N$, and we conclude that

$$p^{\mathsf{dim}\, X} \leq \mathsf{dim}_{k((h))} \mathbb{H}^i\left(\mathbf{P}_h, \overline{\mathcal{K}}^\bullet_{y,a} \overset{L}{\otimes} k_x\right) \leq \sum_{0 \leq n \leq N} \mathsf{dim}_{k((h))} \binom{N}{n} \mathbb{H}^{i-n}(\mathbf{P}_h, \overline{\mathcal{K}}^\bullet_{y,a}(-n)).$$

The right-hand side does not depend on the point $x \in \mathbf{P}_h$. Therefore we can sum up these estimates over all critical $a \in \mathbb{Z}/p\mathbb{Z}$, and conclude that to bound from aboce the number of critical lines, it suffices to prove that for any n, $0 \leq n \leq N$ and i, $0 \geq i \geq -2\,\mathsf{dim}\, X$, we have

$$\sum_{a \in \mathbb{Z}/p\mathbb{Z}} \mathsf{dim}_{k((h))} \mathbb{H}^{i-n}(\mathbf{P}_h, \overline{\mathcal{K}}^\bullet_{y,a}(-n)) \leq M p^{\mathsf{dim}\, X}, \tag{4.4}$$

where M is a positive integer which does not depend on the maximal ideal $\mathfrak{m} \in O$.

Step 4. Using the disjoint union decomposition $\overline{X}_o = \coprod_a \overline{X}_a$, we can rewrite (4.4) as

$$\mathsf{dim}_{k((h))} \mathbb{H}^{i-n}(\mathbf{P}_o, (L^\bullet i^*_y \overline{\mathcal{K}}^\bullet_o)(-n)) \leq M p^{\mathsf{dim}\, X},$$

where $\mathbf{P}_o = \mathbf{P}_y \widehat{\times} S_o \subset \mathbf{P}_Y \widehat{\times} S_o$, and $\overline{\mathcal{K}}^\bullet_o = \mathcal{K}^\bullet(\overline{X}_o, \mathcal{O}_h)$. Moreover, since the scheme S_o is flat over $k[[h]]$ and formation of the kernel $\mathcal{K}^\bullet(-,-)$ commutes with respect to base change, we can apply semicontinuity theorem to $X_o/k[[h]]$; we conclude that to prove (4.4), it suffices to prove that

$$\mathsf{dim}_k \mathbb{H}^{i-n}(\mathbf{P}_y \times S_t, (L^\bullet i^*_y \mathcal{K}^\bullet_t)(-n)) \leq M p^{\mathsf{dim}\, X}, \tag{4.5}$$

where

$$\mathcal{K}^\bullet_t = \mathcal{K}^\bullet(X_o/h, \mathcal{O}_o/h) = \mathcal{K}^\bullet(X^{(1)}_{\mathfrak{m}} \times S_t, \mathcal{O}_{\mathfrak{X}}/t^p).$$

Step 5. It remains to prove (4.5). It explicitly does not depend on the quantization $\mathcal{O}_h$, but still depends on the twistor deformation $\mathfrak{X}_{\mathfrak{m}}$. We first note that the kernel $\mathcal{K}^\bullet_t$ can be rewritten as

$$\mathcal{K}^\bullet_t = \mathcal{K}^\bullet(\mathfrak{X}^{[p]}_{\mathfrak{m}}, \mathsf{Fr}^S_* \mathcal{O}_{\mathfrak{X}_{\mathfrak{m}}})/t^p,$$

where $\mathfrak{X}_{\mathfrak{m}}^{[p]} = \mathfrak{X}_{\mathfrak{m}}^{(1)} \times_{S^{(1)}} S \subset \mathfrak{X}_{\mathfrak{m}}$, and Fr^S is the embedding map (simultaneously, the relative Frobenius map for $\mathfrak{X}/S$). But since the map Fr^S is finite and radical, we may apply (4.2) and write

$$\mathcal{K}_t^{\bullet} = \mathsf{Fr}_*^S \mathcal{K}^{\bullet}/t^p,$$

where $\mathcal{K}^{\bullet} = \mathcal{K}^{\bullet}(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}})$. Since the map $\pi : \mathfrak{X}_{\mathfrak{m}} \to \mathcal{Y}_{\mathfrak{m}}$ is one-to-one over the generic point of S, the kernel $\mathcal{K}^{\bullet}$ is annihilated by t^{M_1} for some integer M_1. Moreover, the map π is generically one-to-one already over our original field K of characteristic 0, so that M_1 does *not* depend on the choice of $\mathfrak{m} \subset O$. We deduce that

$$\mathsf{dim}_k \mathbb{H}^{i-n}(\mathbf{P}_y \times S_t, L^{\bullet} i_y^* \mathcal{K}_t^{\bullet}(-n)) \leq M_1 \, \mathsf{dim}_k \mathbb{H}^{i-n}(\mathbf{P}_y, L^{\bullet} i_y^* (\mathsf{Fr}_* \mathcal{K}^{\bullet})(-n)),$$

where M_1 does not depend on $\mathfrak{m}$. Thus to prove (4.5), it suffice to prove that

$$\mathsf{dim}_k \mathbb{H}^{i-n}(\mathbf{P}_y, L^{\bullet} i_y^* (\mathsf{Fr}_* \mathcal{K}^{\bullet})(-n)) \leq M p^{\mathsf{dim}\, X} \tag{4.6}$$

for some universal constant M.

Step 6. Finally, (4.6) only depends on $X_{\mathfrak{m}}/Y_{\mathfrak{m}}$, not on the quantization $\mathcal{O}_h$, nor on the twistor deformation $\mathfrak{X}$. Thus the symplectic form on X is no longer used. We can compactify X/Y to a birational map $\widetilde{X} \to \widetilde{Y}$ of proper schemes over K, obtain $\widetilde{X}_O$ and $\widetilde{X}_{\mathfrak{m}}$ (possibly changing $O \subset K$), and extend $\mathcal{K}_O^{\bullet} = \mathcal{K}^{\bullet}(X_O, \mathcal{O}_{X_O})$ to some complex $\widetilde{\mathcal{K}}_O^{\bullet}$ of coherent sheaves on $\widetilde{X}_O \times \widetilde{X}_O$. We denote its restriction to $X \times X \subset X_O \times_O X_O$ by $\widetilde{\mathcal{K}}^{\bullet}$, and we denote its restriction to $X_{\mathfrak{m}} \times X_{\mathfrak{m}} \subset X_O \times_O X_O$ by $\widetilde{\mathcal{K}}_{\mathfrak{m}}^{\bullet}$. Choose a projective embedding $\widetilde{X}_O \times_O \widetilde{X}_O \to \mathbf{P}_O^{N_1}$. Localizing O if necessary, we may assume that the Hilbert polynomials of the sheaves $\widetilde{\mathcal{K}}_{\mathfrak{m}}^{\bullet}$ do not depend on $\mathfrak{m} \subset O$ and coincide with the Hilbert polynomials of the sheaves $\widetilde{\mathcal{K}}^{\bullet}$.

We can now remove the restriction functor $L^{\bullet} i_y^*$ from (4.6). To do this, we apply again the Koszul resolution argument of Step 3, and conclude that it suffices to find an estimate of the form

$$\mathsf{dim}_k \mathbb{H}^{i-n}(\widetilde{X}_{\mathfrak{m}}^{(1)} \times \widetilde{X}_{\mathfrak{m}}^{(1)}, (\mathsf{Fr}_* \widetilde{\mathcal{K}}^{\bullet})(-n)) \leq M p^{\mathsf{dim}\, X} \tag{4.7}$$

But the left-hand side is equal to

$$\mathsf{dim}_k \mathbb{H}^{i-n}(\widetilde{X}_{\mathfrak{m}} \times \widetilde{X}_{\mathfrak{m}}, \widetilde{\mathcal{K}}^{\bullet}(-pn)),$$

so that (4.7) is a statement about the Hilbert polynomials of the sheaves $\widetilde{\mathcal{K}}^{\bullet}$ – which are by assumption the same as the Hilbert polynomials of the sheaves $\widetilde{\mathcal{K}}^{\bullet}$. We have to show that the degrees of these polynomials are not greater than $\mathsf{dim}\, X$. Indeed, $\mathcal{K}^{\bullet}$ is supported on $X \times_Y X \subset X \times X$, and choosing the extension $\widetilde{\mathcal{K}}^{\bullet}$ in an appropriate way, we can also insure that $\mathsf{dim}\, \mathsf{Supp}\, \widetilde{\mathcal{K}}^{\bullet} = \mathsf{dim}\, \mathsf{Supp}\, \mathcal{K}^{\bullet} \leq \mathsf{dim}\, X \times_Y X$. Since X/Y is a symplectic resolution, it is semismall by Theorem 1.9, in other words, $\mathsf{dim}\, X \times_Y X = \mathsf{dim}\, X$. Thus $\mathsf{dim}\, \mathsf{Supp}\, \widetilde{\mathcal{K}}^{\bullet} \leq \mathsf{dim}\, X$, which yields (4.7).

4.4 Artin approximation. Using Proposition 4.8, the proof of Theorem 4.2 proceeds as follows. Assume given a symplectic resolution X/Y over a field K of characteristic 0 and a point $y \in Y$. As in the last Subsection, we choose a subalgebra $O \subset K$ of finite type over $\mathbb{Z}$, schemes X_O/Y_O flat, smooth and of finite type over $\operatorname{Spec} O$, and an O-valued point $y_O : \operatorname{Spec} O \to Y_O$ such that $X_O \otimes_O K \cong X$, $Y_O \otimes_O K \cong Y$, $y_O \otimes_O K = y$, and X_O is projective over Y_O. For any maximal ideal $\mathfrak{m} \subset O$ we obtain by reduction schemes $X_{\mathfrak{m}}/Y_{\mathfrak{m}}$ and a point $y_{\mathfrak{m}} \in Y_{\mathfrak{m}}$. By Proposition 4.8, we can choose $\mathfrak{m} \subset O$ in such a way that $X_{\mathfrak{m}}$ admits a regular line in the sense of Definition 4.7. Explicitly, consider the point $y_{\mathfrak{m}}((h)) \in Y_{\mathfrak{m}}((h))$, let $\widehat{Y_{\mathfrak{m}}}$ be the completion of $Y_{\mathfrak{m}}((h))$ near $y_{\mathfrak{m}}((h))$, and let $\widehat{X_{\mathfrak{m}}} = \widehat{Y_{\mathfrak{m}}} \times_{Y_{\mathfrak{m}}} X_{\mathfrak{m}}$; then by Proposition 4.8 we have a vector bundle $\mathcal{E}_{\mathfrak{m}} = \mathcal{E}_a$ on $\widehat{X_{\mathfrak{m}}}$ which is a tilting generator.

Let $\widehat{Y_O}$ be the completion of $Y_O \otimes_O O((h))$ near $y_{\mathfrak{m}}((h))$, and let $\widehat{X_O} = \widehat{Y_O} \times_{Y_O} X_O$. Then $\widehat{X_O}$ is flat and smooth over the completion $\widehat{O}$ of the algebra $O((h))$ with respect to the maximal ideal $\mathfrak{m}((h))$, and the special fiber of $\widehat{X_O}/\widehat{X_O}$ is identified with $\widehat{X_{\mathfrak{m}}}$. Since the vector bundle $\mathcal{E}$ is tilting, it extends uniquely to $\widehat{X_O}$ considered as a a formal scheme – indeed, by standard deformation theory obstructions to this at each level of the adic filtration lie in $\operatorname{Ext}^2(\mathcal{E}_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}})$, the choices of extensions are parametrized by elements of $\operatorname{Ext}^1(\mathcal{E}_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}})$, and both groups are trivial. By [EGA, Théorème 5.4.5], the vector bundle $\mathcal{E}$ therefore extends to a vector bundle $\mathcal{E}_O$ over the actual scheme $\widehat{X_O}/\widehat{Y_O}$. By Nakayama Lemma the vector bundle $\mathcal{E}_O$ is also tilting, and the corresponding kernel $\mathcal{K}^\bullet(\widehat{X_O}, \mathcal{E}nd(\mathcal{E}_O))$ vanishes, so that it is a tilting generator.

By Artin Approximation Theorem [A, Theorem 1.10], there exists a subalgebra $O' \subset \widehat{O}$ of finite type over O, schemes $X_{O'}/Y_{O'}$ of finite type over O', and an O'-valued point $y_{O'} : \operatorname{Spec} O' \to Y_{O'}$ such that $X_{O'} \otimes_{O'} \widehat{O} \cong \widehat{X_O}$, $Y_{O'} \otimes_{O'} \widehat{O} \cong \widehat{Y_O}$, $y_{O'} \otimes_{O'} \widehat{O} \cong \widehat{y_O}$, $\mathfrak{X}_{O'}$ is projective over $Y_{O'}$, $\mathcal{E}_O$ by approximated to a high order by $\mathcal{E}_{O'} \otimes_{O'} \widehat{O}$ for a vector bundle $\mathcal{E}_{O'}$ on $X_{O'}$, and on the other hand, the natural maps $X_{O'} \to X_O \times_O O'$, $Y_{O'} \to Y_O \times_O O'$ are étale, and the second map sends $y_{O'}$ to $y_O \times_O O'$. Again by Nakayama Lemma, we note that if the order of approximation is high enough, then shrinking $Y_{O'}$ if necessary, we can guarantee that the vector bundle $\mathcal{E}_{O'}$ is a tilting generator for $X_{O'}$.

It remains to take a generic point $o \in \operatorname{Spec} O'$ whose residue field K' is a finite extension of our original field K, and notice that, possibly after shrinking $Y_{O'}$ even further, $Y_0 = Y_{O'} \otimes_{O'} K'$ is an étale neghborhood of the point $y \in Y$, and $\mathcal{E} = \mathcal{E}_{O'} \otimes_{O'} K'$ is a tilting generator for $X_0 = X_{O'} \otimes_{O'} K' = Y_0 \times_Y X$.

5 Geometric corollaries.

5.1 Additional results on derived equivalences. One unsatisfactory thing about Theorem 4.2 is the need to fix a point $y \in Y$ and pass to an étale neighborhood Y_0. Of course, one can cover the whole of Y with such étale neighborhoods, but at present, we do not know whether the tilting generators provided by Theorem 4.2 patch together. There very well might be an obstruction to this lying in the Brauer group $\mathsf{Br}(X)$. In practice, this problem is alleviated by the following additional result.

Theorem 5.1 ([K3, Theorem 1.8]). *In the assumptions of Theorem 4.2, assume in addition that Y admits a positive-weight $\mathbb{G}_m$-action. Then this action lifts canonically to a $\mathbb{G}_m$-action on X, and the tilting generator $\mathcal{E}$ provided by Theorem 4.2 extends to a $\mathbb{G}_m$-equivariant tilting generator on the whole of X.* □

In fact, Theorem 5.1 is valid for any tilting generator of the type provided by Theorem 4.2; where it came from is not relevant to the proof. In light of Conjecture 1.8, Theorem 5.1 is potentially very useful.

By general nonsense, the presence of a tilting generator yields strong restrictions on the topology of a resolution X, further extending those given in Theorem 1.9. Namely, we have the following.

Theorem 5.2 ([K3, Theorem 1.9]). *Assume that a smooth manifold X is projective over an affine local Henselian scheme Y/k and admits a tilting generator $\mathcal{E}$. Then the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$ admits a finite resolution by vector bundles of the form $\mathcal{E}_i \boxtimes \mathcal{F}_i$, where $\mathcal{E}_i$, $\mathcal{F}_i$ are some vector bundles on X.* □

Corollary 5.3 ([K3, Corollary 1.10]). *Assume that a smooth manifold X is projective over an affine scheme Y, and let $E \subset X$ be the fiber over a closed point $y \in Y$. Assume that Y admits a positive-weight $\mathbb{G}_m$-action that fixes $y \in Y$, and assume that X admits a tilting generator $\mathcal{E}$. Then the cohomology groups $H^\bullet(E)$ of the scheme E are generated by classes of algebraic cycles.* □

In this Corollary we are deliberately vague as to what particular cohomology groups $H^\bullet(E)$ one may take. In fact, every cohomology theory with the standard weight formalism will suffice; in particular, the statement is true for l-adic cohomology and for analytic cohomology when the base field k is $\mathbb{C}$. The proof is rather standard: one considers the identity endomorphism of the cohomology $H^{\bullet}(E)$ and, using Theorem 5.2, decomposes it as

$$\mathsf{Id}(a) = \sum \eta_i(a)[a_i], \tag{5.1}$$

where η_i are certain linear forms on $H^{\bullet}(E)$, and $[a_i]$ are classes of algebraic cycles. However, there is a complication – since the scheme X is not compact, the natural map from the cohomology $H^{\bullet}_c(X)$ with compact support to the usual cohomology $H^{\bullet}(X)$ is not at all an isomorphism, and the usual way to deduce (5.1) does not work. To overcome this difficulty, we have to require an existence of a $\mathbb{G}_m$-action and work with $\mathbb{G}_m$-equivariant cohomology. This seems much too strong; however, at present, we do not know whether Corollary 5.3 holds without the $\mathbb{G}_m$-action assumption.

Theorem 5.2 itself is a direct corollary of the equivalence

$$D^b_c(X) \cong D^b(R\text{-mod}^{\mathfrak{fg}}), \tag{5.2}$$

where $R = \operatorname{End}(\mathcal{E})$. The non-commutative algebra R is finite over its Henselian center, so that it has a finite number of indecomposable projective modules P_i. The equivalence (5.2) sends R itself to $\mathcal{E}$; every projective module P_i, being a direct summand of R^N for some N, goes to a vector bundle $\mathcal{F}_i$ on X, and it is these vector bundles that appear in the resolution of the diagonal.

The algebra R also has a finite number of irredicible modules; those go to some complexes of coherent sheaves on X supported near the exceptional fiber $E \subset X$. In fact, one can use the equivalence (5.2) to translate the standard t-structure on $D^b(R\text{-mod}^{\mathfrak{fg}})$ to a rather unusual t-structure on $D^b_c(X)$ – it is this "perverse" t-structure on the category of coherent sheaves on X that was discovered in [Br] in $\mathsf{dim}\,3$. The perverse t-structure is Artinian and Noetherian. Its irreducible objects provide a canonical basis in the K-group $K_0'(X)$. It would be very interesting to compute this basis in various particular cases, such as the quiver variety case (Example 1.8).

We note that in our construction of the tilting generator $\mathcal{E}$, there are three choices: we have to choose a maximal ideal $\mathfrak{m} \subset O$ with residue field $k = O/\mathfrak{m}$ of positive characteristic $\mathsf{char}\, k = p$, an ample line bundle L on X, and a regular value $a \in \mathbb{Z}/p\mathbb{Z}$. Since by construction, $\mathcal{E}$ is a vector bundle of rank $p^{\mathsf{dim}\, X}$, it obviously depends at least on the residual characteristic p. However, we venture the following.

Conjecture 5.4. *In the assumption of Theorem 5.2, the perverse t-structure induced on the derived category $D^b_c(X)$ is the same, up to a twist by an autoequivalence of $D^b_c(X)$, for almost all maximal ideals $\mathfrak{m} \subset O$, ample line bundles L on X, and regular values $a \in \mathbb{Z}/p\mathbb{Z}$.*

"Almost all" here means, hopefully, "all but a finite number". Unfortunately, our methods do not yield an easy way to compare the results for different values $a \in \mathbb{Z}/p\mathbb{Z}$, and comparison between different maximal ideals $\mathfrak{m} \subset O$ seems to be completely out of reach.

In the simplest possible example $X = T^*\mathbb{P}^1$, the cotangent bundle to $\mathbb{P}^1$, one can follow through the proof of Theorem 4.2 in an effective way, with the following end result:

$$\mathcal{E}_a \cong \mathcal{O}_X^{\oplus a} \oplus \mathcal{O}_X(1)^{\oplus(p-a)},$$

where $\mathcal{O}_X(1)$ is the pullback of the standard line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ with respect to the projection $X = T^*\mathbb{P}^1 \to \mathbb{P}^1$. Thus every $a \neq 0$ is regular, and the tilting generator $\mathcal{E}_a$ is the sum of two vector bundles $\mathcal{O}_X$, $\mathcal{O}_X(1)$ with different multiplicities depending on $\mathfrak{m} \subset O$ and $a \in \mathbb{Z}/p\mathbb{Z}$. It is easy to see that Conjecture 5.4 is true in this case, with all the tilting generators giving the same t-structure as $\mathcal{O}_X \oplus \mathcal{O}_X(1)$ (which is also a tilting generator, and in a sense, the smallest possible one). We expect that in general, the situation is the same: there is a finite number of indecomposable vector bundles $\mathcal{E}_i$ which generate the t-structure, and all the tilting generators $\mathcal{E}_a$ are obtained by summing up the bundles $\mathcal{E}_i$ with multiplicities depending on $\mathfrak{m} \subset O$, L, and $a \in \mathbb{Z}/p\mathbb{Z}$.

Remark 5.5. There is in fact one more choice in the proof of Theorem 4.2 which we tacitly ignore in the above discussion: when we represent the matrix algebra $\mathcal{O}_a(h^{-1})$ as an endomorphism algebra

$\mathcal{E}nd(\mathcal{E}_a)$, the vector bundle $\mathcal{E}_a$ is only defined by up to a twist by a line bundle. The "twist by an autoequivalence" clause in Conjecture 5.4 is needed to take care of this. To be on the safe side, we do not require this autoequivalence to be a twist by a line bundle. In general, it would be very interesting to study the group of all autoequivalences of the triangulated category $D^b_c(X)$ and its action on various perverse t-structures; however, at present there seems to be no way to do this, at least in the interesting case $\operatorname{dim} X > 2$.

One additional observation is the following.

Proposition 5.6. *In the assumptions of Theorem* 4.2*, every tilting generator $\mathcal{E}$ on X extends uniquely to a tilting generator $\widetilde{\mathcal{E}}$ on the universal deformation $\mathfrak{X}$ provided by Theorem* 2.1*.*

Proof. Standard deformations theory: since $\mathcal{E}$ is tilting, $\operatorname{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for $i = 1, 2$; thus there are no obstructions to deforming it together with X, and no choices are involved in such a deformation. By Nakayam Lemma, the deformed vector bundle $\widetilde{\mathcal{E}}$ is also a tilting generator. □

Thus Theorem 5.1, Theorem 5.2, Corollary 5.3, and all the above discussion apply just as well to the scheme $\mathfrak{X}$.

Remark 5.7. In the case $\operatorname{dim} X = 3$, K_X trivial studied in [Br] and [V1], it is known that Y, being a terminal singularity, must be a so-called cDV point, and the whole X/Y is a one-parameter deformation of a partial resolution X_0/Y_0 of a Du Val quotient singularity $Y_0 = \mathbb{A}^2/G$, $G \subset SL(2, K)$. However, X_0 is usually singular – it is only the total space X of the deformation that is smooth. Thus X/Y is not really of the form $\mathfrak{X}/\mathcal{Y}$ for some 2-dimensional symplectic resolution, and our methods do not apply. It would be very interesting to try to generalize our approach to this situation and compare it with [V1].

Finally, there is a result which compares the derived categories $D^b_c(X)$ for different crepant resultions of the same symplectic singularity Y. This is a generalization of the particular case of [BO1, Section 3, Conjecture] proved by Y. Kawamata in [Ka]: in Kawamata's language, "K-equivalence implies D-equivalence".

Theorem 5.8 ([K3, Theorem 1.6]). *In the assumptions of Theorem* 4.2*, assume given a different resolution X', $\pi' : X \to Y$ of the variety Y with trivial canonical bundle $K_{X'}$. Then for every closed point $y \in Y$, there exists en étale neighborhood $Y_y \to Y$ such that the derived categories $D^b_c(X \times_Y Y_y)$ and $D^b_x(X' \times_Y Y_y)$ are equivalent.*

Sketch of the proof. One checks easily that since $K_{X'}$ is trivial, the resolution X' must also be symplectic. Since a symplectic resolution in $\operatorname{dim} 2$ admits a *unique* symplectic resolution, the rational map $X \dashrightarrow X'$ induces an isomorphism $X_0 = \pi^{-1}(Y_0) \cong X'_0 = (\pi')^{-1}(Y_0)$, where the open subset $Y_0 \subset Y$ is the union of the strata of dimensions $\operatorname{dim} Y$ and $\operatorname{dim} Y - 2$ with respect to the stratification of Theorem 1.6.

Going through the proof of Theorem 4.2, we choose an ample line bundle L on X and obtain a tilting generator $\mathcal{E}_a$ for $X \times_Y Y_y$; repeating the same argument for X' equipped with the strict transform L' of the line bundle L, and possibly changing Y_y, we obtain a tilting vector bundle $\mathcal{E}'_a$ on $X' \times_Y Y_y$.

We can *not* claim that $\mathcal{E}'_a$ is a tilting generator: indeed, unless $X \cong X'$, the line bundle L' is not ample on X'. However, since X'/Y is semismall, the complements $X \setminus X_0 \subset X$, $X' \setminus X'_0 \subset X'$ are of codimension at least 2. Moreover, $H^i(X_0, \mathcal{O}_{X_0}) = 0$ for $i = 1, 2$, and, analysing the proof of Theorem 3.15, we conclude that the quantizations used in the construction of the tilting bundles $\mathcal{E}_1$, $\mathcal{E}'_a$ agree on X_0. Therefore $\mathcal{E}_a \cong \mathcal{E}'_a$ on $X_0 \times_Y Y_y$. Again, since the complement to X_0 is of high codimension both in X and X', we conclude that the algebra $R = \operatorname{End}(\mathcal{E}_a)$ is isomorphic to $R' = \operatorname{End}(\mathcal{E}'_a)$.

In particular, the algebra R' has finite homological dimension, so that the natural functor $D^-(R'\text{-mod}^{\mathrm{fg}}) \to D^-_c(X')$ induces a functor

$$D^b(R'\text{-mod}^{\mathrm{fg}}) \to D^b_c(X').$$

Since $\mathcal{E}'_a$ is tilting, this functor is a fully faithfull embedding with admissible image in the sense of [BO2]. To finish the proof, it suffices to use the following standard trick.

Lemma 5.9. *Assume given an irreducible smooth variety X with trivial canonical bundle K_X equipped with a birational projective map $\pi : X \to Y$ to an affine variety Y. Then any non-trivial admissible full triangulated subcategory in $D^b(X)$ coincides with the whole $D^b(X)$.*

For the proof we refer the reader, for instance, to [BK2, Section 2]. □

We note that in general, Lemma 5.9 gives a quick and easy way to prove that a tilting vector bundle $\mathcal{E}$ is a generator, avoiding all the difficult estimates of Proposition 4.8. However, in order to apply it, one need to know that the algebra $R = \operatorname{End}(\mathcal{E})$ has finite homological dimension. It seems that in general, there is no way to prove it short of proving that $\mathcal{E}$ is a generator.

One notable expection to this is the quotient singularity case $Y = V/G$ considered in [BK2] (this is our Example 1.4). In this case, using a more detailed analysis of quantizations, one shows that there exists a tilting vector bundle $\mathcal{E}$ on X such that $\operatorname{End}(\mathcal{E}) \cong S^\bullet(V^*)\#G$, the smash-product of the algebra of polynomial functions on V and the group algebra of the group G. This algebra obviously has finite homological dimension; therefore $\mathcal{E}$ is a generator. No version of Proposition 4.8 is required, and the description of $D^b_c(X)$ is more explicit than in the general case.

Remark 5.10. One thing that was not done in [BK2] is the analysis of the deformed tilting generator $\widetilde{\mathcal{E}}$ provided by Proposition 5.6. The endomorphism algebra $\widetilde{R} = \operatorname{End}(\widetilde{\mathcal{E}})$ is a flat deformation of the endomorphism algebra $\operatorname{End}(\mathcal{E}) \cong S^\bullet(V^*)\#G$. One expects that $\widetilde{R}$ coincides with the so-called *symplectic reflection algebra* introduced in [EG], but this has never been verified expect in some special cases, see [Go].

5.2 Comparison with quantum groups. To finish the paper, we would like to return to the starting point mentioned in the Introduction and give some speculations on the connections of the present work with Geometric Representation Theory.

The motivation for the research carried out in [BK2], [BK3], [K3] was the paper [BMR], where the authors study the case $X = T^*M$, $M = G/P$, a partial flag variety associated to a semisimple algebraic group G and a parabolic subgroup $P \subset G$. In that case, a particular series of quantizationa of the cotangent bundle $X = T^*M$ is given from the very beginning – one can consider the algebras $\mathcal{D}_{M,L}$ of differential operators on M twisted by a line bundle L. The classic result of A. Beilinson and J. Bernstein [BB] claims that in characteristic 0, the partial flag variety $M = G/P$ is *D-affine* for generic L – that is, the category of sheaves of $\mathcal{D}_{M,L}$-modules is equivalent to the category of modules over the algebra $H^0(M, \mathcal{D}_{M,L})$ of global sections of the sheaf $\mathcal{D}_{M,L}$. In positive characteristic, the statement is no longer true; however, and it has been proved in [BMR], the equivalence does survive on the level of the derived categories: the natural global sections functor induces an equivalence between the derived categories $D^b_c(M, \mathcal{D}_{M,L})$ and $D^b(H^0(M, \mathcal{D}_{M,L})\text{-mod}^{\mathrm{fg}})$. Moreover, the algebra $\mathcal{D}_{M,L}$ acquires a large center, so that sheaves of $\mathcal{D}_{M,L}$-modules on M can be localized to sheaves on $X^{(1)}$. As in our Theorem 4.2, the equivalence can then be lifted back to characteristic 0; the resulting algebra of global sections is closely related to the so-called quantum envelopping algebra at a p-th root of unity.

In general, one can use the sheaves $\mathcal{D}_{M,L}$ for the cotangent bundle T^*M of any algebraic variety M, but this is not expected to be very useful – indeed, the derived D-affine property in positive characteristic would in particular imply that $X = T^*M$ satisfies the assumptions of Conjecture 1.3, so that we are automatically in the situation of [BMR]. Therefore in order to generalize [BMR] to other interesting situations, one has to develop a geometric quantization machinery as in [BK1], [BK3]. One can in fact hope to generalize [BB] as well – the following has been conjectured in [K3].

Conjecture 5.11. *In the assumptions of Theorem* 2.3*, the global sections functor*

$$\mathrm{Shv}(\mathfrak{X}, \mathcal{O}_h) \to \mathrm{Shv}(S[[h]], \pi_*\mathcal{O}_h)$$

from sheaves of finitely generated $\mathcal{O}_h$-modules on $\mathfrak{X}$ to sheaves of finitely generated $\pi_*\mathcal{O}_h$-modules on $S[[h]] = S\widehat{\times}\operatorname{Spec} K[[h]]$ is an equivalence of abelian categories over a dense open subset $U \subset S[[h]]$.

However, from the present perspective, another relation to Geometric Representation Theory seems more promising.

Let us summarize once more the main steps in the construction of a tilting generator on a symplectic resolution X.

(i) We reduce X to a smooth symplectic variety $X_{\mathfrak{m}}$ over a perfect field k of positive characteristic $\mathsf{char}\, k = p > 0$.

(ii) Using quantization theory, we deform the Frobenius map

$$\sigma : \mathcal{O}_{X^{(1)}_{\mathfrak{m}}} \to \mathsf{Fr}_*\mathcal{O}_X$$

to a central algebra map

$$s : \mathcal{O}_{X^{(1)}_{\mathfrak{m}}} \to \mathcal{O}_h.$$

(iii) Using rigidity of tilting vector bundles, we lift the map s to a central algebra map

$$\mathcal{O}_X \to \mathcal{R}, \tag{5.3}$$

where $\mathcal{R} = \mathcal{E}nd(\mathcal{E})$ is a matrix algebra sheaf on X.

In principle, a similar procedure can be applied to a smooth variety X which is only Poisson, not symplectic. The problem is, steps (ii) and (iii) require some rigidity, and one cannot expect them to work nearly as well for arbitrary Poisson varieties. In particular, our approach to quantization is essentially that of Fedosov [F], and it is based on the fact that locally, all symplectic manifolds and all quantizations are the same – this of course breaks down completely in the general Poisson case.

However, there is one more situation where quantization works really well, namely, the case of a semisimple Lie group with a Poisson-Lie structure (see [Dr]). In this case, the necessary rigidity is provided by the fact that G is a group – a quantization becomes essentially unique if one requires it to be compatible with the group structure (see, in particular, [EK]). Motivated by this, we expect, roughly, the following picture in the Lie group case.

> *Assume given a semisimple Lie group G over a field K of characteristic 0. Then the standard Poisson-Lie structure on G canonically extends to a model G_O of the group G over a subalgebra $O \subset K$ of finite type over $\mathbb{Z}$, so that for any maximal ideal $\mathfrak{m} \subset O$, we obtain a Poisson-Lie group $G_{\mathfrak{m}}$ over a finite field $k = O/\mathfrak{m}$. The Poisson-Lie group $\mathcal{G}_{\mathfrak{m}}$ admits a unique Frobenius-constant quantization compatible with the group structure. Moreover, the quantization structure sheaf $\mathcal{O}_h$ on $G^{(1)}_{\mathfrak{m}}$ lifts uniquely to a algebra sheaf $\mathcal{O}_q$ of finite rank on G which is, again, compatible with the group structure.*

This is much too imprecise to be stated even as a Conjecture. In particular, one has to clarify the exact meaning of "compatibility with the group structure" – we expect that it should not be difficult to do this, but at present, this has not been done. In addition, one cannot expect $\mathcal{O}_h(h^{-1})$ to be a matrix algebra, so that step (iii) – lifting to characteristic 0 – will not be automatic, and probably requires the same methods as step (ii). In spite of all this, we can guess what the final result will be – that is, what is the algebra sheaf $\mathcal{O}_q$.

Namely, recall that G. Lusztig – see, e.g., [Lu] – has found a particular form U_q of the quantized enveloping algebra U_h associated in [Dr] to a semisimple Lie group G. The algebra U_q is defined over a much smaller subalgebra $K[q, q^{-1}] \subset K[[h]]$ in the algebra of formal power series in $h = \log q$. Therefore one can actually assign some value to the parameter q. It is known that the resulting algebra is especially interesting when q is a root of 1. In this case, G. Lusztig constructs in addition a so-called *quantum Frobenius map* – an algebra map $U_q \to U$ from U_q to the usual universal enveloping algebra U associated to the group G. The dual picture has been also studied, for instance in [CP]. There instead of quantized enveloping algebra U_q one considers a quantum version $\mathcal{O}_{G,q}$ of the algebra $\mathcal{O}_G$ of algebraic functions on G; if q is a root of 1, one obtains a quantum Frobenius map

$$\mathcal{O}_G \to \mathcal{O}_{G,q},$$

so that $\mathcal{O}_{G,q}$ becomes a sheaf of associative algebras on the group G. This is what we expect our sheaf $\mathcal{O}_q$ to be, for $q = \exp \frac{2\pi\sqrt{-1}}{p}$. The map (5.3) should be the quantum Frobenius map.

Unlike [Lu] and consequently [CP], where U_q and $\mathcal{O}_{G,q}$ are constructed by explicit formulas, it should be possible to obtain $\mathcal{O}_q$ by pure deformation theory, as an essentially unique solution to a deformation problem. We do not know whether it has any real significance for the theory of quantum groups, a subject very well studied already; still, a conceptual explanation of the formulas in [Lu] may be worth trying for.

Conversely, the algebras $R = \mathcal{E}nd(\mathcal{E})$ constructed in Theorem 4.2 should be related to quantum group theory, at least in the cases like Example 1.6 when the symplectic resolution X is related to a semisimple group G. When X is *not* directly related to any group, one could still hope to find in the algebras R some of the rich additional structures known for quantum groups, such as e.g. the so-called *crystal bases*. From this point of view, the most promising case is perhaps Example 1.8, the quiver variety case. Since a quiver variety X is given by a very explicit set of combinatorial data, computing the algebra R explicitly is not perhaps quite out of reach.

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