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Hochschild cohomology of quasiprojective schemes[1]

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Abstract

Three definitions for the Hochschild cohomology of schemes are considered and shown to coincide for quasiprojective schemes. In the smooth case, the associated Hodge spectral sequences are also shown to be isomorphic.

Keywords: Hochschild cohomology; Quasiprojective schemes; Hypercohomology

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Following an idea of Grothendieck [10], Loday [13, 3.4] suggested defining the cyclic homology of schemes by sheafifying the standard complex and taking hyperhomology. This idea has been further developed in [6, Section 4], where a similar definition is also given for Hochschild homology, and in [19]. A closely related definition for the Hochschild cohomology of schemes was given by Gerstenhaber and Schack [8] in terms of a categorical cohomology theory.

For the case of Hochschild cohomology, however, there is a much simpler definition. Recall that if A is an algebra over a field k, the Hochschild cohomology of A is defined to be $H^n(A,M) = \mathrm{Ext}^n_{A^e}(A,M)$ where $A^e = A \otimes_k A^{\mathrm{op}}$ and M is an A-bimodule. We will be interested in the case where A is commutative (so $A^e = A \otimes_k A$) and M is an A-module, i.e. $am = ma$ for $a \in A$, $m \in M$. If X is a separated scheme over k, we can just define the Hochschild cohomology of X, by analogy with the above definition, to be $H^n(\mathcal{O}_X, \mathscr{F}) = \mathrm{Ext}^n_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathscr{F})$ where $\mathscr{F}$ is a sheaf of $\mathcal{O}_{X\times X}$-modules. As above, we will mainly be interested in the case where $\mathscr{F}$ is a sheaf of $\mathcal{O}_X$-modules which is regarded as a sheaf on $X \times X$ in the usual way by identifying $\mathscr{F}$ with $\delta_*\mathscr{F}$ where $\delta: X \to X \times X$ is the diagonal map. This definition was also studied independently by M. Kontsevich.

One of the main objects of this paper is to show that this definition of Hochschild cohomology agrees, for quasiprojective schemes over a field, with the one defined by

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Gerstenhaber and Schack and the one defined by the Grothendieck–Loday method. As a consequence of this we can show that the cyclic cohomology of a projective scheme over a field with coefficients in a coherent sheaf is finite dimensional.

For each definition of Hochschild cohomology, there is a Hodge spectral sequence which, in the smooth case, relates the Hochschild cohomology to the Hodge cohomology of the variety. I will also show that for smooth quasiprojective schemes over a field, the two Hodge spectral sequences coincide. The Gerstenhaber–Schack Hodge theory [7] applies to their definition to show that the Hodge spectral sequence of a smooth projective variety over $\mathbb{C}$ degenerates to the Gerstenhaber–Schack Hodge decomposition of the Hochschild cohomology. It follows that the same is true of the Hodge spectral sequence corresponding to the elementary definition above. This provides a partial answer to the problem of giving a geometric interpretation of this Hodge decomposition. The search for such an interpretation was the original motivation for the work described here. This result was also obtained independently by Kontsevich by a completely different method. See the remarks preceding Corollary 2.6.

In a subsequent paper, I will show how to define Chern classes in Hochschild cohomology which, for smooth projective varieties over $\mathbb{C}$, coincide with the usual topological Chern classes under the isomorphism obtained by comparing the Gerstenhaber–Schack Hodge decomposition with the classical one.

1. Hochschild cohomology

Let X be a separated scheme of finite type over a field k. As indicated above, we define the Hochschild cohomology of X with coefficients in a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules to be $H^n(\mathcal{O}_X, \mathscr{F}) = \operatorname{Ext}^n_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathscr{F})$.

The composition of functors $\operatorname{Hom}_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, -) = \Gamma \circ \underset{\sim}{\operatorname{hom}}_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, -)$ leads to the usual spectral sequence [9]

$$E_2^{pq} = H^p(X \times X,\ \underset{\sim}{\operatorname{ext}}^q_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathscr{F})) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathscr{F}) = H^{p+q}(\mathcal{O}_X, \mathscr{F}).$$

Since $\underset{\sim}{\operatorname{ext}}^q_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathscr{F})$ is supported on the diagonal, this can be rewritten as

$$E_2^{pq} = H^p(X,\ \underset{\sim}{\operatorname{ext}}^q_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathscr{F})) \Rightarrow H^{p+q}(\mathcal{O}_X, \mathscr{F}), \tag{1.1}$$

which I will refer to as the Hodge spectral sequence for Hochschild cohomology since in the smooth case, following [8], we can rewrite it in a form analogous to the usual Hodge spectral sequence

$$E_1^{pq} = H^p(X, \Omega^q) \Rightarrow H^{p+q}_{dR}(X, \mathbb{C}) \tag{1.2}$$

of a complex manifold. This is done as follows.

If $\mathscr{F}$ is locally free, we have $\underset{\sim}{\operatorname{ext}}^q_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathscr{F}) \approx \underset{\sim}{\operatorname{ext}}^q_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathcal{O}_X) \otimes \mathscr{F}$.

If X is smooth, $\underset{\sim}{\mathrm{ext}}^q_{\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathcal{O}_X) \approx \bigwedge^q \mathcal{T}$ where $\mathcal{T}$ is the tangent bundle of X [12]. Let $\mathcal{K} = \Omega^d_{X/k}$ where $d = \dim X$. Since $\Omega^q \otimes \Omega^{d-q} \to \Omega^d = \mathcal{K}$ and $\Omega^q \otimes \bigwedge^q \mathcal{T} \to \mathcal{O}_X$ are dual pairings, we have $\mathcal{K} \otimes \bigwedge^q \mathcal{T} \approx \Omega^{d-q}$ so, replacing $\mathcal{F}$ by $\mathcal{K} \otimes \mathcal{F}$ in the spectral sequence (1.1), we get

$$E_2^{pq} = H^p(X, \Omega_X^{d-q} \otimes \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{O}_X, \mathcal{K} \otimes \mathcal{F}). \tag{1.3}$$

Except for the indexing and the presence of $\mathcal{K}$, this resembles the usual Hodge spectral sequence (1.2).

For a smooth projective variety over $\mathbb{C}$, ordinary Hodge theory shows that (1.2) degenerates to give the Hodge decomposition $H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^p(X, \Omega^q)$. A similar result for (1.3) will be proved in Section 2.

2. The hyperext definition

Recall that if A is an algebra over a field k, we define $A^e = A \otimes_k A$ and $A^e \xrightarrow{\varepsilon} A$ by $a \otimes b \mapsto ab$. The bar construction on A is defined to be $B_n(A) = A \otimes_k A^{\otimes n} \otimes_k A$. It is an A^e-module and $B_\bullet(A) \xrightarrow{\varepsilon} A$ is a free A^e-resolution of A as an A^e-module so that the usual Hochschild cohomology is given by $H^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M) = H^n(\mathrm{Hom}_{A^e}(B_\bullet(A), M))$ [16].

Remark. If k is not a field, $B_\bullet(A)$ is only relatively projective over A^e so that Hochschild cohomology should be defined as a relative Ext in this more general case (see e.g. [8]).

If M is an A-module, then $\mathrm{Hom}_{A^e}(B_\bullet(A), M) = \mathrm{Hom}_A(C_\bullet(A), M)$ where $C_\bullet(A) = A \otimes_{A^e} B_\bullet(A)$ so we have $H^n(A, M) = H^n(\mathrm{Hom}_A(C_\bullet(A), M))$. Note that $C_n(A) = A \otimes_k A^{\otimes n}$ and $C_\bullet(A)$ is a chain complex of A-modules. It is, of course, no longer acyclic.

For our purposes, the following definition of the hyperext will suffice.

Definition. Let $\mathcal{A}_\bullet$ be a chain complex of sheaves of $\mathcal{O}$-modules which is bounded below, i.e. $\mathcal{A}_n = 0$ for $n \ll 0$. Let $\mathcal{G}$ be an $\mathcal{O}$-module, choose an injective resolution $0 \to \mathcal{G} \to \mathcal{I}^\bullet$, and define $\mathbb{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathcal{A}_\bullet, \mathcal{G}) = H^n(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}_\bullet, \mathcal{I}^\bullet))$.

Standard arguments of homological algebra show that this is well defined and is an exact δ-functor in $\mathcal{A}_\bullet$ and in $\mathcal{G}$. The more general hyperext in which $\mathcal{G}$ is also a complex will not be needed here.

Let X be a scheme over k. Define a presheaf on X by letting $C_\bullet(U) = C_\bullet(\Gamma(U, \mathcal{O}_X))$. Let $\mathcal{C}_\bullet$ be the associated sheaf. It is a sheaf of modules over $\mathcal{O}_X$. The Grothendieck–Loday type definition of the Hochschild cohomology can now be formulated as follows.

Definition. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define $HH^n(X,\mathscr{F}) = \mathbb{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathscr{C}_\bullet,\mathscr{F})$.

The following is one of the main results of this paper. It will be proved in Section 10.

Theorem 2.1. *Let X be a quasiprojective scheme over a field k. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there is a natural isomorphism of δ-functors $H^n(\mathcal{O}_X,\mathscr{F}) \approx HH^n(X,\mathscr{F})$.*

A similar definition can be given for cyclic cohomology. Let $D_\bullet(A)$ be the total complex of Connes double complex [15]. As above we sheafify $D_\bullet(A)$ getting a complex of sheaves $\mathscr{D}_\bullet$ and define $HC^n(X,\mathscr{F}) = \mathbb{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathscr{D}_\bullet,\mathscr{F})$. One has the usual exact sequence $0 \to C_\bullet(A) \to D_\bullet(A) \to D_\bullet(A)[-2] \to 0$ leading to $0 \to \mathscr{C}_\bullet \to \mathscr{D}_\bullet \to \mathscr{D}_\bullet[-2] \to 0$ and therefore to Connes' exact sequence.

$$\cdots \to HH^{n-1}(X,\mathscr{F}) \to HC^{n-2}(X,\mathscr{F}) \to HC^n(X,\mathscr{F}) \to HH^n(X,\mathscr{F}) \to \cdots \quad (2.1)$$

Corollary 2.2. *Let X be a projective scheme over a field k. Let $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then for all n, $HH^n(X,\mathscr{F})$ and $HC^n(X,\mathscr{F})$ are finite dimensional over k.*

For $HH^n(X,\mathscr{F})$, this follows from Theorem 2.1 since $H^n(\mathcal{O}_X,\mathscr{F})$ is clearly finite dimensional. It then follows for $HC^n(X,\mathscr{F})$ by induction on n using (2.1).

Remark. Note that HH^n and HC^n are 0 for $n < 0$ in contrast to the homology groups HH_n and HC_n [20, 1.1].

We can also define a Hodge spectral sequence associated to $HH^n(X,\mathscr{F})$. The construction can be done quite generally as follows.

Lemma 2.3. *Let $\mathscr{A}_\bullet$ be a chain complex of sheaves of $\mathcal{O}$-modules which is bounded below. Then there is a spectral sequence*

$$E_2^{pq} = \mathrm{Ext}^p_{\mathcal{O}}(H_q(\mathscr{A}_\bullet),\mathscr{G}) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathcal{O}}(\mathscr{A}_\bullet,\mathscr{G}).$$

We need only filter $\mathrm{Hom}(\mathscr{A}_\bullet,\mathscr{I}^\bullet)$ by the degree of $\mathscr{I}^\bullet$. I will refer to this spectral sequence as the hyperext sequence. In particular, we can take $\mathscr{A}_\bullet = \mathscr{C}_\bullet$. Defining $\mathscr{H}_q = H_q(\mathscr{C}_\bullet)$, we get the Hodge spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathscr{H}_q,\mathscr{F}) \Rightarrow HH^{p+q}(X,\mathscr{F}). \quad (2.2)$$

One of the difficulties in dealing with the complex of sheaves $\mathscr{C}_\bullet$ is that it is not quasicoherent in general. There is no problem, however, with $\mathscr{H}_\bullet$. We first recall some standard facts concerning Hochschild homology.

It is well known that Hochschild homology commutes with localization [2, 5]. More generally, if $A \to B$ is flat, then $B \otimes_A H_n(A,M) \xrightarrow{\approx} H_n(B, B \otimes_A M)$ for any A-module M [14, 1.1.17; 18, 9.1.8]. This follows from the fact that Tor commutes with flat base change since $A^e \to B^e$ is also flat. In particular, this applies if Spec $A \subset$ Spec B is an open embedding of affine schemes.

Write $HH_n(A)$ for the Hochschild homology $H_n(A, A)$. Most of the following lemma is contained in [2, Cor. 1; 6, 0.4; II.8].

Lemma 2.4. (1) *The sheaves* $\mathscr{H}_q$ *are coherent.*

(2) $\Gamma(U, \mathscr{H}_q) = HH_q(\Gamma(U, \mathscr{O}_X))$ *if* U *is affine.*

(3) *If* X *is smooth,* $\mathscr{H}_q \approx \Omega^q$ *for all* q.

(4) *If* $X =$ Spec A, *then* $\delta^* \mathscr{B}_\bullet \xrightarrow{\simeq} \mathscr{C}_\bullet$ *is a homology equivalence where* $\mathscr{B}_\bullet$ *is the sheaf on* $X \times X$ *defined by* $B_\bullet(A)$.

Proof. The sheaf $\mathscr{C}_q$ is associated to the presheaf $U \mapsto C_q(\Gamma(U, \mathscr{O}_X))$ so $\mathscr{H}_q$ is associated to the presheaf $U \mapsto HH_q(\Gamma(U, \mathscr{O}_X))$. Suppose that $X =$ Spec A is affine. The sets $X_s =$ Spec A_s are a base of open sets and $HH_q(\Gamma(X_s, \mathscr{O}_X)) = HH_q(A_s) = HH_q(A)_s$ by Corollary 2.8. Therefore $\mathscr{H}_q$ is the sheaf corresponding to the A-module $HH_q(A)$. This proves (2). Also $\mathrm{Tor}_q^{A^e}(A, A)$ is a finitely generated module over A^e and therefore over A which proves (1). Suppose now that X is smooth. If $X =$ Spec A is affine, then by [12], we have $HH_q(A) = \bigwedge^q HH_1(A)$ and $HH_1(A) = \Omega^1$. It follows that, for any affine open set U of X, $\mathscr{H}_q | U = \Omega^q | U$. Since these isomorphisms are natural, they patch to give the isomorphism of (3). Finally for (4), note that $A_s \otimes_{A_s^e} B_\bullet(A_s) = C_\bullet(A_s)$ so that $H_q(\delta^* \mathscr{B}_\bullet) \to H_q(\mathscr{C}_\bullet)$ is induced by the maps $H_q(C_\bullet(A)_s) \xrightarrow{\approx} H_q(C_\bullet(A_s))$. $\square$

We therefore get the following Hodge spectral sequence for smooth varieties.

$$E_2^{pq} = \mathrm{Ext}^p_{\mathscr{O}_X}(\Omega^q, \mathscr{F}) \Rightarrow HH^{p+q}(X, \mathscr{F}). \tag{2.3}$$

Since X is smooth, Ω^q is locally free so $\underset{\sim}{\mathrm{ext}}^j_{\mathscr{O}_X}(\Omega^q, \mathscr{F}) = 0$ for $j > 0$ and the spectral sequence $E_2^{ij} = H^i(X, \underset{\sim}{\mathrm{ext}}^j_{\mathscr{O}_X}(\Omega^q, \mathscr{F})) \Rightarrow \mathrm{Ext}^{i+j}_{\mathscr{O}_X}(\Omega^q, \mathscr{F})$ reduces to an isomorphism $\mathrm{Ext}^p_{\mathscr{O}_X}(\Omega^q, \mathscr{F}) = H^p(X, \underset{\sim}{\mathrm{hom}}_{\mathscr{O}_X}(\Omega^q, \mathscr{F}))$. Now $\underset{\sim}{\mathrm{hom}}_{\mathscr{O}_X}(\Omega^q, \mathscr{F}) = \check{\Omega}^q \otimes \mathscr{F}$ where we set $\check{\Omega} = \underset{\sim}{\mathrm{hom}}_{\mathscr{O}_X}(\Omega^q, \mathscr{O}_X)$. As in Section 1, the dual pairing $\Omega^q \otimes \Omega^{d-q} \to \mathscr{K}$ shows that $\check{\Omega}^q \otimes \mathscr{F} \approx \Omega^{d-q} \otimes \check{\mathscr{K}} \otimes \mathscr{F}$ so, after substituting $\mathscr{F} \otimes \mathscr{K}$ for $\mathscr{F}$, we get the spectral sequence

$$E_2^{pq} = H^p(X, \Omega^{d-q} \otimes \mathscr{F}) \Rightarrow HH^{p+q}(X, \mathscr{K} \otimes \mathscr{F}) \tag{2.4}$$

with the same E_2 and E_∞ terms as (1.3). This leads to our second main result which will also be proved in Section 10.

Theorem 2.5. *If* X *is a smooth quasiprojective variety over a field* k, *the spectral sequences* (1.3) *and* (2.4) *are naturally isomorphic. More generally, the spectral sequences* (1.1) *and* (2.2) *are isomorphic if the sheaves* $\mathscr{H}_q$ *are all locally free.*

Remark. The local freeness condition cannot be omitted here. For example, if X is affine, the E_2 term of (1.1) is $H^p(X, \underset{\sim}{\mathrm{ext}}^q_{\mathscr{O}_{X \times X}}(\mathscr{O}_X, \mathscr{F}))$ which is 0 for $p \neq 0$, while that of (2.2) is $\mathrm{Ext}^p_{\mathscr{O}_X}(\mathscr{H}_q, \mathscr{F})$ which will only be 0 for all $\mathscr{F}$ and all $p \neq 0$ if $\mathscr{H}_q$ is locally free.

The following result was also proved independently by Kontsevich using quite different methods. He does not reduce to the theorem of Gerstenhaber and Schack but instead gives a direct proof by taking the formal completion of $X \times X$ along the diagonal and using complexes of sheaves of differential operators to compute Hochschild cohomology. His method has the advantage of not requiring X to be quasi-projective.

Corollary 2.6. *Let X be a smooth quasiprojective variety over $\mathbb{C}$. Then the spectral sequence* (1.3) *degenerates to give the canonical Hodge decomposition of Gerstenhaber and Schack,*

$$H^n(\mathcal{O}_X, \mathcal{K} \otimes \mathcal{F}) = \bigoplus_{p+q=n} H^p(X, \Omega^{d-q} \otimes \mathcal{F}).$$

Proof. It is sufficient to prove this for (2.4). Gerstenhaber and Schack [7] have shown that there is a Hodge decomposition for Hochschild cohomology in a characteristic 0. This is obtained from a natural decomposition of the complex $C_\bullet(A)$ into a direct sum of subcomplexes $C_\bullet(A) = \bigoplus C_\bullet^{(p)}(A)$. This leads to a decomposition $H_n(A) = \bigoplus_{p+q=n} H_{pq}(A)$ where $H_{p*}(A)$ is the homology of $C_\bullet^{(p)}(A)$. Using the results of [12], they show [8] that if A is smooth over k then $H_{pq}(A) = 0$ for $q \neq 0$. (Gerstenhaber and Schack actually do this for Hochschild cohomology but the homology version follows with no further effort because $H_n(A) = \Omega^n_{A/k}$ is projective so $H^n(A, A) = \mathrm{Hom}_A(H_n(A), A)$.) Therefore the complex $\mathscr{C}_\bullet$ of sheaves splits into a direct sum $\mathscr{C}_\bullet = \bigoplus \mathscr{C}_\bullet^{(p)}$ where $\mathscr{C}_\bullet^{(p)}$ has non-trivial homology in one dimension only. This leads to a natural decomposition of the spectral sequence (2.2) into the sequences obtained from the double complexes $\mathrm{Hom}(\mathscr{C}_\bullet^{(p)}, \mathscr{I}^\bullet)$. These have $E_2^{ij} = 0$ for $i \neq p$ and so degenerate to isomorphisms. □

3. The Gerstenhaber–Schack definition

I will show here that the definition of Hochschild cohomology given by Gerstenhaber and Schack [8] agrees with that given in Section 2. We first recall the definition.

Let X be a separated scheme of finite type over k. Let $\mathbb{A}$ be the category of affine open sets of X and inclusion maps. In this section, a presheaf will mean a contravariant functor on $\mathbb{A}$ (not on the category of all open sets of X). Since $\mathbb{A}$ is a base of open sets and is closed under finite intersection, the usual relations with sheaves on X still hold, i.e. the category $\mathscr{S}$ of sheaves can be identified with a full subcategory of the category $\mathscr{P}$ of presheaves and the inclusion $i: \mathscr{S} \hookrightarrow \mathscr{P}$ has a left adjoint $a: \mathscr{P} \to \mathscr{S}$, the associated sheaf functor. Also a is exact so i preserves injectives.

Let $\mathcal{O}^e$ be the presheaf $U \mapsto \Gamma(U, \mathcal{O}_X) \otimes_k \Gamma(U, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_X)^e$. Any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules can be thought of as a presheaf over $\mathcal{O}^e$ using the map $\Gamma(U, \mathcal{O}_X)^e \to \Gamma(U, \mathcal{O}_X)$. Gerstenhaber and Schack define the Hochschild cohomology of $\mathscr{F}$ to be $\mathrm{Ext}^n_{\mathcal{O}^e}(\mathcal{O},\mathscr{F})$ in the category of presheaves of $\mathcal{O}^e$-modules. It is easy to see that this agrees with the definition given in [8, 20.1, 28] except for the terminology and notation. The next theorem shows that it also agrees with the hyperext definition of Section 2.

Theorem 3.1. $\mathrm{Ext}^n_{\mathcal{O}^e}(\mathcal{O},\mathscr{F}) \approx HH^n(X,\mathscr{F})$.

Proof. Let $\mathscr{B}_{\bullet}$ be the chain complex of $\mathcal{O}^e$-modules defined by $U \mapsto B_{\bullet}(U) = B_{\bullet}(\Gamma(U,\mathcal{O}_X))$. Each $B_{\bullet}(U)$ is projective over $\mathcal{O}^e(U)$ but, in general, $\mathscr{B}_{\bullet}$ is not projective over $\mathcal{O}^e$. To remedy this, Gerstenhaber and Schack [8, 19, 20] take a further resolution $\mathscr{S}_{\bullet}\mathscr{B}_{\bullet}$ of $\mathscr{B}_{\bullet}$ which is projective over $\mathcal{O}^e$. I will use a variant of this construction here. This, like the $\mathscr{S}_{\bullet}$ resolution, can be defined for an arbitrary presheaf of rings $\mathscr{A}$.

As in [8, 19], the functor $R:\mathrm{Mod}\,\mathscr{A} \to \prod \mathrm{Mod}\,\mathscr{A}(U)$ has a left adjoint L. For a presheaf $\mathscr{M}$ of $\mathscr{A}$-modules, we define $P(\mathscr{M}) = LR(\mathscr{M})$. Explicitly, $P(\mathscr{M})(U) = \coprod_{V \supset U} \mathscr{A}(U) \otimes_{\mathscr{A}(V)} \mathscr{M}(V)$. Since R is exact, it follows that $P(\mathscr{M})$ is a projective presheaf of $\mathscr{A}$-modules if each $\mathscr{M}(U)$ is projective over $\mathscr{A}(U)$. We have an adjunction map $\varepsilon: P(\mathscr{M}) \to \mathscr{M}$ and $P(\mathscr{M})(U) \to \mathscr{M}(U)$ splits for each U since $\mathscr{A}(U) \otimes_{\mathscr{A}(U)} \mathscr{M}(U) = \mathscr{M}(U)$ is a summand of $P(\mathscr{M})(U)$. Let $Q(\mathscr{M})$ be the kernel of ε. Then $0 \to Q(\mathscr{M})(U) \to P(\mathscr{M})(U) \to \mathscr{M}(U) \to 0$ is split exact for each U. In particular, if all $\mathscr{M}(U)$ are projective, so are all $Q(\mathscr{M})(U)$. By splicing $0 \to Q(\mathscr{M}) \to P(\mathscr{M}) \to \mathscr{M} \to 0$, $0 \to Q^2(\mathscr{M}) \to PQ(\mathscr{M}) \to Q(\mathscr{M}) \to 0$, etc., we get the required resolution

$$\cdots \to P_2(\mathscr{M}) \to P_1(\mathscr{M}) \to P_0(\mathscr{M}) \to \mathscr{M} \to 0$$

where $P_n(\mathscr{M}) = PQ^n(\mathscr{M})$. For each U,

$$\cdots \to P_2(\mathscr{M})(U) \to P_1(\mathscr{M})(U) \to P_0(\mathscr{M})(U) \to \mathscr{M}(U) \to 0$$

is split exact. Also, if all $\mathscr{M}(U)$ are projective then all $P_n(\mathscr{M})$ are projective. We write $P^{\mathscr{A}}(\mathscr{M})$, etc. if it is necessary to specify the presheaf of rings $\mathscr{A}$. □

Lemma 3.2. *If $A \to \mathscr{B}$ is a map of presheaves of rings, then $\mathscr{B} \otimes_{\mathscr{A}} P^{\mathscr{A}}(\mathscr{M}) \xrightarrow{\approx} P^{\mathscr{B}}(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{M})$, $\mathscr{B} \otimes_{\mathscr{A}} Q^{\mathscr{A}}(\mathscr{M}) \xrightarrow{\approx} Q^{\mathscr{B}}(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{M})$, and $\mathscr{B} \otimes_{\mathscr{A}} P^{\mathscr{A}}_n(\mathscr{M}) \xrightarrow{\approx} P^{\mathscr{B}}_n(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{M})$ for all n.*

Proof. The first assertion is clear from the explicit form of P since $\mathscr{B}(U) \otimes_{\mathscr{A}(U)} \mathscr{A}(U) \otimes_{\mathscr{A}(V)} \mathscr{M}(V) = \mathscr{B}(U) \otimes_{\mathscr{A}(V)} \mathscr{M}(V) = \mathscr{B}(U) \otimes_{\mathscr{B}(V)} \mathscr{B}(V) \otimes_{\mathscr{A}(V)} \mathscr{M}(V)$. The second statement follows from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{B} \otimes_{\mathscr{A}} Q^{\mathscr{A}}(\mathscr{M}) & \longrightarrow & \mathscr{B} \otimes_{\mathscr{A}} P^{\mathscr{A}}(\mathscr{M}) & \longrightarrow & \mathscr{B} \otimes_{\mathscr{A}} \mathscr{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \approx & & \downarrow = & & \\
0 & \longrightarrow & Q^{\mathscr{B}}(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{M}) & \longrightarrow & P^{\mathscr{B}}(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{M}) & \longrightarrow & \mathscr{B} \otimes_{\mathscr{A}} \mathscr{M} & \longrightarrow & 0.
\end{array}$$

Note that the top sequence is exact since its evaluation at each U splits. The third statement now follows from the first two.

Returning now to the case of $\mathcal{O}^e$, we see that $P.\mathscr{B}.$ is a projective double complex and its associated total complex is a projective resolution of $\mathcal{O}_X$ over $\mathcal{O}^e$. Therefore $\mathrm{Ext}^n_{\mathcal{O}^e}(\mathcal{O}_X, \mathscr{F}) = H^n(\mathrm{Hom}_{\mathcal{O}^e}(P.\mathscr{B}., \mathscr{F}))$. Let $\mathscr{F} \to \mathscr{I}^\cdot$ be an injective resolution of sheaves of $\mathcal{O}_X$-modules. Since $P.\mathscr{B}.$ is projective, $\mathrm{Hom}_{\mathcal{O}^e}(P.\mathscr{B}., \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}^e}(P.\mathscr{B}., \mathscr{I}^\cdot)$ (where the symbol $\xrightarrow{\sim}$ denotes a homology equivalence). Since $\mathscr{I}^\cdot$ is a complex of $\mathcal{O}_X$-modules, $\mathrm{Hom}_{\mathcal{O}^e}(P^{\mathcal{O}^e}\mathscr{B}., \mathscr{I}^\cdot) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{\mathcal{O}^e} P^{\mathcal{O}^e}_{\cdot}\mathscr{B}., \mathscr{I}^\cdot) = \mathrm{Hom}_{\mathcal{O}_X}(P^{\mathcal{O}_X}_{\cdot}(\mathcal{O}_X \otimes_{\mathcal{O}^e} \mathscr{B}.), \mathscr{I}^\cdot) = \mathrm{Hom}_{\mathcal{O}_X}(P^{\mathcal{O}_X}_{\cdot}(\mathscr{D}.), \mathscr{I}^\cdot)$ where $\mathscr{D}.$ is the presheaf $U \mapsto C.(\Gamma(U, \mathcal{O}_X))$. Since $\mathscr{I}^\cdot$ is injective as a sheaf and therefore as a presheaf, and $P^{\mathcal{O}_X}_{\cdot}(\mathscr{D}.) \xrightarrow{\sim} \mathscr{D}.$, it follows that $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{D}., \mathscr{I}^\cdot) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(P^{\mathcal{O}_X}_{\cdot}(\mathscr{D}.), \mathscr{I}^\cdot)$. Now $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{D}., \mathscr{I}^\cdot) = \mathrm{Hom}_{\mathcal{O}_X}(a\mathscr{D}., \mathscr{I}^\cdot) = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{C}., \mathscr{I}^\cdot)$, where a is the associated sheaf functor. Therefore we have recovered the definition of $HH^n(X, \mathscr{F})$ given in Section 2. □

The Gerstenhaber–Schack Hodge decomposition considered at the end of Section 2 clearly agrees with the one given in [8] under this isomorphism. We could also define a Hodge spectral sequence in the Gerstenhaber–Schack theory by taking an injective resolution $\mathscr{F} \to \mathscr{I}^\cdot$ of sheaves of $\mathcal{O}_X$-modules and filtering $\mathrm{Hom}_{\mathcal{O}^e}(P.\mathscr{B}., \mathscr{I}^\cdot)$ by the degree of $\mathscr{I}^\cdot$. It is clear that this agrees with the hyperext spectral sequence constructed in Section 2.

4. Locally free resolutions

The proof of Theorem 2.1 is much simpler in the affine case since we can use the fact that $B.(A)$ is a projective resolution of A over A^e. In the non-affine case, however, it is not even clear how to define an analogue of $B.(A)$ as a complex of sheaves on $X \times X$. The proof will therefore be given in three steps. First we consider the case in which $\mathscr{C}.$ is replaced by a sheaf $\delta^*\mathscr{L}.$ where $\mathscr{L}.$ is a locally free resolution of $\delta_*\mathcal{O}_X$ on $X \times X$. Secondly, we use standard approximation techniques to generalize to the case where $\mathscr{L}.$ is only assumed flat. Finally we use a Čech patching technique to find such a complex which approximates the bar resolution near the diagonal. In the present section I will consider the first step. This can be done more generally for a closed embedding $Y \hookrightarrow X$ rather than just for the special case $X \hookrightarrow X \times X$.

I will prove first some basic facts used in the proof.

Lemma 4.1. *Let $\mathscr{A}.$ and $\mathscr{B}.$ be chain complexes of sheaves which are bounded below. Then a homology equivalence $\mathscr{A}. \xrightarrow{\sim} \mathscr{B}.$ induces an isomorphism between the spectral sequences*

$$\mathrm{Ext}^p_{\mathcal{O}_X}(H_q(\mathscr{A}.), \mathscr{G}) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathcal{O}_X}(\mathscr{A}., \mathscr{G})$$

and

$$\mathrm{Ext}^p_{\mathcal{O}_X}(H_q(\mathcal{B}.),\mathcal{G}) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathcal{O}_X}(\mathcal{B}.,\mathcal{G})$$

of Lemma 2.3. *Homotopic maps* $\mathcal{A}. \to \mathcal{B}.$ *induce the same map* $\mathbb{E}\mathrm{xt}^n(\mathcal{B}.,\mathcal{G}) \to \mathbb{E}\mathrm{xt}^n(\mathcal{A}.,\mathcal{G})$ *and the same map of spectral sequences.*

Proof. The first statement is clear from Lemma 2.3. For the second, if $h \simeq 0$, then h factors through the mapping cone $\mathcal{M}.$ of the identity map of $\mathcal{A}.$. Since $H.(\mathcal{M}.) = 0$, the first statement implies that the spectral sequence for $\mathcal{M}.$ is 0. □

Lemma 4.2. *Let* $\mathcal{L}.$ *be a loclly free chain complex of sheaves and let* $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$ *be cochain complexes of sheaves. Assume* $\mathcal{L}.$, $\mathcal{A}^\bullet$, *and* $\mathcal{B}^\bullet$ *are bounded below. If* $\mathcal{A}^\bullet \xrightarrow{\sim} \mathcal{B}^\bullet$ *then* $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}}(\mathcal{L}.,\mathcal{A}^\bullet) \xrightarrow{\sim} \underset{\sim}{\mathrm{hom}}_{\mathcal{O}}(\mathcal{L}.,\mathcal{B}^\bullet)$.

Proof. Since $\mathcal{L}.$ is locally free, filtering on the degree in $\mathcal{L}.$ gives a map of spectral sequences which is an isomorphism at the E_1 level. □

Lemma 4.3. *If* $\mathcal{F}$ *is a flat* $\mathcal{O}$*-module and* $\mathcal{G}$ *is an injective* $\mathcal{O}$*-module then the sheaf* $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ *is injective.*

Proof. We have $\mathrm{Hom}_{\mathcal{O}}(\mathcal{M}, \underset{\sim}{\mathrm{hom}}_{\mathcal{O}}(\mathcal{F},\mathcal{G})) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{M}\otimes_{\mathcal{O}}\mathcal{F},\mathcal{G})$. This is an exact functor in $\mathcal{M}$ by the hypotheses. □

Lemma 4.4. *Let* $\mathcal{A}^\bullet$ *and* $\mathcal{B}^\bullet$ *be cochain complexes of flabby sheaves which are bounded below. If* $\mathcal{A}^\bullet \xrightarrow{\sim} \mathcal{B}^\bullet$ *then* $\Gamma\mathcal{A}^\bullet \xrightarrow{\sim} \Gamma\mathcal{B}^\bullet$.

Proof. Let $\mathcal{C}^\bullet$ be the mapping cone of $\mathcal{A}^\bullet \xrightarrow{\sim} \mathcal{B}^\bullet$. Then $\mathcal{C}^\bullet$ is a flabby resolution of the zero sheaf so $H^\bullet(\Gamma\mathcal{C}^\bullet) = 0$. □

Lemma 4.5. *Let* $i: Y \hookrightarrow X$ *be closed. Then* $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathcal{A}, i_*\mathcal{B}) = i_*\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathcal{A},\mathcal{B})$.

Proof. For any $\mathcal{F}$, we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathcal{A}, i_*\mathcal{B})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{A}, i_*\mathcal{B}) = \mathrm{Hom}_{\mathcal{O}_Y}(i^*(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{A}),\mathcal{B}) = \mathrm{Hom}_{\mathcal{O}_Y}((i^*\mathcal{F}\otimes_{\mathcal{O}_Y} i^*\mathcal{A}),\mathcal{B}) = \mathrm{Hom}_{\mathcal{O}_Y}(i^*\mathcal{F}, \underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathcal{A},\mathcal{B})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_*\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathcal{A},\mathcal{B}))$. □

Proposition 4.6. *Let* $i: Y \hookrightarrow X$ *be a closed embedding. Let* $\mathcal{L}.$ *be a locally free chain complex on X which is bounded below. Let* $\mathcal{S}$ *be a sheaf of* $\mathcal{O}_Y$*-modules. Then there is a natural isomorphism of* δ*-functors in* $\mathcal{S}$, $\mathbb{E}\mathrm{xt}^n_{\mathcal{O}_X}(\mathcal{L}., i_*\mathcal{S}) \approx \mathbb{E}\mathrm{xt}^n_{\mathcal{O}_Y}(i^*\mathcal{L}.,\mathcal{S})$.

Proof. Let $\mathcal{S} \to \mathcal{I}^\bullet$ and $i_*\mathcal{S} \to \mathcal{J}^\bullet$ be injective resolutions. Since i_* is exact, $i_*\mathcal{S} \to i_*\mathcal{I}^\bullet$ is exact and we can find a map $i_*\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ over $i_*\mathcal{S}$. By Lemma 4.2, $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathcal{L}., i_*\mathcal{I}^\bullet) \xrightarrow{\sim} \underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathcal{L}.,\mathcal{J}^\bullet)$. By Lemma 4.5, the left side is $i_*\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathcal{L}.,\mathcal{I}^\bullet)$. By Lemma 4.3, we see that $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathcal{L}.,\mathcal{I}^\bullet)$ is injective so $i_*\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathcal{L}.,\mathcal{I}^\bullet)$ is flabby.

Also $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathcal{L}_\cdot, \mathcal{I}^\cdot)$ is injective. By Lemma 4.4, $\Gamma(X, i_*\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathcal{L}_\cdot, \mathcal{I}^\cdot)) \xrightarrow{\sim} \Gamma(X, \underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathcal{L}_\cdot, \mathcal{I}^\cdot))$. This can be written $\mathrm{Hom}_{\mathcal{O}_Y}(i^*\mathcal{L}_\cdot, \mathcal{I}^\cdot)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}_\cdot, \mathcal{I}^\cdot))$ and the result follows by taking cohomology.

If we have a short exact sequence $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$, we can choose exact sequences of injective resolutions $0 \to \mathcal{I}'^\cdot \to \mathcal{I}^\cdot \to \mathcal{I}'' \to 0$, $0 \to \mathcal{J}'^\cdot \to \mathcal{J}^\cdot \to \mathcal{J}''^\cdot + 0$, and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i^*\mathcal{J}'^\bullet & \longrightarrow & i^*\mathcal{J}^\bullet & \longrightarrow & i^*\mathcal{J}''^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{J}'^\bullet & \longrightarrow & \mathcal{J}^\bullet & \longrightarrow & \mathcal{J}''^\bullet & \longrightarrow & 0. \end{array}$$

This leads to an exact ladder of cohomology showing that our isomorphism is one of δ-functors. □

Corollary 4.7. *Let $\mathcal{L}_\cdot$ be a locally free chain complex on $X \times X$ which is bounded below and such that $H_0(\mathcal{L}_\cdot) = \delta_*\mathcal{O}_X$ and $H_p(\mathcal{L}_\cdot) = 0$ for $p \neq 0$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then $H^n(\mathcal{O}_X, \mathcal{F}) = \mathbb{E}\mathrm{xt}^n_{\mathcal{O}_X}(\delta^*\mathcal{L}_\cdot, \mathcal{F})$.*

Proof. The right-hand side is $\mathbb{E}\mathrm{xt}^n_{\mathcal{O}_{X\times X}}(\mathcal{L}_\cdot, \delta_*\mathcal{F})$ by Proposition 4.6. By Lemma 4.1, we can replace $\mathcal{L}_\cdot$ by $\mathcal{O}_X$. □

5. Spectral sequences

In this section I will prove an analogue of Corollary 4.7 for the Hodge spectral sequences. We first recall some standard results about Cartan–Eilenberg resolutions [3]. I will use the following terminology. All complexes here are assumed to be bounded below.

(1) A monomorphism $i: A^\cdot \to B^\cdot$ of cochain complexes is a CE-monomorphism if $i_*: H^*(A^*) \to H^*(B^*)$ is also a monomorphism.

(2) An exact sequence $C'^\cdot \xrightarrow{f} C^\cdot \xrightarrow{g} C''^\cdot$ is CE-exact if $\mathrm{im}\, f \to C^\cdot$ is a CE-monomorphism.

(3) $I^\cdot$ is CE-injective if whenever $i: A^\cdot \to \mathrm{B}^\cdot$ is a CE-monomorphism, any $A^\cdot \to I^\cdot$ extends to $B^\cdot \to I^\cdot$.

(4) A CE-resolution is a CE-exact sequence $0 \to A^\cdot \to C^{0\cdot} \to C^{1\cdot} \to \cdots$ where the $C^{n\cdot}$ are CE-injective.

If we assume the existence of enough injectives, every cochain complex admits a CE-monomorphism into a CE-injective complex. Therefore CE-resolutions exist. The usual mapping theorem shows that any map of cochain complexes $A^\cdot \to B^\cdot$ extends to a map of their CE-resolutions and this extension is unique up to homotopy. If F is an additive functor, the hypercohomology spectral sequences of F are defined to be the spectral sequences associated to the double complex $F(C^{\cdot\cdot})$ where $C^{\cdot\cdot}$ is a CE-resolution of $A^{\cdot}$ [3]. In the construction of these sequences it is shown that $H^q_{II}(F(C^{\cdot\cdot}) = F(H^q_{II}(C^{\cdot\cdot}))$ where H_{II} denotes the cohomology with respect to the differential in the complexes $C^{n\cdot}$.

We will mainly be interested here in the case where $A^{\cdot}$ is a cochain complex of sheaves and $F = \Gamma$ which leads to the hypercohomology spectral sequence $H^p(X, H^q(\mathscr{A}^{\cdot})) \Rightarrow \mathbb{H}^{p+q}(X, \mathscr{A}^{\cdot})$.

Lemma 5.1. *Let $\mathscr{M}_{\cdot}$ be a chain complex of locally free sheaves on a scheme Y which is bounded below and such that $H_q(\mathscr{M}_{\cdot})$ is locally free for all q. Let $\mathscr{S}$ be a sheaf of $\mathscr{O}_Y$-modules. Then there is a natural isomorphism* $H^q(\underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{S})) \xrightarrow{\approx} \underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(H^q(\mathscr{M}_{\cdot}), \mathscr{S})$.

Proof. If F is any contravariant left exact functor and $C_{\cdot}$ is a chain complex we can define a natural map $H^q(F(C_{\cdot})) \to F(H_q(C_{\cdot}))$ as follows. If $Z'_q = \text{ckr}[C_{q+1} \to C_q]$, then $F(Z'_q) = \ker[F(C_q) \to F(C_{q+1})]$. Now $0 \to H_q \to Z'_q \to C_{q-1}$ is exact so $F(Z'_q) \to F(H_q)$ annihilates the image of $F(C_{q-1})$ and so factors through $\text{ckr}[F(C_{q-1}) \to F(Z'_q)] = H^q(F(C_{\cdot}))$. This gives us a natural map $H^q(\underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{S})) \to \underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(H_q(\mathscr{M}_{\cdot}), \mathscr{S})$. It is sufficient to show it is an isomorphism locally. Locally, $\mathscr{M}_{\cdot}$ is the complex of sheaves associated to a complex $M_{\cdot}$ of projective modules whose homology is also projective. It is well known that such a complex is isomorphic to a direct sum of elementary complexes. For example, if $M_n = 0$ for $n < 0$, then $M_0 \to H_0$ is onto so we can write $M_0 = M'_0 \oplus H_0$ and split off the complex $0 \to H_0 \to 0$. If $H_0 = 0$, then $M_1 \to M_0$ is onto so we can write $M_1 = M'_1 \oplus M_0$ and split off the complex $0 \to M_0 \xrightarrow{1} M_0 \to 0$. In this way we split $\mathscr{M}_{\cdot}$ (locally) into a direct sum of complexes of the form $0 \to \mathscr{M} \to 0$ and $0 \to \mathscr{M} \xrightarrow{1} \mathscr{M} \to 0$. The lemma is clearly true for such complexes. □

Corollary 5.2. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is exact and $\mathscr{M}_{\cdot}$ satisfies the hypotheses of Lemma 5.1 then $0 \to \underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{F}') \to \underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{F}) \to \underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{F}'') \to 0$ is CE-exact.*

Lemma 5.3. *Let $\mathscr{M}_{\cdot}$ be a chain complex of locally free sheaves on a scheme Y which is bounded below and such that $H_q(\mathscr{M}_{\cdot})$ is locally free for all q. Let $\mathscr{S}$ be a sheaf of $\mathscr{O}_Y$-modules. Then the hyperext spectral sequence*

$$\text{Ext}^p_{\mathscr{O}_Y}(H_q(\mathscr{M}_{\cdot}), \mathscr{S}) \Rightarrow \mathbb{E}\text{xt}^{p+q}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{S})$$

is isomorphic to the hypercohomology spectral sequence

$$H^p(Y, H^q(\underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{S}))) \Rightarrow \mathbb{H}^{p+q}(Y, \underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{S}))$$

of the complex $\underset{\sim}{\text{hom}}_{\mathscr{O}_Y}(\mathscr{M}_{\cdot}, \mathscr{S})$.

Note that if $\mathscr{S} \to \mathscr{I}^{\bullet}$ is an injective resolution then $\underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{S}) \xrightarrow{\sim} \underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{I}^{\bullet})$ so the hypercohomology sequence of Lemma 5.3 is also the hypercohomology sequence of $\underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{I}^{\bullet})$.

Proof. Let $\mathscr{S} \to \mathscr{I}^{\bullet}$ be an injective resolution of $\mathscr{S}$. Then

$$0 \to \underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{S}) \to \underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{I}^{0}) \to \underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{I}^{1}) \to \cdots$$

is CE-exact by Corollary 5.2. Let $\underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{S}) \to \mathscr{J}^{\bullet\bullet}$ be a CE-resolution. Then there is a map f, unique up to homotopy, making the diagram

$$\begin{array}{ccc} \underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{S}) & \longrightarrow & \underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{I}^{\bullet}) \\ \big\downarrow {\scriptstyle =} & & \big\downarrow {\scriptstyle f} \\ \underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{S}) & \longrightarrow & \mathscr{J}^{\bullet\bullet} \end{array}$$

commute. Applying Γ gives $\mathrm{Hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{I}^{\bullet}) = \Gamma(\underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{I}^{\bullet})) \to \Gamma(\mathscr{J}^{\bullet\bullet})$. By filtering $\mathrm{Hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{I}^{\bullet})$ by the degree of $\mathscr{I}^{\bullet}$ we get the first spectral sequence of Lemma 5.3. The corresponding filtration of $\Gamma(\mathscr{J}^{\bullet\bullet})$ gives the second spectral sequence of Lemma 5.3 so it will suffice to show that the induced map on E_2 terms is an isomorphism. This map on the E_1 terms is $\mathrm{Hom}_{\mathcal{O}_Y}(H_q(\mathscr{M}_{\bullet}), \mathscr{I}^{\bullet}) \to \Gamma(H^{\bullet}_{II}(\mathscr{J}^{\bullet\bullet}))$ which is obtained by applying Γ to $\underset{\sim}{\hom}_{\mathcal{O}_Y}(H_q(\mathscr{M}_{\bullet}), \mathscr{I}^{\bullet}) \to H^{\bullet}_{II}(\mathscr{J}^{\bullet\bullet})$. By Lemma 4.3, $\underset{\sim}{\hom}_{\mathcal{O}_Y}(H_q(\mathscr{M}_{\bullet}), \mathscr{I}^{\bullet})$ is an injective resolution of $\underset{\sim}{\hom}_{\mathcal{O}_Y}(H_q(\mathscr{M}_{\bullet}), \mathscr{S})$. By the properties of CE-resolutions, $H^{\bullet}_{II}(\mathscr{J}^{\bullet\bullet})$ is an injective resolution of $H^q(\underset{\sim}{\hom}_{\mathcal{O}_Y}(\mathscr{M}_{\bullet}, \mathscr{S}))$ which is equal to $\underset{\sim}{\hom}_{\mathcal{O}_Y}(H_q(\mathscr{M}_{\bullet}), \mathscr{S})$ by Lemma 5.1. It follows that the induced map of E_2 terms is an isomorphism. □

Lemma 5.4. *Let $i: Y \hookrightarrow X$ be a closed embedding. Let $\mathscr{A}^{\bullet}$ be a cochain complex of sheaves on Y which is bounded below. Then the hypercohomology spectral sequences*

$$H^p(Y, H^q(\mathscr{A}^{\bullet})) \Rightarrow \mathbb{H}^{p+q}(Y, \mathscr{A}^{\bullet})$$

and

$$H^p(X, H^q(i_*\mathscr{A}^{\bullet})) \Rightarrow \mathbb{H}^{p+q}(X, i_*\mathscr{A}^{\bullet})$$

are isomorphic.

Proof. Choose CE-resolutions $\mathscr{A}^{\bullet} \to \mathscr{I}^{\bullet\bullet}$ and $i_*\mathscr{A}^{\bullet} \to \mathscr{J}^{\bullet\bullet}$. Since i_* is exact, $i_*\mathscr{A}^{\bullet} \to i_*\mathscr{I}^{\bullet\bullet}$ is exact and so is $H^p(i_*\mathscr{A}^{\bullet}) \to H^p_{II}(i_*\mathscr{I}^{\bullet\bullet})$. Also there is a map f unique up to homotopy making the diagram

$$\begin{array}{ccc} i_*\mathscr{A}^{\bullet} & \longrightarrow & i_*\mathscr{I}^{\bullet\bullet} \\ \big\downarrow {\scriptstyle =} & & \big\downarrow {\scriptstyle f} \\ i_*\mathscr{A}^{\bullet} & \longrightarrow & \mathscr{J}^{\bullet\bullet} \end{array}$$

commute. Applying $\Gamma(X,-)$ to f gives $\Gamma(Y,\mathscr{I}^{\cdot\cdot}) = \Gamma(X,i_*\mathscr{I}^{\cdot\cdot}) \to \Gamma(X,\mathscr{J}^{\cdot\cdot})$ where the double complex $\Gamma(Y,\mathscr{I}^{\cdot\cdot})$ gives the first spectral sequence of Lemma 5.4 and the double complex $\Gamma(X,\mathscr{J}^{\cdot\cdot})$ gives the second. It is enough to check that the map of E_2 terms is an isomorphism but this map is $H^p(Y,H^q(\mathscr{A}^\cdot)) = H^p(X,i_*H^q(\mathscr{A}^\cdot)) \xrightarrow{\approx} H^p(X,H^q(i_*\mathscr{A}^\cdot))$. □

Proposition 5.5. *Let $i: Y \hookrightarrow X$ be a closed embedding. Let $\mathscr{L}_\cdot$ be a locally free chain complex on X which is bounded below. Assume that $H_0(\mathscr{L}_\cdot) = \mathscr{T}$ and $H_q(\mathscr{L}_\cdot) = 0$ for $q \neq 0$. Suppose also that $H_q(i^*\mathscr{L}_\cdot)$ is locally free on Y for all q. Let $\mathscr{S}$ be sheaf of $\mathcal{O}_Y$-modules. Then the spectral sequences*

$$\mathrm{Ext}^p_{\mathcal{O}_Y}(H_q(i^*\mathscr{L}_\cdot),\mathscr{S}) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathcal{O}_Y}(i^*\mathscr{L}_\cdot,\mathscr{S})$$

and

$$H^p(X, \underset{\sim}{\mathrm{ext}}^q_{\mathcal{O}_X}(\mathscr{T},i_*,\mathscr{S})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathscr{T},i^*\mathscr{S})$$

are isomorphic.

Proof. The second spectral sequence is the one associated to the composition of functors $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{T},-) = \Gamma(\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathscr{T},-))$ applied to $i_*\mathscr{S}$. Therefore if $i_*\mathscr{S} \to \mathscr{J}^\cdot$ is an injective resolution of $\mathscr{S}$, then the second sequence is the hypercohomology spectral sequence of the complex $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathscr{T},\mathscr{J}^\cdot)$. Let $\mathscr{M}_n = \mathscr{L}_n$ for $n > 0$, $\mathscr{M}_0 = Z_0(\mathscr{L}_\cdot)$, and $\mathscr{M}_n = 0$ for $n < 0$. Then $\mathscr{L}_\cdot \xleftarrow{\sim} \mathscr{M}_\cdot \xrightarrow{\sim} \mathscr{T}$ so $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathscr{L}_\cdot,\mathscr{J}^\cdot) \xrightarrow{\sim} \underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathscr{M}_\cdot,\mathscr{J}^\cdot) \xleftarrow{\sim} \underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathscr{T},\mathscr{J}^\cdot)$. Therefore these complexes have the same hypercohomology spectral sequence. Let $\mathscr{S} \to \mathscr{I}^\cdot$ be an injective resolution on Y. Since i_* is exact, $i_*\mathscr{S} \to i_*\mathscr{I}^\cdot$ is exact and we can find a map $i_*\mathscr{I}^\cdot \to \mathscr{J}^\cdot$ over $i_*\mathscr{S}$. This map is clearly a homology equivalence and $\mathscr{L}_\cdot$ is locally free so $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathscr{L}_\cdot,i_*\mathscr{I}^\cdot) \xrightarrow{\sim} \underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathscr{L}_\cdot,\mathscr{J}^\cdot)$. By Lemma 4.5, $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_X}(\mathscr{L}_\cdot,i_*\mathscr{I}^\cdot) \approx i_*\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathscr{L}_\cdot,\mathscr{I}^\cdot)$. The second spectral sequence of Proposition 5.5 is isomorphic to the hypercohomology spectral sequence of this which, by Lemma 5.4, is isomorphic to the hypercohomology spectral sequence of the complex $\underset{\sim}{\mathrm{hom}}_{\mathcal{O}_Y}(i^*\mathscr{L}_\cdot,\mathscr{I}^\cdot)$ on Y. By Lemma 5.3 this is isomorphic to the first spectral sequence. □

Corollary 5.6. *Let X be a separable scheme of finite type over a field. Let $\mathscr{L}_\cdot$ be a locally free chain complex on $X \times X$ which is bounded below. Assume that $H_0(\mathscr{L}_\cdot) \approx \delta_*\mathcal{O}_X$ and $H_q(\mathscr{L}_\cdot) = 0$ for $q \neq 0$. Suppose also that $H_q(\delta^*\mathscr{L}_\cdot)$ is locally free on X for all q. Then the Hodge spectral sequences*

$$\mathrm{Ext}^p_{\mathcal{O}_X}(H_q(\delta^*\mathscr{L}_\cdot),\mathscr{F}) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathcal{O}_X}((\delta^*\mathscr{L}_\cdot),\mathscr{F})$$

and

$$H^p(X, \underset{\sim}{\mathrm{ext}}^q_{\mathcal{O}_{X\times X}}(\mathcal{O}_X,\mathscr{F})) \Rightarrow H^{p+q}(\mathcal{O}_X,\mathscr{F})$$

are isomorphic.

6. Flat resolutions

In this section we show that the local freeness hypothesis on $\mathscr{L}_\bullet$ in Sections 4 and 5 can be replaced by a flatness condition. We begin with some well-known results on approximating complexes by locally free ones.

Lemma 6.1. *Let X be a quasiprojective scheme over a field. Let $\mathscr{F} \twoheadrightarrow \mathscr{G}$ be an epimorphism of quasicoherent sheaves of $\mathcal{O}_X$-modules. If $\mathscr{G}$ is coherent, then there is a locally free $\mathscr{L}$ and a map $\mathscr{L} \to \mathscr{F}$ such that the composition $\mathscr{L} \to \mathscr{F} \to \mathscr{G}$ is an epimorphism.*

Proof. By [11, Ch. II, Ex. 5.15e] $\mathscr{F}$ is a filtered union of coherent subsheaves. One of these will map onto $\mathscr{G}$ so we can assume that $\mathscr{F}$ is coherent. Extend $\mathscr{F}$ to a projective closure of X [11, Ch. II, Ex. 5.15]. If $\mathscr{F}(n)$ is the usual Serre twist of $\mathscr{F}$, then $\mathscr{F}(n)$ is generated by global sections for $n \gg 0$ [11, Ch. II, Cor. 5.18] so we have an epimorphism $\mathcal{O}_X(-n)^N \twoheadrightarrow \mathscr{F}$. □

Lemma 6.2. [1, I.2.10, II.1.1(c)]. *Let $\mathscr{K}_\bullet$ be a chain complex of quasicoherent sheaves on a quasiprojective scheme. Assume that $\mathscr{K}_\bullet$ is bounded below. Suppose that $H_i(\mathscr{K}_\bullet)$ is coherent for all i. Then there is a chain complex $\mathscr{L}_\bullet$ of locally free sheaves, also bounded below, and a homology equivalence $\mathscr{L}_\bullet \xrightarrow{\sim} \mathscr{K}_\bullet$.*

Proof. Suppose that $\mathscr{L}_i$ has been constructed for $i \le p$ in such a way that $H_i(\mathscr{L}_\bullet) \to H_i(\mathscr{K}_\bullet)$ is an isomorphism for $i < p$ and is onto for $i = p$. Then $Z_p(\mathscr{L}_\bullet) \to Z_p(\mathscr{K}_\bullet) \to H_p(\mathscr{K}_\bullet)$ is onto where Z denotes the sheaf of cycles. Let $\mathscr{P}$ be the pullback in

$$\begin{array}{ccc} \mathscr{P} & \longrightarrow & Z_p(\mathscr{L}_\bullet) \\ \big\downarrow & & \big\downarrow \\ \mathscr{K}_{p+1} & \longrightarrow & Z_p(\mathscr{K}_\bullet). \end{array}$$

Since $Z_p(\mathscr{L}_\bullet)$ maps onto $H_p(\mathscr{K}_\bullet)$, it is easy to check that the cokernel of $\mathscr{P} \to Z_p(\mathscr{L}_\bullet)$ maps isomorphically to $H_p(\mathscr{K}_\bullet)$ and the image $\mathscr{B}_p$ of $\mathscr{P} \to Z_p(\mathscr{L}_\bullet)$ is coherent. By Lemma 6.1 we can find a locally free $\mathscr{L}'_{p+1}$ and a map $\mathscr{L}'_{p+1} \to \mathscr{P}$ so that $\operatorname{im}[\mathscr{L}'_{p+1} \to Z_p(\mathscr{L}_\bullet)] = \mathscr{B}_p$. By Lemma 6.1 choose $\mathscr{L}''_{p+1}$ locally free with $\mathscr{L}''_{p+1} \to Z_{p+1}(\mathscr{K}_\bullet) \to H_{p+1}(\mathscr{K}_\bullet)$ onto. Let $\mathscr{L}_{p+1} = \mathscr{L}'_{p+1} \oplus \mathscr{L}''_{p+1}$ and map it to $\mathscr{L}_p$ by $\mathscr{L}'_{p+1} \to \mathscr{P} \to Z_p(\mathscr{L}_\bullet) \to \mathscr{L}_p$ and by 0 on $\mathscr{L}''_{p+1}$. Let $\mathscr{L}_{p+1}$ map to $\mathscr{K}_{p+1}$ by $\mathscr{L}'_{p+1} \to \mathscr{P} \to \mathscr{K}_{p+1}$ and $\mathscr{L}''_{p+1} \to Z_{p+1}(\mathscr{K}_\bullet) \to \mathscr{K}_{p+1}$. This satisfies our conditions for $p + 1$ so we can iterate the construction. □

Lemma 6.3. *Suppose we have two homology equivalences $\mathscr{L}'_\bullet \xrightarrow{\sim} \mathscr{K}_\bullet$ and $\mathscr{L}''_\bullet \xrightarrow{\sim} \mathscr{K}_\bullet$ which satisfy the conclusions of Lemma 6.2. Then there is a chain complex $\mathscr{L}_\bullet$ of locally* *free sheaves which is bounded below and a diagram of homology equivalences*

$$\begin{array}{ccc} \mathscr{L}_\bullet & \longrightarrow & \mathscr{L}'_\bullet \\ \downarrow & & \downarrow \\ \mathscr{L}''_\bullet & \longrightarrow & \mathscr{K}_\bullet \end{array}$$

which commutes up to homotopy.

Proof. Let $\mathscr{M}_\bullet$ be the mapping cone of the identity map of $\mathscr{K}_\bullet$. After a shift of dimension we have an epimorphism $\mathscr{M}_\bullet \twoheadrightarrow \mathscr{K}_\bullet$. Let $\mathscr{G}_\bullet$ be the kernel in $0 \to \mathscr{G}_\bullet \to \mathscr{L}'_\bullet \oplus \mathscr{L}''_\bullet \oplus \mathscr{M}_\bullet \to \mathscr{K}_\bullet \to 0$. Then $\mathscr{G}_\bullet \xrightarrow{\sim} \mathscr{L}'_\bullet$. By Lemma 6.2, we can find $\mathscr{L}_\bullet \xrightarrow{\sim} \mathscr{G}_\bullet$. The diagram

$$\begin{array}{ccc} \mathscr{L}_\bullet & \longrightarrow & \mathscr{L}'_\bullet \\ \downarrow & & \downarrow \\ \mathscr{L}''_\bullet \oplus \mathscr{M}_\bullet & \longrightarrow & \mathscr{K}_\bullet \end{array}$$

commutes. Since $\mathscr{M}_\bullet$ is contractible, the diagram obtained by omitting $\mathscr{M}_\bullet$ is commutative up to homotopy. □

Lemma 6.4. *Let $f: \mathscr{F}_\bullet \to \mathscr{G}_\bullet$ be a map of chain complexes of sheaves over a sheaf of rings $\mathscr{A}$. Assume that $\mathscr{F}_\bullet$ and $\mathscr{G}_\bullet$ are bounded below and that $\mathscr{F}_\bullet$ and $\mathscr{G}_\bullet$ are flat over $\mathscr{A}$. Let $\mathscr{A} \to \mathscr{B}$ be a map of sheaves of rings. If f is a homology equivalence, then so is $1 \otimes f: \mathscr{B} \otimes_{\mathscr{A}} \mathscr{F}_\bullet \to \mathscr{B} \otimes_{\mathscr{A}} \mathscr{G}_\bullet$.*

Proof. Let $\mathscr{M}_\bullet$ be the mapping cone of f. Let $\mathscr{M}_i = 0$ for $i < p$ so that $\mathscr{M}_\bullet$ has the form $\cdots \to \mathscr{M}_{p+1} \to \mathscr{M}_p \to 0$. Break $\mathscr{M}_\bullet$ into short exact sequences $0 \to Z_{p+1} \to \mathscr{M}_{p+1} \to \mathscr{M}_p \to 0$, $0 \to Z_{p+2} \to \mathscr{M}_{p+2} \to Z_{p+1} \to 0$, etc. Induction shows that all Z_i are flat over $\mathscr{A}$ so we can tensor these sequences with $\mathscr{B}$ and reassemble them to see that $H_\bullet(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{M}_\bullet) = 0$. □

Proposition 6.5. (1) *Let X be a quasiprojective scheme over a field. Let $\mathscr{G}_\bullet$ be a flat chain complex of quasicoherent $\mathcal{O}_{X \times X}$-modules which is bounded below. Assume that $H_0(\mathscr{G}_\bullet) = \delta_* \mathcal{O}_X$ and $H_q(\mathscr{G}_\bullet) = 0$ for $q \neq 0$. Then $H^n(\mathcal{O}_X, \mathscr{F}) = \mathbb{E}\mathrm{xt}^n_{\mathcal{O}_X}(\delta^* \mathscr{G}_\bullet, \mathscr{F})$.*

(2) *Suppose also that $H_q(\delta^* \mathscr{G}_\bullet)$ is locally free for all q. Then the spectral sequences*

$$\mathrm{Ext}^n_{\mathcal{O}_X}(H_q(\delta^* \mathscr{G}_\bullet), \mathscr{F}) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathcal{O}_X}((\delta^* \mathscr{G}_\bullet), \mathscr{F})$$

and

$$H^p(X, \underset{\sim}{\mathrm{ext}}^q_{\mathcal{O}_{X \times X}}(\mathcal{O}_X, \mathscr{F})) \Rightarrow H^{p+q}(\mathcal{O}_X, \mathscr{F})$$

are isomorphic.

Proof. By Lemma 6.2 we can find a locally free complex $\mathscr{L}.$ with $\mathscr{L}.\xrightarrow{\sim}\mathscr{G}..$ By Lemma 6.4, $\delta^*\mathscr{L}.\xrightarrow{\sim}\delta^*\mathscr{G}..$ Therefore, by Lemma 4.1 and Corollary 4.7, $\mathbb{E}\mathrm{xt}^n_{\mathcal{O}_X}(\delta^*\mathscr{G}.,\mathscr{F})\xrightarrow{\approx}\mathbb{E}\mathrm{xt}^n_{\mathcal{O}_X}(\delta^*\mathscr{L}.,\mathscr{F}) = H^n(\mathcal{O}_X,\mathscr{F})$. If we choose a different $\mathscr{L}'.\xrightarrow{\sim}\mathscr{G}.$, Lemma 6.3 shows that we get the same isomorphism. The same argument applies to (2) using Corollary 5.6. □

The special case of Theorems 2.1 and 2.5 where X is affine follows immediately from this.

Corollary 6.6. *Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over a field. Then Theorem 2.1 holds for X. If X is smooth, Theorem 2.5 also holds.*

We need only take $\mathscr{G}.$ to be the complex $\mathscr{B}.$ defined in Lemma 2.4(4).

Remark. If $\mathscr{F} = M^\sim$ for an A-module M, then $H^n(\mathcal{O}_X,\mathscr{F}) = H^n(A,M)$ and the Hodge spectral sequence (1.1) degenerates to an isomorphism $\Gamma(\underset{\sim}{\mathrm{ext}}^n_{\mathcal{O}_{X\times X}}(\mathcal{O}_X,\mathscr{F})) = H^n(A,M)$. This follows from the standard fact that on an affine scheme $Y = \operatorname{Spec} B$ we have $\mathrm{Ext}^n_{\mathcal{O}_Y}(M^\sim, N^\sim) = \mathrm{Ext}^n_B(M,N)$. Also, if B is noetherian and M is finitely generated, then $\underset{\sim}{\mathrm{ext}}^n_{\mathcal{O}_Y}(M^\sim, N^\sim) = \mathrm{Ext}^n_B(M,N)^\sim$ which is quasicoherent.

7. Sheaf theoretic lemmas

I will recall here a few fairly general facts about sheaves on schemes which will be useful in the proofs of Theorems 2.1 and 2.5. Presumably they are all well known.

The following lemma will be needed to verify the flatness assumptions of Section 6.

Lemma 7.1. *Let M be a presheaf of modules over the presheaf of rings R. Let $\mathscr{M}$ and $\mathscr{R}$ be the associated sheaves. If $M(U)$ is flat over $R(U)$ for all open sets U then $\mathscr{M}$ is flat over $\mathscr{R}$.*

Proof. We first observe that if N is another presheaf of modules over R with associated sheaf $\mathscr{N}$ then $\mathscr{M}\otimes_{\mathscr{R}}\mathscr{N}$ is the sheaf associated to the presheaf $U \mapsto M(U)\otimes_{R(U)}N(U)$. This is easily checked by showing that the map $M\otimes_R N \to \mathscr{M}\otimes_{\mathscr{R}}\mathscr{N}$ induces an isomorphism of stalks. The functor $\mathscr{M}\otimes_{\mathscr{R}}-$ is right exact in any case. By the above remark, $\mathscr{M}\otimes_{\mathscr{R}}\mathscr{F}$ is the sheaf associated to the presheaf $U \mapsto M(U)\otimes_{R(U)}\Gamma(U,\mathscr{F})$. This functor is left exact in $\mathscr{F}$ and therefore so is $\mathscr{M}\otimes_{\mathscr{R}}\mathscr{F}$. □

We will also need the following simple observation.

Lemma 7.2. *Let $f: X \to Y$ be a flat affine morphism. Let $\mathscr{F}$ be quasicoherent and flat on X. Then $f_*\mathscr{F}$ is flat (and quasicoherent) on Y.*

Proof. We can clearly assume that Y and therefore X is affine so we can write $f: \operatorname{Spec} B \to \operatorname{Spec} A$. Let $\mathscr{F}$ correspond to the B-module M. Then $f_*\mathscr{F}$ corresponds to M considered as an A-module. Since M is flat over B and B is flat over A, M is flat over A. □

Lemma 7.3. *Let $f: X \to Y$ be an affine morphism. Then f_* preserves quasicoherent sheaves and $f_*: q\text{-}Coh(X) \to q\text{-}Coh(Y)$ is exact.*

Remark. In particular, this applies if Y is separated and X is an open set of Y.

Proof. The assertion is local on Y so we can assume that Y is affine. Therefore f has the form $f: \operatorname{Spec} B \to \operatorname{Spec} A$. If $\mathscr{F}$ in $q\text{-}Coh(X)$ corresponds to the B-module M, then $f_*(\mathscr{F})$ corresponds to M considered as an A-module. The assertion is obvious in this case. □

Lemma 7.4. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram with f affine. Then there is a natural isomorphism $g^*f_* \xrightarrow{\approx} f'_*g'^*: q\text{-}Coh(X) \to q\text{-}Coh(S')$.

Proof. The base change map $g^*f_* \to f'_*g'^*$ is defined for any commutative diagram. Its construction commutes with localization on S and S', i.e. if S and S' are replaced by U and U' where $g(U') \subset U$ and X and X' are replaced by $f^{-1}(U)$ and $f'^{-1}(U')$. Therefore we can assume that S and S' are affine so the diagram takes the form

$$\begin{array}{ccc} \operatorname{Spec} B' & \xrightarrow{g'} & \operatorname{Spec} B \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A' & \xrightarrow{g} & \operatorname{Spec} A. \end{array}$$

If $\mathscr{F}$ corresponds to the B-module M, then $g^*f_*\mathscr{F} \to f'_*g'^*\mathscr{F}$ corresponds to the map of A'-modules $A' \otimes_A M \to B' \otimes_B M$. Since the diagram is cartesian, $B' = A' \otimes_A B$ so $B' \otimes_B M = A' \otimes_A B \otimes_B M = A' \otimes_A M$ as required. □

The following is a very simple case of flat base-change. Note that the maps i and j here are open embeddings so the functors i^* and j^* are just restrictions to an open set.

Lemma 7.5. *Let $f: X \to S$ and let U be an open set of S. Consider the following cartesian diagram.*

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{j} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & S. \end{array}$$

If $\mathscr{F}$ is a sheaf on X then $i^*f_*\mathscr{F} = g_*j^*\mathscr{F}$ on U.

Proof. If W is an open set of U, one calculates immediately that $\Gamma(W, i^*f_*\mathscr{F}) = \Gamma(f^{-1}W, \mathscr{F}) = \Gamma(W, g_*j^*\mathscr{F})$. $\square$

8. Presheaves of sheaves

In Section 9 we will consider the Čech complex of alternating cochains for presheaves with values in an abelian category, specifically, in a category of sheaves. This idea occurs in [11, Ch. III, Section 4] which was the inspiration for most of this section. Since this is not a very familiar situation, I will review here the few facts needed.

Let Q be a presheaf on a topological space X with values in an abelian category $\mathscr{A}$. If $V \subset U$, the notation $\rho: Q(U) \to Q(V)$ will denote the restriction map. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be a finite open covering of X. Assume that the index set I has been simply ordered. Define $U_{i_0 \cdots i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$ and let $C^n(\mathscr{U}, Q) = \prod_{i_0 < \cdots < i_n} Q(U_{i_0 \cdots i_n})$. Let $p_{i_0 \cdots i_n}: C^n(\mathscr{U}, Q) \to Q(U_{i_0 \cdots i_n})$ be the projection. Define $\varepsilon: Q(X) \to C^0(\mathscr{U}, Q)$ by $p_i \circ \varepsilon = \rho: Q(X) \to Q(U_i)$. Define $\delta_\nu: C^n(\mathscr{U}, Q) \to C^{n+1}(\mathscr{U}, Q)$ by $p_{i_0 \cdots i_{n+1}} \circ \delta_\nu = \rho \circ p_{i_0 \cdots \hat{\imath}_\nu \cdots i_{n+1}}$ *where* $\rho: Q(U_{i_0 \cdots \hat{\imath}_\nu \cdots i_{n+1}}) \to Q(U_{i_0 \cdots i_n})$. Define $d = \sum(-1)^\nu \delta_\nu$. Then $dd = 0$ and $d\varepsilon = 0$. To see this we can apply the embedding theorem for abelian categories [4, 17] to reduce to the classical case of presheaves of abelian groups. If $\mathscr{A}$ is a category of sheaves we can just use the stalk functors in place of the embedding theorem.

It will be convenient to extend the definition of the projections by letting $p_{i_0 \cdots i_n} = 0$ if $i_\mu = i_\nu$ for some $\mu \neq \nu$ and defining $p_{\sigma i_0 \cdots \sigma i_n} = \mathrm{sgn}(\sigma) p_{i_0 \cdots i_n}$ for any permutation σ of $\{i_0 \cdots i_n\}$.

Suppose $\mathscr{V} = \{V_j\}_{j \in J}$ refines $\mathscr{U}$. Choose $\alpha: J \to I$ such that $V_j \subset U_{\alpha(j)}$ and define $\alpha^*: C^n(\mathscr{U}, Q) \to C^n(\mathscr{V}, Q)$ by requiring $p_{j_0 \cdots j_n} \circ \alpha^* = \rho \circ p_{\alpha(j_0) \cdots \alpha(j_n)}$ where $\rho: Q(U_{\alpha(j_0) \cdots \alpha(j_n)}) \to Q(V_{j_0 \cdots j_n})$. Then, as above, we can check that α^* is a cochain map and commutes with ε.

In particular, if $\mathscr{V} = \mathscr{U}$ and $J = I$ with a different ordering, we can choose $\alpha = \mathrm{id}$ and one verifies immediately that α^* is an isomorphism. Therefore, up to isomorphism, the complex $C^*(\mathscr{U}, Q)$ is independent of the ordering.

Lemma 8.1. *Suppose that $U_i = X$ for some $i \in I$. Then*

$$0 \to Q(X) \xrightarrow{\varepsilon} C^0(\mathscr{U}, Q) \xrightarrow{d} C^1(\mathscr{U}, Q) \xrightarrow{d} \cdots$$

is exact.

Proof. We can assume that $i = 0$ is the least element of I. Define $S: C^n(\mathscr{U}, Q) \to C^{n-1}(\mathscr{U}, Q)$ as follows. If $i_1 < \cdots < i_n$, let $p_{i_1 \cdots i_n} \circ S = 0$ if $i_1 = 0$. If $i_1 > 0$, let $p_{i_1 \cdots i_n} \circ S = p_{0 i_1 \cdots i_n}$. Note that $Q(U_{0 i_1 \cdots i_n}) = Q(U_{i_1 \cdots i_n})$ so there is no problem with restriction. One checks easily, as before, that $dS + Sd = 1$ in positive degrees and that $dS + Sd = 1 - \varepsilon p_0$ in degree 0. □

We now consider a specific example from [11, Ch. III, Section 4].

Definition. If $\mathscr{F}$ is a sheaf on X we define a presheaf $P_X\mathscr{F}$ on X with values in the category of sheaves on X by $P_X\mathscr{F}\{U\} = i_{U*}i_U^*(\mathscr{F})$ where $i_U: U \hookrightarrow X$. Similarly, if $W \subset V \subset X$ let $P_V\mathscr{F}\{W\} = j_*(\mathscr{F}|W)$ where $j: W \hookrightarrow V$.

Note that $\Gamma(W, P_X\mathscr{F}\{U\}) = \Gamma(U \cap W, \mathscr{F})$ so if $V \subset U$, restriction from $U \cap W$ to $V \cap W$ gives us a map $P_X\mathscr{F}\{U\} \to P_X\mathscr{F}\{V\}$.

Lemma 8.2. *If $V \subset X$ is open then $P_X\mathscr{F}\{U\}|V = P_V\mathscr{F}\{U \cap V\}$.*

Proof. This is an immediate consequence of Lemma 7.5 taking f and i to be the inclusion maps of U and V in X. □

Now let $\mathscr{U} = \{U_i\}_{i \in I}$ be a finite open covering of X. If V is an open set of X we write $\mathscr{U} \cap V = \{U_i \cap V\}$.

Corollary 8.3. $C^\cdot(\mathscr{U}, P_X\mathscr{F})|V = C^\cdot(\mathscr{U} \cap V, P_V\mathscr{F})$.

The following is proved in [11, Ch. III, Lemma 4.2].

Lemma 8.4. *If $\mathscr{U}$ is a covering of X then $H^0(C^\cdot(\mathscr{U}, P_X\mathscr{F})) = \mathscr{F}$ and $H^i(C^\cdot(\mathscr{U}, P_X\mathscr{F})) = 0$ for $i \neq 0$.*

In other words we have an exact sequence

$$0 \to \mathscr{F} \xrightarrow{\varepsilon} C^0(\mathscr{U}, P_X\mathscr{F}) \to C^1(\mathscr{U}, P_X\mathscr{F}) \to \cdots$$

where ε is induced by the maps $\mathscr{F} = P_X\mathscr{F}\{X\} \to P_X\mathscr{F}\{U_i\}$.

Proof. It is sufficient to show the exactness locally since all terms are sheaves on X. If V lies in some set of $\mathscr{U}$ then Lemma 8.1 applies to $\mathscr{U} \cap V$, so by Corollary 8.3 the restriction of our sequence to V is exact. □

Corollary 8.5. *If $\mathscr{F}^\bullet$ is a cochain complex of sheaves then $\varepsilon:\mathscr{F}^\bullet \to C^\bullet(\mathscr{U}, P_X\mathscr{F}^\bullet)_{\mathrm{tot}}$ is a homology equivalence.*

Proof. Filter $C^\bullet(\mathscr{U}, P_X\mathscr{F}^\bullet)$ by the degree of $P_X\mathscr{F}^\bullet$ and filter $\mathscr{F}^\bullet$ by degree. By Lemma 8.4, $E_1^{pq}(\mathscr{F}^\bullet) \to E_1^{pq}(C^\bullet(\mathscr{U}, P_X\mathscr{F}^\bullet))$ is an isomorphism. If $\mathscr{U}$ has $d+1$ sets then $C^p(\mathscr{U},-) = 0$ for $p > d$ so the spectral sequence converges and the result follows. □

Note that $\mathscr{F}^\bullet$ is not required to be bounded below here because $\mathscr{U}$ is finite.

Lemma 8.6. *Let $\mathscr{U} = \{U_i\}$ be a finite open covering of X. Let $M^\bullet$ and $N^\bullet$ be cochain complexes of presheaves. If $M^\bullet(U) \xrightarrow{\sim} N^\bullet(U)$ is a homology equivalence for all U then so is $C^\bullet(\mathscr{U}, M^\bullet) \to C^\bullet(\mathscr{U}, N^\bullet)$.*

Proof. Filter by the degree of $C^\bullet(\mathscr{U},-)$. The resulting spectral sequences have E_1^{pq}-terms $C^p(\mathscr{U}, H^q(M^\bullet))$ and $C^p(\mathscr{U}, H^q(N^\bullet))$ so the map of E_1-terms is an isomorphism. As in the proof of Corollary 8.5, the spectral sequences converge and we are done. □

9. The Čech patching trick

Let X be a quasicompact separated scheme. Suppose for each affine open set $U \subset X$ we are given a sheaf $\mathscr{S}_U$ of $\mathscr{O}_U$-modules. Suppose also for $V \subset U$ we are given $\rho_{UV}:\mathscr{S}_U|V \to \mathscr{S}_V$ such that $\rho_{UU} = id$ and $\rho_{UW} = \rho_{VW}(\rho_{UV}|W)$ if $W \subset V \subset U$. If the ρ_{UV} are isomorphisms, we can patch the sheaves $\mathscr{S}_U$ together to get a sheaf of $\mathscr{O}_X$-modules. If the $\mathscr{S}_U$ are complexes and the ρ_{UV} are just assumed to be homology equivalences, we will construct a similar patching up to homology equivalence using the method of Lemma 8.4.

As in Section 8, let $P_X\mathscr{S}\{U\} = j_{U*}\mathscr{S}_U$ where $j_U: U \hookrightarrow X$. Then $\Gamma(W, P_X\mathscr{S}\{U\}) = \Gamma(U \cap W, \mathscr{S}_U)$. If $V \subset U$, we define $P_X\mathscr{S}\{U\} \to P_X\mathscr{S}\{V\}$ to be given by $\Gamma(U \cap W, \mathscr{S}_U) \to \Gamma(V \cap W, \mathscr{S}_U|V) \xrightarrow{\rho_{UV}} \Gamma(V \cap W, \mathscr{S}_V)$. This makes $U \mapsto P_X\mathscr{S}\{U\}$ a presheaf. We recover the situation of Section 8 if $\mathscr{S}_U = \mathscr{S}|U$ for a sheaf of $\mathscr{O}_X$-modules $\mathscr{S}$. If $W \subset V \subset X$, we define similarly $P_V\mathscr{S}\{W\} = j_*\mathscr{S}_W$ where $j: W \hookrightarrow V$.

Lemma 9.1. *If the $\mathscr{S}_{\cdot U}$ are chain complexes of quasicoherent sheaves of $\mathscr{O}_U$-modules and each $\rho_{UV}:\mathscr{S}_{\cdot U}|V \to \mathscr{S}_{\cdot V}$ is a homology equivalence, then there is a natural homology equivalence $P_X\mathscr{S}_\cdot\{U\}|V \xrightarrow{\sim} P_V\mathscr{S}_\cdot\{U \cap V\}$.*

Proof. Applying Lemma 7.5 to

$$\begin{array}{ccc} U \cap V & \xrightarrow{i'} & U \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ V & \xrightarrow{i} & X \end{array}$$

shows that $P_X\mathcal{S}.\{U\}|V = i^*j_*(\mathcal{S}._U) = j'_*(\mathcal{S}._U|U\cap V) \xrightarrow{\rho} j'_*(\mathcal{S}._{U\cap V}) = P_V\mathcal{S}.\{U\cap V\}$. By Lemma 7.3, this is a homology equivalence.

Let $\mathcal{U}$ be a finite open affine covering of X and consider the double complex $C^\cdot(\mathcal{U}, P_X\mathcal{S}.)$. The total complex of this is well behaved since $\mathcal{U}$ is finite.

Corollary 9.2. *Let $\mathcal{S}.$ be as in Lemma 9.1. Then there is a natural homology equivalence of total complexes $C^\cdot(\mathcal{U}, P_X\mathcal{S}.)|V \xrightarrow{\sim} C^\cdot(\mathcal{U}\cap V, P_V\mathcal{S}.)$.*

Proof. Let $I = \{i_0, \dots, i_n\}$ and $U_I = U_{i_0,\dots,i_n}$. Then $C^n(\mathcal{U}, P_X\mathcal{S}.)|V = \prod P_X\mathcal{S}.\{U_I\}|V \xrightarrow{\sim} \prod P_V\mathcal{S}.\{U_I\cap V\} = C^n(\mathcal{U}\cap V, P_V\mathcal{S}.)$ by Lemma 9.1. Therefore we get an isomorphism of E_1-terms. □

If V is affine, we have an augmentation $\varepsilon: P_V\mathcal{S}.\{V\} = \mathcal{S}._V \to C^0(\mathcal{U}\cap V, P_V\mathcal{S}.)$.

Lemma 9.3. *Let $\mathcal{S}.$ be as in Lemma 9.1 and let V be an affine open set of X. Then $\varepsilon: \mathcal{S}._V \to C^\cdot(\mathcal{U}\cap V, P_V\mathcal{S}.)$ is a homology equivalence.*

Proof. As in the proof of Lemma 8.4, it is enough to prove this locally. Let W be an affine open set lying in some set of the covering $\mathcal{U}$. Then we have

$$\begin{array}{ccc} \mathcal{S}._V|W & \longrightarrow & C^\cdot(\mathcal{U}\cap V, P_V\mathcal{S}.)|W \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathcal{S}._W & \longrightarrow & C^\cdot(\mathcal{U}\cap W, P_W\mathcal{S}.). \end{array}$$

The bottom arrow is a homology equivalence, since if we filter $C^\cdot(\mathcal{U}\cap W, P_W\mathcal{S}.$ and $\mathcal{S}._W$ by the degree of $\mathcal{S}._W$ we get an isomorphism of E_1-terms by Lemma 8.1. □

Remark. The homology equivalences $\mathcal{S}._V \xrightarrow{\sim} C^\cdot(\mathcal{U}\cap V, P_V\mathcal{S}.) \xleftarrow{\sim} C^\cdot(\mathcal{U}, P_X\mathcal{S}.)|V$ show that $C^\cdot(\mathcal{U}, P_X\mathcal{S}.)$ does give the required patching up to homology equivalence.

These results apply in particular to the sheaves $\mathcal{S}._V = C.(\Gamma(V, \mathcal{O}_V))^\sim$ on a quasicompact separated scheme X. The condition that $\mathcal{S}._U|V \to \mathcal{S}._V$ be a homology equivalence is satisfied since $\Gamma(V,\mathcal{O}_V)\otimes_{\Gamma(U,\mathcal{O}_U)} C.(\Gamma(U,\mathcal{O}_U)) \xrightarrow{\sim} C.(\Gamma(V,\mathcal{O}_V))$ by the remarks preceding Lemma 2.4.

Let $\mathcal{C}.$ be the sheaf on X associated to the presheaf $U \mapsto C.(\Gamma(U,\mathcal{O}_X))$. If U is affine, we have a map $\mathcal{S}._U = C.(\Gamma(U,\mathcal{O}_X))^\sim \to \mathcal{C}.|U$ induced by the maps $C.(\Gamma(U,\mathcal{O}_X))_s \to C.(\Gamma(U_s,\mathcal{O}_X))$ where $U_s \subset U$ is the open set where s is invertible. By Lemma 2.4(4) applied to U, $\mathcal{S}._U \xrightarrow{\sim} \mathcal{C}.|U$ is a homology equivalence.

Lemma 9.4. *If $\mathcal{U}$ is a finite open affine covering of X then $C^\cdot(\mathcal{U}, P_X\mathcal{S}.) \to C^\cdot(\mathcal{U}, P_X\mathcal{C}.)$ is a homology equivalence.*

Proof. The obvious approach, applying Lemma 8.6, fails because $\mathscr{C}_\bullet$ is not quasicoherent so we cannot apply Lemma 7.3 to show that $P_X\mathscr{S}_\bullet\{U\} = j_*\mathscr{S}_{\bullet U} \to j_*(\mathscr{C}_\bullet | U) = P_X\mathscr{C}_\bullet\{U\}$ is a homology equivalence. Instead we proceed as follows. It will suffice to show that $C^\bullet(\mathscr{U}, P_X\mathscr{S}_\bullet)|V \to C^\bullet(\mathscr{U}, P_X\mathscr{C}_\bullet)|V$ is a homology equivalence for all affine open V which are contained in some set of $\mathscr{U}$. We have, by Corollary 9.2, Lemma 9.3, Corollary 8.3, and Lemma 8.4,

$$\begin{array}{ccc}
C^\bullet(\mathscr{U},P_X\mathscr{S}_\bullet)|V & \longrightarrow & C^\bullet(\mathscr{U},P_X\mathscr{C}_\bullet)|V \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle=} \\
C^\bullet(\mathscr{U}\cap V,P_V\mathscr{S}_\bullet) & \longrightarrow & C^\bullet(\mathscr{U}\cap V,P_V\mathscr{C}_\bullet) \\
\uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\
\mathscr{S}_{\bullet V} & \xrightarrow{\sim} & \mathscr{C}_\bullet|V
\end{array}$$

and the result follows. □

10. Proof of the main theorems

Let X be a quasicompact separated scheme over a field k. If $U = \operatorname{Spec} A$ is an affine open set of X then $U \times U = \operatorname{Spec} A^e$. Let $\mathscr{B}_{\bullet U}$ be the complex of quasicoherent sheaves on $U \times U$ associated to the complex of A^e-modules $B_\bullet(A)$. Then $\mathscr{B}_{\bullet U}$ is a resolution of $\delta_{U*}\mathscr{O}_U$ by flat quasicoherent sheaves of $\mathscr{O}_{U\times U}$-modules. By Lemma 2.4, we have a map $\delta_U^*\mathscr{B}_{\bullet U} \xrightarrow{\sim} \mathscr{C}_\bullet | U$ which is a homology equivalence. The aim is to patch together the sheaves $\mathscr{B}_{\bullet U}$ (up to homology equivalence) to produce a global complex having similar properties. We do this by a variant of the construction of Section 9.

Define $\mathscr{E}_\bullet\{U\} = i_*\mathscr{B}_{\bullet U}$ where $i: U \times U \hookrightarrow X \times X$. Note that $\mathscr{E}_\bullet\{U\}$ is quasicoherent by Lemma 7.3. If U and $V \subset U$ are affine, the map $B_\bullet(\Gamma(U,\mathscr{O}_X)) \to B_\bullet(\Gamma(V,\mathscr{O}_X))$ induces a map $\mathscr{B}_{\bullet U}| V \times V \to \mathscr{B}_{\bullet V}$. Therefore, we get $\mathscr{E}_\bullet\{U\} \to \mathscr{E}_\bullet\{V\}$ and it is easily verified that $\mathscr{E}_\bullet$ is a presheaf on the category of affine open sets of X (with values in the category of sheaves on $X \times X$). Let $\mathscr{U}$ be a finite affine open covering of X and define $\mathscr{F}_\bullet^\bullet = C^\bullet(\mathscr{U}, \mathscr{E}_\bullet)$. Let $\mathscr{S}_{\bullet U}$ be the sheaf associated to $C_\bullet(\Gamma(U,\mathscr{O}_X))$ as in Section 9. Since $C_\bullet(A) = A \otimes_{A^e} B_\bullet(A)$ we have $\mathscr{S}_{\bullet U} = \delta_U^*\mathscr{B}_{\bullet U}$ and therefore $P_X\mathscr{S}_\bullet\{U\} = j_*\mathscr{S}_{\bullet U} = j_*\delta_U^*\mathscr{B}_{\bullet U}$.

By Lemma 7.4 applied to the diagram

$$\begin{array}{ccc}
U & \xrightarrow{\delta_U} & U \times U \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\
X & \xrightarrow{\delta} & X \times X
\end{array}$$

we see that $j_*\delta_U^*\mathscr{B}_{\bullet U} \approx \delta^* i_*\mathscr{B}_{\bullet U} = \delta^*\mathscr{E}_\bullet\{U\}$. Therefore $\delta^*\mathscr{E}_\bullet\{U\} = P_X\mathscr{S}_\bullet\{U\}$ and Lemma 9.4 shows that $C^\bullet(\mathscr{U}, \delta^*\mathscr{E}_\bullet) \to C^\bullet(\mathscr{U}, P_X\mathscr{C}_\bullet)$ is a homology equivalence. Since

$\mathscr{U}$ is finite, it is clear that $C^{\bullet}(\mathscr{U},\delta^*\mathscr{E}_{\bullet}) = \delta^*C^{\bullet}(\mathscr{U},\mathscr{E}_{\bullet}) = \delta^*\mathscr{F}^{\bullet}_{\bullet}$. By Corollary 8.5, $\mathscr{C}_{\bullet} \xrightarrow{\varepsilon} C^{\bullet}(\mathscr{U}, P_X\mathscr{C}_{\bullet})$ is a homology equivalence. Let $\mathscr{F}_{\bullet}$ denote the total complex of $\mathscr{F}^{\bullet}_{\bullet}$ and let $\mathscr{G}_{\bullet}$ denote the total complex of $C^{\bullet}(\mathscr{U}, P_X\mathscr{C}_{\bullet})$. We now have homology equivalences $\delta^*\mathscr{F}_{\bullet} \to \mathscr{G}_{\bullet} \xleftarrow{\varepsilon} \mathscr{C}_{\bullet}$. Therefore, $H_q(\delta^*\mathscr{F}_{\bullet}) \approx H_q(\mathscr{C}_{\bullet}) = \mathscr{H}_q$.

Now $\mathscr{B}_{\bullet U} \to \delta_{U^*}(\mathscr{O}_U)$ is a homology equivalence and therefore, by Lemma 7.3, so is $\mathscr{E}_{\bullet}\{U\} \to i_*\delta_{U^*}(\mathscr{O}_U)$. Also $i_*\delta_{U^*}(\mathscr{O}_U) = \delta_* j_*(\mathscr{O}_U) = \delta_* P_X\mathscr{O}_X\{U\}$. By Lemmas 8.6 and 8.4, we get homology equivalences $\mathscr{F}^{\bullet}_{\bullet} = C^{\bullet}(\mathscr{U},\mathscr{E}_{\bullet}) \to C^{\bullet}(\mathscr{U}, \delta_* P_X\mathscr{O}_X) \xleftarrow{\varepsilon} \delta_*\mathscr{O}_X$ showing that $H_0(\mathscr{F}_{\bullet}) = \delta_*\mathscr{O}_X$ and $H_i(\mathscr{F}_{\bullet}) = 0$ if $i \neq 0$. Now $\mathscr{F}_{\bullet}$ is flat on $X \times X$ since $\mathscr{B}_{\bullet U}$ is flat and therefore $\mathscr{E}_{\bullet}\{U\}$ is flat by Lemma 7.2. By Proposition 6.5(1) we have, for a sheaf of $\mathscr{O}_X$-modules $\mathscr{M}$, $H^n(\mathscr{O}_X, \mathscr{M}) = \mathbb{E}\mathrm{xt}^n_{\mathscr{O}_X}(\delta^*\mathscr{F}_{\bullet}, \mathscr{M})$. Therefore $H^n(\mathscr{O}_X, \mathscr{M}) = \mathbb{E}\mathrm{xt}^n_{\mathscr{O}_X}(\mathscr{C}_{\bullet}, \mathscr{M})$ using the homology equivalences $\delta^*\mathscr{F}_{\bullet} \to \mathscr{G}_{\bullet} \xleftarrow{\varepsilon} \mathscr{C}_{\bullet}$. This proves Theorem 2.1.

For Theorem 2.5, $H_q(\delta^*\mathscr{F}_{\bullet}) \approx \mathscr{H}_q$ is locally free by hypothesis so Proposition 6.5(2) shows that the spectral sequences

$$\mathrm{Ext}^p_{\mathscr{O}_X}(H_q(\delta^*\mathscr{F}_{\bullet}), \mathscr{M}) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathscr{O}_X}(\delta^*\mathscr{F}_{\bullet}, \mathscr{M})$$

and

$$H^p(X, \underset{\sim}{\mathrm{ext}}^q_{\mathscr{O}_{X\times X}}(\mathscr{O}_X, \mathscr{M})) \Rightarrow H^{p+q}(\mathscr{O}_X, \mathscr{M})$$

are isomorphic. Using Lemma 4.1 and the homology equivalences $\delta^*\mathscr{F}_{\bullet} \to \mathscr{G}_{\bullet} \xleftarrow{\varepsilon} \mathscr{C}_{\bullet}$ again we see that the first spectral sequence is isomorphic to

$$\mathrm{Ext}^p_{\mathscr{O}_X}(\mathscr{H}_q, \mathscr{M}) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathscr{O}_X}(\mathscr{C}_{\bullet}, \mathscr{M}),$$

proving the theorem.

The resulting isomorphisms are independent of the choice of the covering $\mathscr{U}$. To see this, it is sufficient to compare $\mathscr{U}$ with a refinement $\mathscr{V}$. In this case we have a map $C^{\bullet}(\mathscr{U},-) \to C^{\bullet}(\mathscr{V},-)$ which is compatible with ε. We have $\mathscr{F}_{\bullet} = C^{\bullet}(\mathscr{U},\mathscr{E}_{\bullet}) \to \mathscr{F}'_{\bullet} = C^{\bullet}(\mathscr{V},\mathscr{E}_{\bullet})$, and $\mathscr{G}_{\bullet} = C^{\bullet}(\mathscr{U}, P_X\mathscr{C}_{\bullet}) \to \mathscr{G}'_{\bullet} = C^{\bullet}(\mathscr{V}, P_X\mathscr{C}_{\bullet})$. This gives us a commutative diagram

$$\begin{array}{ccccc}
\delta^*\mathscr{F}_{\bullet} & \longrightarrow & \mathscr{G}_{\bullet} & \xleftarrow{\ \varepsilon\ } & \mathscr{C}_{\bullet} \\
\downarrow & & \downarrow & & \downarrow{=} \\
\delta^*\mathscr{F}'_{\bullet} & \longrightarrow & \mathscr{G}'_{\bullet} & \xleftarrow{\ \varepsilon\ } & \mathscr{C}_{\bullet}
\end{array}$$

from which it follows easily that the isomorphism is the same for $\mathscr{U}$ as for $\mathscr{V}$.

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