Del Pezzo surfaces and non-commutative geometry

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• Joint work with V. Ginzburg (Univ. of Chicago). No definitive results yet, just some observations and questions.

• Motivation: trying to understand recent works of M. Gualtieri ("Branes on Poisson varieties", arXiv:0710.2719) and N.J. Hitchin ("Bihermitian metrics on Del Pezzo surfaces", math.DG/0608213).

• Generalized complex manifolds (Gualtieri, arXiv.math/0703298, Hitchin, math.DG/0503432).

$\langle M, J\rangle$, M a C^∞-mfld, $\mathcal{J} : T(M) \oplus T^*(M) \to T(M) \oplus T^*(M)$ orthogonal w.r.t the indefinite metric and s.t. $\mathcal{J}^2 = -\,\mathrm{id}$, plus some integrability conditions.

$\dim_{\mathbb{R}} M = 2n$ is even (from orthogonality). Examples:

1. $\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$, $J : TM \to TM$ a complex structure on M.

2. $\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$, ω a symplectic form on M.

The general case is an interpolation between the two. But there is a third explicit example.

3. $\mathcal{J} = \begin{pmatrix} -J & P \\ 0 & J^* \end{pmatrix}$, $J : TM \to TM$ a complex structure on M,
   P the imaginary part of a holomorphic Poisson bivector Θ on M.

Thus a holomorphic Poisson manifold $\langle X, \Theta\rangle$, $\Theta \in \Lambda^2\mathcal{T}(X)$ is generalized complex. Example is a Del Pezzo surface: Θ is a section of the (ample) anticanonical bundle $K_X^* = \Lambda^2\mathcal{T}(X)$.

• There is also a notion of "generalized Kähler" (two commuting generalized comp.str. $\mathcal{J}_1$, $\mathcal{J}_2$, plus some positivity). This is equivalent to "Bihermitian": $\langle M, g, I^+, I^-, b\rangle$, g a metric, I^+, I^- complex structures, b a 2-form, $d^c_- \omega_- = db = -d^c_+ \omega_+$.

Hitchin showed that this structure exists for Del Pezzo surfaces $\langle X, \Theta\rangle$ s.t. the divisor $E \subset X$ of zeroes of Θ is smooth (automatically an elliptic curve). Method: fix $I = I^+$, use Θ to obtain a Hamiltonian flow, flow I to I_-. $E \subset X$ is fixed by the flow (but the flow on E non-trivial – a translation w.r.t. to the group structure of E). The argument is very non-trivial!

Hitchin: generalized Kähler is to "holomorphic Poisson" what hyperkähler is to "holomorphically sympletic".

• Generalized Kähler manifolds appear in physics as "targets for $(2,2)$ SUSY string theories", whatever it means. At least, expect an A_∞-category associated to M. In the three examples:

1. $\mathcal{D}^b(X)$, the derived category of coherent sheaves on $X = \langle M, J\rangle$.
2. The Fukaya category of $\langle M, \omega\rangle$.
3. Not known ("the category of generalized branes"). Should be some "non-commutative" deformation of Ex. 1.

• <u>Our goal</u>: in the Poisson case $\langle X, \Theta\rangle$, to understand those parts of the story that could be understood algebraically, esp. when X is a Del Pezzo surface. We believe that the appropriate context is that of "non-commutative algebraic geometry". So far, we are mostly concerned with deformation theory of $\langle X, \Theta\rangle$, both commutative and non-commutative ("quantizations"). "Generalized branes" are certainly more interesting, but we are too stupid.

• Recollection on the symplectic case.

• A holomorphically symplectic mfld $\langle X, \Omega\rangle$ is Poisson, $\Theta = \Omega^{-1}$. There are very interesting compact examples – K3 surfaces, their Hilbert schemes, etc. These have been studied from the generalized complex point of view (e.g. by D. Huybrechts), but they are dissimilar to Del Pezzos in that for a projective hol. sympl. X, we have $H^2(X, \mathcal{O}_X) \neq 0$. It is better to consider the following case.

Def. A holomorphically symplectic X is a *symplectic resolution* if the natural map $X \to Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$ is projective and birational.

Examples: minimal resolutions of Kleinian singularities ($Y = \mathbb{C}^2/\Gamma$), cotangent bundles T^*G/P, etc.

Deformations are controlled by the period map

$$X \mapsto [\Omega_X] \in H^2_{DR}(X).$$

Every symplectic resolution X has a universal formal deformation $\widetilde{X}/S$, S is the formal neighborhood of $[\Omega_X] \in H^2_{DR}(X)$ (—,M. Verbitsky, math.AG/0005007). So, we have a "local Torelli Theorem". If Y admits a dilating $\mathbb{C}^*$-action (that is, $\lambda^*\Omega = \lambda \cdot \Omega$ for all $\lambda \in \mathbb{C}^*$), then $[\Omega_X] = 0$, and the deformation can be made global.

• There are also non-commutative deformations.

Def. A *quantization* of a Poisson alg. variety X is a sheaf of associative $\mathbb{C}[[h]]$-algebras $\mathcal{O}_h$ s.t. $\mathcal{O}_h/h \cong \mathcal{O}_X$, and

$$fg - gf = h\{f, g\} \mod h^2.$$

In the symplectic case, existence of quantization proved by B. Fedosov (in the algebraic setting, R. Bezrukavnikov, —, following the holomorphic case done by B. Tsygan and R. Nest, math.AG/0309290 resp. math.AG/9906020). Moreover, we have a classification.

Thm. 1. X a symplectic resolution, $\widetilde{X}/S$ its universal deformation. Then there exists a quantization $\widetilde{\mathcal{O}}_h$ of $\widetilde{X}$ s.t. any quantization of X is obtained by restriction via a unique section

$$\Delta \to \Delta \times S$$

of the projection $\Delta \times S \to \Delta$, where $\Delta = \operatorname{Spec} \mathbb{C}[[h]]$ is the formal disc.

In other words, there is a universal deformation with $b_2(X) + 1$ parameters, $b_2(X)$ directions commutative, one non-commutative.

Thm. 2 (—, math.AG/03121134). X Poisson, $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2, 3$. L line bundle on X. Then there exists a canonical one-parameter Poisson deformation $X_t/\operatorname{Spec} \mathbb{C}[[t]]$ s.t. L extends to a Poisson line bundle on X_t, and for any local section l, we have

$$\{t, l\} = l.$$

If X is a symplectic resolution, the period map gives a line $[\Omega] = [\Omega_X] + tc_1(L)$, and if in addition L is ample, X_t is affine over $\mathbb{C}((t))$.

Thm. 3. $\mathcal{O}_h$ a quantization of X, $s : \Delta \to \Delta \times S$ the classifying map. Then L_t can be quantized to a $\mathcal{O}_h$-$\mathcal{O}_{h'}$-bimodule, where the classifying map $s' : \Delta \to \Delta \times S$ of the quantization $\mathcal{O}_{h'}$ is

$$s'(h) = s(h) + hc_1(L).$$

Thus there is additional symmetry for derived categories $\mathcal{D}^b(\mathcal{O}_h)$: $\mathcal{D}^b(\mathcal{O}_h) \cong \mathcal{D}^b(\mathcal{O}_{h'})$, $\operatorname{Pic}(X)$ acts by translations on S preserving $\mathcal{D}^b$.

• A note on proofs. Poisson deformations are controlled by the *Poisson complex* $\langle \Lambda^{\cdot}\mathcal{T}(X), \delta\rangle$, with $\delta(a) = [a, \Theta]$ w.r.t. the Schouten bracket (Brylinski). Θ gives a map

$$\Omega^{\cdot}_{DR}(X) \to \Lambda^{\cdot}\mathcal{T}(X).$$

If X is symplectic, this is an isomorphism! – thus the deformations are rigid, we have T_1-lifting principle, etc. If X is only Poission, $c_1(L) \in H^1(X, \Omega^1_X)$ still defines a deformation class in $H^1(X, \mathcal{T}(X))$.

• The Poisson case.

• Now let $\langle X, \Theta \rangle$ be a general Poisson variety.

Thm. 2 is completely general (it even holds for singular X)! But the map $\Omega^{\cdot}_{DR}(X) \to \Lambda^{\cdot}\mathcal{T}(X)$ is not a quasiisomorphism, so there is no rigidity. The base S of the universal Poisson deformation $\widetilde{X}/S$ can be very bad. Quantizations still correspond to sections of $\Delta \times S \to \Delta$ by Kontsevich Formality, at least for affine X. But this is very inexplicit and depends on a choice (of "Drinfeld associator").

But there is one situation which is not so bad.

Def. A *log-symplectic mfld* X is a pair $\langle X, D \rangle$, $D \subset X$ a NC divisor, equipped with a Poisson bivector Θ which induces an isomorphism

$$\Omega^1_X\langle D \rangle \to \mathcal{T}_D(X),$$

$\mathcal{T}_D(X) \subset \mathcal{T}(X)$ the subbundle of vector fields tangent to D. (In particular, the defining ideal $\mathcal{J}_D \subset \mathcal{O}_X$ of $D \subset X$ is Poisson.)

• Assume that X is log-symplectic. Then the map

$$\Omega^{\cdot}_X\langle D \rangle \to \Lambda^{\cdot}\mathcal{T}_D(X)$$

is a quasiisomorphism. $\Lambda^{\cdot}\mathcal{T}_D$ controls deformations of the pair $\langle X, \mathcal{J}_D \subset \mathcal{O}_X \rangle$. It is well-known that $H^{\cdot}(X, \Omega^{\cdot}_X\langle D \rangle) \cong H^{\cdot}_{DR}(X \setminus D)$. Therefore the deformation functors of $\langle X, \mathcal{J}_D \rangle$ and $X \setminus D$ are the same:

$$\mathrm{Def}_{Pois}(\langle X, \mathcal{J}_D \rangle) \cong \mathrm{Def}_{Pois}(X \setminus D).$$

Thus we have the same rigidity for $\langle X, \mathcal{J}_D \rangle$ as for $X \setminus D$, and the local Torelli holds: deformations of $\langle X, \mathcal{J}_D \rangle$ are controlled by the period map

$$\langle X, \mathcal{J}_D \rangle \mapsto [\Theta^{-1}] \in H^2_{DR}(X \setminus D)$$

(Θ^{-1} has poles, but they are logarithmic).

• As for quantizations, some are provided by a procedure of Tsygan and Nest (in the holomorphic case). Same picture: quantizations correspond to sections

$$s : \Delta \to \Delta \times S,$$

$S = H^2_{DR}(X \setminus D)$.

Method is the same: Fedosov connection, but now with logarithmic poles.

Conjecturally, this procedure gives all quantization of the pair $\langle X, \mathcal{J}_D \rangle$ ($\mathcal{J}_D$ quantizes to a two-sided ideal in $\mathcal{O}_h$).

• A still more special case: D is smooth. Then the embedding

$$\Lambda^{\cdot}\mathcal{T}_D(X) \to \Lambda^{\cdot}\mathcal{T}(X)$$

is a quasiisomorphism (easy local computation). So, all quantizations and all Poisson deformations preserve D! Of course, a Poisson deformation can move it to some other $D' \subset X$. (Not surprising, since D is the zero locus of the anticanonical section Θ^{top}.)

• An important special case of Thm. 2: assume that $L = \mathcal{O}(D)$. Then $c_1(L)$ restricts to 0 in $H^2_{DR}(X \setminus D)$, so that the canonical deformation X_t is trivial. Under quantization, L becomes a $\mathcal{O}_h$-bimodule, so we can take its tensor powers.

The algebra

$$A^{\cdot}_h = \bigoplus_n H^0(X \times X, L_h^{\otimes n})$$

is then a deformation of the algebra

$$A^{\cdot} = \bigoplus_n H^0(X, L^{\otimes n}).$$

If L is ample, we obtain a quantization of the projective coordinate ring of X – that is, a non-commutative deformation in the sense of M. Artin, M. Van den Bergh, J.T. Stafford et al.

• <u>The Del Pezzo case.</u>

Now assume $X = X_r$ is a Del Pezzo surface, that is, the blowup of $\mathbb{P}^2$ in r sufficiently general points, $0 \leq r \leq 8$. Assume fixed a section $\Theta \in H^0(X, K^{-1})$, so that X is Poisson. Assume that the divisor of zeroes of Θ is a smooth elliptic curve $E \subset X$. If X_r is a blowup of X_0, then the image of E in X_0 contains all the blown-up points.

• Non-commutative deformations of the anticanonical coordinate ring $\bigoplus_i H^0(X, K^{-n})$ of X are well-studied (Van den Bergh et al.) The starting point is a choice of Θ and a choice of a translation along the elliptic curve E. Same parameters in Hitchin's bihermitian structure. $E \subset X$ also deforms. Since $\Theta|_E = 0$, the deformation of E is commutative. But K deforms to a non-trivial sheaf K_h on $E \times E$ – the graph of the translation. The algebra

$$\bigoplus_n H^0(E \times E, K_h^{-n})$$

is known as *Sklyanin algebra* (at least for $X = X_0$).

Also, Van den Bergh has a "non-commutative blowup procedure" (works because subscheme $E \subset X$ stays commutative and contains all the blown-up points).

• From our point of view:

Def$_{Pois}(X, \Theta)$ smooth, a formal neighb. in $H^2_{DR}(X \setminus E)$.

$b_2(X \setminus E) = b_2(X) + b_1(E) - 1$

Since E is smooth, all quantizations and Poisson deformations are logarithmic w.r.t. E.

K quantizes to a bimodule because $K_{X \setminus E}$ is trivial.

The translation: apply to Θ the residue map $\Omega^2_X\langle E\rangle \to \Omega^1_E$, or rather,

$$\Lambda^2 \mathcal{T}_E(X) \to \mathcal{T}(E).$$

• Globalization and symmetries.

In the symplectic resolution case X/Y, deformations are global if Y has a $\mathbb{C}^*$-action. Example: Kleinian singularity $Y = \mathbb{C}^2/\Gamma$.

And in the Kleinian case, there is also an additional symmetry: $H^2_{DR}(X)$ has a simply laced root lattice inside it, the Weyl groups W acts on $S = H^2_{DR}(X)$, the quotient S/W is smooth and is the base for a universal Poisson deformation of Y. The affine Weyl group $W_{aff} = \mathrm{Pic}(X) \rtimes W$ acts on $D^b(X)$.

In the Del Pezzo case, the global picture is not clear.

$H^2(X_r \setminus E)$ is the sum of $H^1(E)$ and orthogonal to $[K]$ in $H^2(X)$. $H^1(E)$ has an $SL_2(\mathbb{Z})$-action. The orthogonal has a root lattice of type $A_1, A_1 \times A_1, A_2 \times A_1, A_4, D_5, E_6, E_7, E_8$, so there is a Weyl group W. So, we can consider W, W_{aff}, $W_{aff} \times SL_2(\mathbb{Z})$, etc.

But if any of them act on the deformation space, what is the fixed point? Similarly, if $\mathbb{C}^*$ acts on the deformation space with positive weights, what is the center?

X_r without Θ has deformations for $r \geq 5$, has automorphisms for $r \leq 3$, is rigid for $r = 4$. For $r = 6, 7, 8$, can deform to a cone over E (elliptic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, also appeared in K. Saito's work). This can even further degenerate to a rational log-canonical singularity. So, we have a positive-weight $\mathbb{C}^*$-action. But our theory does not work well for singular X!

For small r, probably E should degenerate rather than X_r. To have a residual action of $\mathbb{C}^*$, E should be, for example, a triangle in $X_0 = \mathbb{P}^2$. But then $H^2(X \setminus E)$ drops! so, our theory not sufficient: logarithmic deformations can never smooth out a singular NC divisor.