Lecture 2.

Second bicomplex for cyclic homology. Connes' differential. Cyclic homology and the de Rham cohomology in the HKR case. Homology of small categories. Simplicial vector spaces and homology of the category Δ^{opp}.

2.1 Second bicomplex for cyclic homology.

Recall that in the end of the last lecture, we have defined cyclic homology $HC_{\bullet}(A)$ of an associative unital algebra A over a field k as the homology of the total complex of a certain explicit bicomplex (1.8) constructed from A and its tensor powers[1]. This definition is very *ad hoc*. Historically, it was arrived at as a result of a certain computation of the homology of Lie algebras of matrices over A; it is not clear at all what is the invariant meaning of this explicit bicomplex. Next several lectures will be devoted mostly to various alternative definitions of cyclic homology. Unfortunately, all of them are *ad hoc* to some degree, and none is completely satisfactory and should be regarded as final. No really good explanation of what is going on exists to this day. But we can at least do computations.

The first thing to do is to notice that not only we know that the even-numbered columns $C'_{\bullet}(A)$ of the cyclic bicomplex (1.8) are acyclic, but we actually have a contracting homotopy h for them given by $h(a_0 \otimes \dots \otimes a_n) = 1 \otimes a_0 \otimes \dots \otimes a_n$. This can be used to remove these acyclic columns entirely. The result is the *second bicomplex for cyclic homology* which has the form

$$
\begin{array}{ccccccc}
 & & & & & & A \\
 & & & & & & \uparrow b \\
 & & & & A & \xrightarrow{B} & A^{\otimes 2} \\
 & & & & \uparrow b & & \uparrow b \\
 & & A & \xrightarrow{B} & A^{\otimes 2} & \xrightarrow{B} & A^{\otimes 3} \\
 & & \uparrow b & & \uparrow b & & \uparrow b \\
A & \xrightarrow{B} & A^{\otimes 2} & \xrightarrow{B} & A^{\otimes 3} & \xrightarrow{B} & A^{\otimes 4}, \\
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b
\end{array}
\tag{2.1}
$$

with the horizontal differential $B : A^{\otimes n} \to A^{\otimes(n+1)}$ given by

$$B = (\mathsf{id} - \tau) \circ h \circ (\mathsf{id} + \tau + \dots + \tau^{n-1}).$$

This differential B is known as the *Connes' differential*, or the *Connes-Tsygan differential*, or the *Rinehart differential*. In the commutative case, it was discovered by G. Rinehart back in the 1960es; then it was forgotten, and rediscovered independently by A. Connes and B. Tsygan in about 1982 (in the general associative case).

Lemma 2.1. *The diagram* (2.1) *is a bicomplex whose total complex is quasiisomorphic to the total complex of* (1.8).

[1]By the way, a good reference for everything related to cyclic homology is J.-L. Loday's book *Cyclic homology*, Springer, 1998. Personally, I find also very useful an old overview article B. Feigin, B. Tsygan, *Additive K-theory*, in Lecture Notes in Math, vol. 1289.

Proof. This is a general fact from linear algebra which has nothing to do with the specifics of the situation. Assume given a bicomplex $K_{\bullet,\bullet}$ with differentials $d_{1,0}$, $d_{0,1}$, and assume given a contracting homotopy h for the complex $\langle K_{i,\bullet}, d_{0,1}\rangle$ for every odd $i \geq 1$. Define the diagram $\langle K'_{\bullet,\bullet}, d'_{1,0}, d'_{0,1}\rangle$ by

$$K'_{i,j} = K_{2i,j-i}, \quad d'_{0,1} = d_{0,1}, \quad d'_{1,0} = d_{1,0} \circ h \circ d_{1,0}.$$

Then $d'_{1,0} \circ d'_{1,0} = d_{1,0} \circ h \circ d^2_{1,0} \circ h \circ d_{1,0} = 0$, and

$$\begin{aligned} d'_{1,0} \circ d'_{0,1} + d'_{0,1} \circ d'_{1,0} &= d_{1,0} \circ h \circ d_{1,0} \circ d_{0,1} + d_{0,1} \circ d_{1,0} \circ h \circ d_{1,0} \\ &= -d_{1,0} \circ h \circ d_{0,1} \circ d_{1,0} - d_{1,0} \circ d_{0,1} \circ h \circ d_{1,0} \\ &= -d_{1,0} \circ (h \circ d_{0,1} + d_{0,1} \circ h) \circ d_{1,0} = -d_{1,0} \circ d_{1,0} = 0, \end{aligned}$$

so that $K'_{\bullet,\bullet}$ is indeed a bicomplex, and one checks easily that the map

$$\bigoplus_i (-1)^i \operatorname{id} \oplus (-1)^{i+1}(h \circ d_{1,0}) : \bigoplus_i K'_{i,\bullet-i} = \bigoplus_i K_{2i,\bullet-2i} \to \bigoplus_i K_{i,\bullet-i}$$

is a chain homotopy equivalence between the total complexes of $K_{\bullet,\bullet}$ and $K'_{\bullet,\bullet}$. □

Exercise 2.1. *Check this.*

2.2 Comparison with de Rham cohomology.

The main advantage of the complex (2.1) with respect to (1.8) is that it allows the comparison with the usual de Rham cohomology in the commutative case.

Proposition 2.2. *In the assumptions of the Hochschild-Kostant-Rosenberg Theorem, denote* $n = \operatorname{dim} \operatorname{Spec} A$*, and assume that* $n!$ *is invertible in the base field* k *(thus either* $\operatorname{char} k = 0$*, or* $\operatorname{char} k > n$*). Then the HKR isomorphism* $HH_\bullet(A) \cong \Omega^\bullet_A$ *extends to a quasiisomorphism between the bicomplex* (2.1) *and the bicomplex*

$$\begin{array}{ccccccc} & & & & & & A \\ & & & & & & \uparrow{\scriptstyle 0} \\ & & & & A & \xrightarrow{d} & \Omega^2_A \\ & & & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \\ & & A & \xrightarrow{d} & \Omega^2_A & \xrightarrow{d} & \Omega^3_A \\ & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \\ A & \xrightarrow{d} & \Omega^2_A & \xrightarrow{d} & \Omega^3_A & \xrightarrow{d} & \Omega^4_A, \\ \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \end{array}$$

where the vertical differential is 0*, and the horizontal differential is the de Rham differential* d*.*

Proof. First we show that under the additional assumption of the Proposition, the HKR isomorphism extends to a canonical quasiisomorphism P between the Hochschild complex and the complex $\langle \Omega^\bullet_A, 0\rangle$. This quasiisomorphism P is given by

$$P(a_0 \otimes a_1 \otimes \cdots \otimes a_i) = \frac{1}{i!} a_0 da_1 \wedge \cdots \wedge a_i.$$

This is obviously a map of complexes: indeed, since $d(a_1 a_2) = a_1 da_2 + a_2 da_1$ by the Leibnitz rule, the expression for $P(b(a_0 \otimes \cdots \otimes a_i))$ consists of terms of the form

$$a_0 a_j da_1 \wedge \cdots \wedge da_{j-1} \wedge da_{j+1} \wedge \cdots \wedge da_i,$$

every such term appears exactly twice, and with opposite signs. Thus P induces a map $p : HH_\bullet(A) \to \Omega^\bullet_A$. By HKR, both sides are isomorphic flat finitely generated A-modules; by Nakayama Lemma, to prove that p an isomorphism, it suffices to prove that it is surjective. This is clear – since A is commutative, the alternating sum

$$\sum_\sigma \mathsf{sgn}(\sigma) a_0 \otimes \sigma(a_1 \otimes \cdots \otimes a_i)$$

over all the permutations σ of the indices $1, \dots, i$ is a Hochschild cycle for any $a_0, \dots, a_i \in A$, and we have

$$P\left(\sum_\sigma \mathsf{sgn}(\sigma) a_0 \otimes \sigma(a_1 \otimes \cdots \otimes a_i)\right) = a_0 da_1 \wedge \cdots \wedge a_i.$$

So, p is an isomorphism, and P is indeed a quasiisomorphism. It remains to prove that it sends the Connes-Tsygan differential B to the de Rham differential d – that is, we have $P \circ B = d \circ P$. This is also very easy to see. Indeed, every term in $B(a_0 \otimes \dots a_i)$ contains 1 as one of the factors. Since 1 is annihilated by the de Rham differential d, the only non-trivial contribution to $P(B(a_0 \otimes \dots a_i))$ comes from the terms which contain 1 as the first factor, so that we have

$$\begin{aligned} P(B(a_0 \otimes \dots a_i)) &= \sum_{j=0}^{i-1} P(h(\tau^j(a_0 \otimes \cdots \otimes a_i))) = \frac{1}{i!} \sum_{j=0}^{i-1} \tau^j(da_0 \wedge \cdots \wedge da_i) \\ &= \frac{1}{(i-1)!} da_0 \wedge \cdots \wedge da_i, \end{aligned}$$

which is exactly $d(P(a_0 \otimes \cdots \otimes a_i))$. □

Corollary 2.3. *In the assumptions of Proposition* 2.2, *we have a natural isomorphism*

$$HP_\bullet(A) \cong H^\bullet_{DR}(\operatorname{Spec} A)((u)).$$

Proof. Clear. □

Remark 2.4. For example, the Connes-Tsygan differential B in the lowest degree, $B : A \to A^{\otimes 2}$, is given by

$$B(a) = 1 \otimes a + a \otimes 1,$$

which is very close to the formula $a \otimes 1 - 1 \otimes a$ which gives the universal differential $A \to \Omega^1(A)$ into the module of Kähler differentials $\Omega^1(A)$ for a commutative algebra A. The difference in the sign is irrelevant because of the HKR identification of $HH_1(A)$ and $\Omega^1(A)$ – if one works out explicitly the identification given in Lecture 1, one checks that $1 \otimes a$ goes to 0, so that it does not matter with which sign we take it. The comparison map P in the lowest degree just sends $a \otimes b$ to adb, so that $P(B(a)) = da$.

2.3 Generalities on small categories.

Our next goal is to give a slightly less *ad hoc* definition of cyclic homology also introduced by A. Connes. This is based on the techniques of the so-called homology of small categories. Let us describe it.

For any small category Γ and any base field k, the category $\mathrm{Fun}(\Gamma, k)$ of functors from Γ to k-vector spaces is an abelian category, and the direct limit functor $\lim_{\overrightarrow{\Gamma}}$ is right-exact. Its derived functors are called *homology functors* of the category Γ and denoted by $H_{\bullet}(\Gamma, E)$ for any $E \in \mathrm{Fun}(\Gamma, k)$. For instance, if Γ is a groupoid with one object with automorphism group G, then $\mathrm{Fun}(\Gamma, k)$ is the category of k-representations of the group G; the homology $H_{\bullet}(\Gamma, -)$ is then tautologically the same as the group homology $H_{\bullet}(G, -)$. Analogously, the inverse limit functor $\lim_{\overleftarrow{\Gamma}}$ is left-exact, and its derived functors $H^{\bullet}(\Gamma, -)$ are the cohomology functors of the category Γ. In the group case, this corresponds to the usual cohomology of the group. By definition of the inverse limit, we have

$$H^{\bullet}(\Gamma, E) = \mathrm{Ext}^{\bullet}(k^{\Gamma}, E),$$

where k^{Γ} denotes the constant functor from Γ to k-Vect. In particular, $H^{\bullet}(\Gamma, k^{\Gamma}) = \mathrm{Ext}^{\bullet}(k^{\Gamma}, k^{\Gamma})$ is an algebra, and the homology $H_{\bullet}(\Gamma, k^{\Gamma})$ with constant coefficients is a module over this algebra.

In general, it is not easy to compute the homology of a small category Γ with arbitrary coefficients $E \in \mathrm{Fun}(\Gamma, k)$. One way to do it is to use resolutions by the *representable functors* $k_{[a]}$, $[a] \in \Gamma$ – these are by definition given by

$$k_{[a]}([b]) = k[\Gamma([a], [b])]$$

for any $[b] \in \Gamma$, where $\Gamma([a], [b])$ is the set of maps from $[a]$ to $[b]$ in Γ, and $k[-]$ denotes the k-linear span. By Yoneda Lemma, we have $\mathsf{Hom}(k_{[a]}, E') = E'([a])$ for any $E' \in \mathrm{Fun}(\Gamma, k)$; therefore $k_{[a]}$ is a projective object in $\mathrm{Fun}(\Gamma, k)$, higher homology groups $H_i(\Gamma, k_{[a]})$, $i \geq 1$ vanish, and again by Yoneda Lemma, we have

$$\mathsf{Hom}(\lim_{\overrightarrow{\Gamma}} k_{[a]}, k) \cong \mathsf{Hom}(k_{[a]}, k^{\Gamma}) \cong k^{\Gamma}([a]) = k, \tag{2.2}$$

so that $H_0(\Gamma, k_{[a]}) = k$. Every functor $E \in \mathrm{Fun}(\Gamma, k)$ admits a resolution by sums of representable functors — for example, we have a natural adjunction map

$$\bigoplus_{[a] \in \Gamma} E([a]) \otimes k_{[a]} \to E,$$

and this map is obviously surjective. Analogously, for cohomology, we can use *co-representable functors* $k^{[a]}$ given by

$$k^{[a]}([b]) = k[\Gamma([b], [a])]^*;$$

they are injective, $H^0(\Gamma, k^{[a]}) \cong k$, and every $E \in \mathrm{Fun}(\Gamma, k)$ has a resolution by products of functors of this type.

One can also think of functors in $\mathrm{Fun}(\Gamma, k)$ as "presheaves of k-vector spaces on Γ^{opp}". This is of course a very complicated name for a very simple thing, but it is useful because it brings to mind familiar facts about sheaves on topological spaces or étale sheaves on schemes. Most of these facts hold for functor categories as well, and the proofs are actually much easier. Specifically, it is convenient to use a version of Grothendieck's "formalism of six functors". Namely, if we are given two small categories Γ, Γ', and a functor $\gamma : \Gamma \to \Gamma'$, then we have an obvious restriction functor $\gamma^* : \mathrm{Fun}(\Gamma', k) \to \mathrm{Fun}(\Gamma, k)$. This functor has a left-adjoint $\gamma_!$ and a right-adjoint f_*, called the left and right *Kan extensions*. (If you cannot remember which is left and which is right, but are familiar with sheaves, then the notation $\gamma_!$, γ_* will be helpful.)

The direct and inverse limit over a small category Γ are special cases of this construction – they are Kan extensions with respect to the projection $\Gamma \to \mathsf{pt}$ onto the point category pt. The representable and co-representable functors $k_{[a]}$, $k^{[a]}$ are obtained by Kan extensions with respect to the embedding $\mathsf{pt} \to \Gamma$ of the object $[a] \in \Gamma$. Given three categories Γ, Γ', Γ'', and two functors $\gamma : \Gamma \to \Gamma'$, $\gamma' : \Gamma' \to \Gamma''$, we obviously have $(\gamma' \circ \gamma)^* \cong \gamma^* \circ \gamma'^*$, which implies by adjunction $\gamma'_! \circ \gamma_! \cong (\gamma' \circ \gamma)_!$ and $\gamma'_* \circ \gamma_* \cong (\gamma' \circ \gamma)_*$. In general, the Kan extensions $\gamma_!$, γ_* have derived functors $L^{\bullet} f_!$, $R^{\bullet} f_*$; just as in the case of homology and cohomology, one can compute them by using resolutions by representable resp. corepresentable functors.

2.4 Homology of the category Δ^{opp}.

Probably the first useful fact about homology of small categories is a description of the homology of the category Δ^{opp}, the opposite to the category Δ of finite non-empty totally ordered sets. We denote by $[n] \in \Delta^{opp}$ the set of cardinality n. Objects $E \in \operatorname{Fun}(\Delta^{opp}, k)$ are known as *simplicial k-vector spaces*. Explicitly, such an object is given by k-vector spaces $E([n])$, $n \geq 1$, and various maps between them, among which one traditionally distinguishes the *face maps* $d_n^i : E([n+1]) \to E_{[n]}$, $0 \leq i \leq n$ – the face map d_n^i corresponds to the injective map $[n] \to [n+1]$ whose image does not contain the $(i+1)$-st element in $[n+1]$.

Lemma 2.5. *For any simplicial vector space $E \in \operatorname{Fun}(\Delta^{opp}, k)$, the homology $H_{\bullet}(\Delta^{opp}, E)$ can be computed by the* standard complex $E_{\bullet}$ *given by $E_n = E([n+1])$, $n \geq 0$, with differential $d : E_n \to E_{n-1}$, $n \geq 1$, equal to*

$$d = \sum_{0 \leq i \leq n} (-1)^i d_n^i.$$

Proof. By definition, we have a map $E_0 = E([1]) \to H_0(\Delta^{opp}, E)$, which obviously factors through the cokernel of the differential d, and this is functorial in E.

Denote by $H'_{\bullet}(\Delta^{opp}, E)$ the homology groups of the standard complex $E_{\bullet}$. Then every short exact sequence of simplicial vector spaces induces a long exact sequence of $H'_{\bullet}(\Delta^{opp}, -)$, so that $H'_{\bullet}(\Delta^{opp}, -)$ form a δ-functor. Moreover, $H'_0(\Delta^{opp}, E)$ is by definition the cokernel of the map $d = d_1^0 - d_1^1 : E([2]) \to E([1])$. This is the same as the direct limit of the diagram

$$E([2]) \underset{d_1^1}{\overset{d_1^0}{\rightrightarrows}} E([1])$$

of two k-vector spaces $E([2])$, $E([1])$ and two maps d_1^0, d_1^1 between them (a direct limit of this type is called a *coequalizer*). Since this diagram has an obvious map to Δ^{opp}, we have a natural map

$$H'_0(\Delta^{opp}, E) \to \lim_{\overrightarrow{\Delta^{opp}}} E = H_0(\Delta^{opp}, E),$$

and by the universal property of derived functors, it extends to a canonical map

$$H'_{\bullet}(\Delta^{opp}, E) \to H_{\bullet}(\Delta^{opp}, E) \tag{2.3}$$

of δ-functors. We have to prove that it is an isomorphism. Since every $E \in \operatorname{Fun}(\Delta^{opp}, k)$ admits a resolution by sums of representable functors $k_{[n]}$, $[n] \in \Delta^{opp}$, it suffices to prove that the map (2.3) is an isomorphism for all $E = k_{[n]}$ (this is known as *the method of acyclic models*). This is clear: $H_i(\Delta^{opp}, k_{[n]})$ is k for $i = 0$ and 0 otherwise, and the left-hand side of (2.3) is the homology of the standard complex of an n-simplex, which is also k in degree 0 and 0 in higher degrees. □

Exercise 2.2. *Compute the cohomology $H^{\bullet}(\Delta^{opp}, E)$. Hint: compute $k^{[1]}$.*