Lecture 3.

Connes' cyclic category Λ. Cyclic homology as homology of the category Λ. Yet another bicomplex, and a definition of cyclic homology using arbitrary resolutions.

3.1 Connes' category Λ.

For applications to cyclic homology, A. Connes introduced a special small category known as *the cyclic category* and denoted by Λ. Objects $[n]$ of Λ are indexed by positive integers n, just as for Δ^{opp}. Maps between $[n]$ and $[m]$ can be defined in various equivalent ways; we give two of them.

Topological description. The object $[n]$ is thought of as a "wheel" – the circle S^1 with n distinct marked points, called *vertices*. A continuous map $f : [n] \to [m]$ is *good* if it sends marked points to marked points, has degree 1, and is *monotonous* in the following sense: for any connected interval $[a, b] \subset S^1$, the preimage $f^{-1}([a, b]) \subset S^1$ is connected. Morphisms from $[n]$ to $[m]$ in the category Λ are homotopy classes of good maps $f : [n] \to [m]$.

Combinatorial description. Consider the category Cycl of linearly ordered sets equipped with an order-preserving endomorphism σ. Let $[n] \in$ Cycl be the set $\mathbb{Z}$ with the natural linear order and endomorphism $\sigma : \mathbb{Z} \to \mathbb{Z}$, $\sigma(a) = a + n$. Let $\Lambda_\infty \subset$ Cycl be the full subcategory spanned by $[n]$, $n \geq 1$ – in other words, for any n, m, let $\Lambda_\infty([n], [m])$ be the set of all maps $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a) \leq f(b) \quad \text{whenever } a \leq b, \qquad f(a+n) = f(a) + m, \tag{3.1}$$

for any $a, b \in \mathbb{Z}$. For any $[n], [m] \in \Lambda_\infty$, the set $\Lambda_\infty([n], [m])$ is acted upon by the endomorphism σ (on the left, or on the right, by definition it does not matter). We define the set of maps $\Lambda([n], [m])$ in the category Λ by $\Lambda([n], [m]) = \Lambda_\infty([n], [m])/\sigma$.

Here is the correspondence between the two definitions. First of all, we note that homotopy classes of continuous monotonous maps from $\mathbb{R}$ to itself whcih send integral points into integral points are obviously in one-to-one correspondence with non-descreaing maps from $\mathbb{Z}$ to itself. Now, in the topological description above, we may assume that if we consider S^1 as the unit disc in the complex plane $\mathbb{C}$, then the marked points are placed at the roots of unity. Then the universal cover of S^1 is $\mathbb{R}$, and after rescaling, we may assume that exactly the integral points are marked. Thus any good map $f : S^1 \to S^1$ induces a map $\mathbb{R} \to \mathbb{R}$ which sends integral points into integral points, or in other words, a non-decreasing map from $\mathbb{Z}$ to itself. Such a map $\mathbb{R} \to \mathbb{R}$ comes from a map $S^1 \to S^1$ if and only if the corresponding map $\mathbb{Z} \to \mathbb{Z}$ commutes with σ.

There is also an explicit description of maps in Λ by generators and relations which we will not need; an interested reader can find it, for instance, in Chapter 6 of Loday's book.

Given an object $[n] \in \Lambda$, it will be convenient to denote by $V([n])$ the set of vertices of the wheel $[n]$ (in the topological description), and it will be also convenient to denote by $E([n])$ the set of *edges* of the wheel – that is, the clock-wise intervals $(s, s') \subset S^1$ between the two neighboring vertices $s, s' \in V([n])$.

Lemma 3.1. *The category Λ is self-dual: we have $\Lambda \cong \Lambda^{opp}$.*

Proof. In the combinatorial description, define a map $\Lambda_\infty([m], [n]) \to \Lambda_\infty([n], [m])$ by $f \mapsto f^o$, $f^o : \mathbb{Z} \to \mathbb{Z}$ given by

$$f^o(a) = \min\{b \in \mathbb{Z} | f(b) \geq a\}.$$

This is obviously compatible with compositions and bijective, so that we get an isomorphism $\Lambda_\infty \cong \Lambda_\infty^{opp}$. Being compatible with σ, it descends to Λ.

In the topological description, note that for any map $f : [n] \to [n]$ and any edge $e = (s, s') \in E([m])$, the preimage $f^{-1}(e) \subset S^1$ with respect to the corresponding good map $f : S^1 \to S^1$ lies entirely within a single edge $e' \in E([m'])$, Thus we get a natural map $f^o : E([m]) \to E([n])$. We leave it to the reader to check that this extends to a duality functor $\Lambda \to \Lambda^o$ which interchanges $V([n])$ and $E([n])$. □

If we only consider those maps in (3.1) which send $0 \in \mathbb{Z}$ to 0, then the resulting subcategory in Λ_∞ is equivalent to Δ^{opp}.

Exercise 3.1. *Check this. Hint: use the duality* $\Lambda \cong \Lambda^o$.

This gives a canonical embedding $j : \Delta^{opp} \to \Lambda_\infty$, and consequently, an embedding $j : \Delta^{opp} \to \Lambda$ (this is injective on maps). Functors in $\operatorname{Fun}(\Lambda, k)$ are called *cyclic k-vector spaces*. Any cyclic k-vector space E defines by restriction a simplicial k-vector space $j^*E \in \operatorname{Fun}(\Delta^{opp}, k)$.

3.2 Homology of the category Λ.

The category Λ conveniently encodes the maps m_i and τ between various tensor powers $A^{\otimes n}$ used in the complex (1.8): m_i corresponds to the map $f \in \Lambda([n+1], [n])$ given by

$$f(a(n+1)+b) = \begin{cases} an+b, & b \leq i, \\ an+b-1, & b > i, \end{cases}$$

where $0 \leq b \leq n$, and τ is the map $a \mapsto a+1$, twisted by the sign (alternatively, one can say that m_i are obtained from face maps in Δ^{opp} under the embedding $\Delta^{opp} \subset \Lambda_p$). The relations $m_{i+1} \circ \tau = \tau \circ m_i$, $0 \leq i \leq n-1$, and $m_0 \circ \tau = (-1)^n m_n$ between these maps which we used in the proof of Lemma 1.4 are encoded in the composition laws of the category Λ. Thus for any object $E \in \operatorname{Fun}(\Lambda, k)$ – they are called *cyclic vector spaces* – one can form the bicomplex of the type (1.8):

$$(3.2) \qquad \begin{array}{ccccccc}
\dots \longrightarrow & E([1]) & \xrightarrow{\mathsf{id}} & E([1]) & \xrightarrow{\mathsf{id}-\tau} & E([1]) \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots \longrightarrow & E([2]) & \xrightarrow{\mathsf{id}+\tau} & E([2]) & \xrightarrow{\mathsf{id}-\tau} & E([2]) \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots & \dots & & \dots & & \dots \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots \longrightarrow & E([n]) & \xrightarrow{\mathsf{id}+\tau+\dots+\tau^{n-1}} & E([n]) & \xrightarrow{\mathsf{id}-\tau} & E([n]) \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b}
\end{array}$$

(where b and b' are obtained from m_i and τ by the same formulas as in (1.8)). We also have the periodic version, the Connes' exact sequence and the Hodge-to-de Rham spectral sequence (where the role of Hochschild homology is played by the homology $H_\bullet(\Delta^{opp}, j^*E)$).

Lemma 3.2. *For any $E \in \operatorname{Fun}(\Lambda, k)$, the homology $H_\bullet(\Lambda, E)$ can be computed by the bicomplex* (1.8).

Proof. As in Lemma 2.5, we use the method of acyclic models. We denote by $H'_\bullet(\Lambda, E)$ the homology of the total complex of the bicomplex (3.2). Just as in Lemma 2.5, we have a natural map $H'_0(\Lambda, E) \to H_0(\Lambda, E)$, we obtain an induced functorial map

$$H'_\bullet(\Lambda, E) \to H_\bullet(\Lambda, E),$$

and we have to prove that it is an isomorphism for $E = k_{[n]}$, $[n] \in \Lambda$. We know that for such E, in the right-hand side we have k in degree 0 and 0 in higher degrees. On the other hand, the action of the cyclic group $\mathbb{Z}/m\mathbb{Z}$ generated by $\tau \in \Lambda([m],[m])$ on $\Lambda([n],[m])$ is obviously free, and we have

$$\Lambda([n],[m])/\tau \cong \Delta^{opp}([n],[m])$$

– every $f : \mathbb{Z} \to \mathbb{Z}$ can be uniquely decomposed as $f = \tau^j \circ f_0$, where $0 \leq j < m$, and f_0 sends 0 to 0. The rows of the complex (1.8) compute

$$H_\bullet(\mathbb{Z}/m\mathbb{Z}, k_{[n]}([m])) \cong k\left[\Delta^{opp}([n],[m])\right],$$

and the first term in the corresponding spectral sequence is the standard complex for the simplicial vector space $k^\Delta_{[n]} \in \operatorname{Fun}(\Delta^{opp}, k)$ represented by $[n] \in \Delta^{opp}$. Therefore this complex computes $H_\bullet(\Delta^{opp}, k^\Delta_{[n]})$, and we are done by Lemma 2.5. □

There is one useful special case where the computation of $H_\bullet(\Lambda, E)$ is even easier.

Definition 3.3. A cyclic vector space $E \in \operatorname{Fun}(\Lambda, k)$ is *clean* if for any $[n] \in \Lambda$, the homology $H_i(\mathbb{Z}/n\mathbb{Z}, E([n]))$ with respect to the $\mathbb{Z}/n\mathbb{Z}$-action on $E([n])$ given by τ is trivial for all $i \geq 1$.

In practice, a cyclic vector space can be clean for two reasons. First, $E([n])$ might be a free $k[\mathbb{Z}/n\mathbb{Z}]$-module for any n. Second, the base field k might have characteristic 0, so that finite groups have no higher homology with any coeffients. In any case, for a clean $E \in \operatorname{Fun}(\Lambda, k)$, computing the homology of the rows of the bicomplex (3.2) reduces to taking the coinvariants $E([n])_\tau$ with respect to the autmorphism τ, and the whole (3.2) reduces to a complex

$$\dots \xrightarrow{b} E([n])_\tau \xrightarrow{b} \dots \xrightarrow{b} E([2])_\tau \xrightarrow{b} E([1])_\tau, \tag{3.3}$$

with the differential induced by the differential b of (3.2). We note that the coinvariants $E([n])_\tau$, $n \geq 1$, do not form a simplicial vector space; nevertheless, the differential b is well-defined.

3.3 The small category definition of cyclic homology.

Assume now again given an associative unital algebra A over a field k. To define cyclic homology $HC_\bullet(A)$ as homology of the cyclic category Λ, one constructs a cyclic k-vector space $A_\#$ in the following way: for any $[n] \in \Lambda$, $A_\#([n]) = A^{\otimes n}$, where we think of the factors A in the tensor product as being numbered by vertices of the wheel $[n]$, and for any map $f : [n] \to [m]$, the corresponding map $A_\#(f) : A^{\otimes n} \to A^{\otimes m}$ is given by

$$A_\#(f)\left(\bigotimes_{i \in V([n])} a_i\right) = \bigotimes_{j \in V([m])} \prod_{i \in f^{-1}(j)} a_i, \tag{3.4}$$

where $V([n])$, $V([m])$ are the sets of vertices of the wheel $[n],[m] \in \Lambda$. We note that for any $j \in V([m])$, the finite set $f^{-1}(j)$ has a natural total order given by the clockwise order on the circle S^1. Thus, although A need not be commutative, the product in the right-hand side is well-defined. If $f^{-1}(j)$ is empty for some $j \in V([m])$, then the right-hand side involves a product numbered by the empty set; this is defined to be the unity element $1 \in A$.

As an immediate corollary of Lemma 3.2, we obtain the following.

Proposition 3.4. *We have a natural isomorphism $HC_\bullet(A) \cong H_\bullet(\Lambda, A_\#)$.* □

This isomorphism is also obviously compatible with the periodicity, the Connes' exact sequence, and the Hodge-to-de Rham spectral sequence. In particular, the standard complex for the simplicial k-vector space $j^*A_\#$ is precisely the Hochschild homology complex, so that we have $HH_\bullet(A) = H_\bullet(\Delta^{opp}, j^*A_\#)$.

Exercise 3.2. *Show that the Hochschild homology complex which computes $HH_\bullet(A, M)$ for an A-bimodule M also is the standard complex for a simplicial k-vector space. Does it extend to a cyclic vector space?*

3.4 Example: yet another bicomplex for cyclic homology.

The definition of cyclic homology using small categories may seem too abstract at first, but this is actually a very convenient technical tool: it allows to control the combinatorics of various complexes in a quite efficient way. As an illustration of this, let me sketch, in $\operatorname{char} 0$, yet one more description of cyclic homology by an explicit complex (this definition has certain advantages explained in the next subsection).

For any k-vector space V equipped with a non-zero covector $\eta \in V^*$, $\eta : V \to k$, contraction with η defines a differential $\delta : \Lambda^{\bullet+1}V \to \Lambda^\bullet V$ on the exterior algebra $\Lambda^\bullet V$, and the complex $\langle \Lambda^\bullet V, \delta\rangle$ is acyclic, so that $\Lambda^{\geq 1}V$ is a resolution of $k = \Lambda^0 V$. This construction depends functorially on the pair $\langle V, \eta\rangle$, so that it can be applied poinwise to the representable functor $k_{[1]} \in \operatorname{Fun}(\Lambda, k)$ equipped with the natural map $\eta : k_{[1]} \to k^\Lambda$. The result is a resolution $\Lambda^\bullet k_{[1]}$ of the constant cyclic vector space $k^\Lambda \in \operatorname{Fun}(\Lambda, k)$.

Here is another description of the exterior powers $\Lambda^\bullet k_{[1]}$. Consider a representable functor $k_{[i]}$ for some $i \geq 1$, and let $\overline{k}_{[i]} \in \operatorname{Fun}(\Lambda, k)$ be its quotient given by

$$\overline{k}_{[i]}([n]) = k\left[\Lambda([i],[n])\right]/\{f \in \Lambda([i],[n]) \mid f \text{ not injective}\};$$

in other words, $\overline{k}_{[i]}([n])$ is spanned by injective maps from $[i]$ to $[n]$. Then $k_{[i]}$ is acted upon by the cyclic group $\mathbb{Z}/i\mathbb{Z}$ of automorphisms of $[i] \in \Lambda$, this action descends to the quotient $\overline{k}_{[i]}$, and we have

$$\Lambda^i k_{[i]} = \left(\overline{k}_{[i]}\right)_\tau,$$

where $\tau : \overline{k}_{[i]} \to \overline{k}_{[i]}$ is the generator of $\mathbb{Z}/i\mathbb{Z}$ twisted by $(-1)^{i+1}$. The differential δ lifts to a differential $\delta : \overline{k}_{[i]} \to \overline{k}_{[i-1]}$ given by the alternating sum of the maps $\overline{k}_{[i]} \to \overline{k}_{[i-1]}$ induced by the i injective maps $[i-1] \to [i]$. We note, however, that the complex $\overline{k}_{[\bullet]}$ is no longer a resolution of k^Λ.

Lemma 3.5. *For any $i \geq 1$, we have $H_j(\Lambda, \overline{k}_{[i]}) = 0$ if $j \neq i-1$, and k if $j = i-1$. The $\mathbb{Z}/i\mathbb{Z}$-action on $k = H_{i-1}(\Lambda, \overline{k}_{[i]})$ by the $\mathbb{Z}/i\mathbb{Z}$-action on $\overline{k}_{[i]}$ is given by the sign representation. Moreover, for any $E \in \operatorname{Fun}(\Lambda, k)$, we have $H_\bullet(\Lambda, \overline{k}_{[i]} \otimes E) \cong H_{\bullet+i-1}(\Delta^{opp}, E)$, with the sign action of $\mathbb{Z}/i\mathbb{Z}$.*

Proof. The cyclic object $\overline{k}_{[i]} \in \operatorname{Fun}(\Lambda, k)$ is clean, and the corresponding complex (3.3) is the quotient of the standard complex of the elementary $(i-1)$-simplex by the subcomplex spanned by all faces of dimension less than $i-1$. In other words, $H_\bullet(\Lambda, \overline{k}_{[i]})$ is the reduced homology of the $(i-1)$-sphere. This proves the first claim. The second claim is obvious: the term of degree $i-1$ in the complex (3.3) is isomorphic to $(k[\mathbb{Z}/i\mathbb{Z}])_\tau$, and this is the sign representation by the definition of τ. The third claim now follows immediately from the well-known Künneth formula, which says that for any simplicial vector spaces $V, W \in \operatorname{Fun}(\Delta^{opp}, k)$, the standard complex of the product $V \otimes W$ is naturally quasiisomorphic to the product of the standard complexes for V and W. □

Now, for any cyclic k-vector space $E \in \operatorname{Fun}(\Lambda, k)$, and any $[i] \in \Lambda$, the product $E \otimes \overline{k}_{[i]} \in \operatorname{Fun}(\Lambda, k)$ is clean, so that it makes sense to consider the complex (3.3). Then the differential $\delta : \overline{k}_{[i]} \to \overline{k}[i-1]$ induces a map between these complexes, and we can form a bicomplex $K_{\bullet,\bullet}(E)$ given by

$$K_{i,j}(E) = \left(\overline{k}_i([j+1]) \otimes E([j+1])\right)_\tau, \tag{3.5}$$

where the horizontal differential $K_{\bullet+1,\bullet}(E) \to K_{\bullet,\bullet}(E)$, henceforth denoted by $\widetilde{B}$, is induced by δ, and the vertical differential is the Hochschild differential b, as in (3.3).

Lemma 3.6. *Assume that* $\operatorname{\mathsf{char}} k = 0$. *Then for any* $E \in \operatorname{Fun}(\Lambda, k)$, *the total complex of the bicomplex* (3.5) *computes the homology* $H_\bullet(\Lambda, k)$.

Proof. Since $\operatorname{\mathsf{char}} k = 0$, every cyclic vector space is clean, and we can compute cyclic homology by using the complex (3.3). Since $\langle \Lambda^{\geq 1} k_{[1]}, \delta\rangle$ is a resolution of the constant cyclic vector space k^Λ, we have

$$H_\bullet(\Lambda, E) \cong \mathbb{H}_\bullet(\Lambda, K_\bullet \otimes E),$$

where $K_\bullet \cong \Lambda^{\bullet+1} k_{[1]}$, and the differential in $K_\bullet \otimes E$ is induced by δ. Applying (3.3) to the right-hand side *almost* gives the bicomplex (3.5) – the difference is that we take $K_i = (\overline{k}_{[i+1]})_\tau$ instead of $\overline{k}_{[i+1]}$. Thus it suffices to prove that the natural map

$$H_\bullet(\Lambda, \overline{k}_{[i]} \otimes E) \to H_\bullet\left(\Lambda, \left(\overline{k}_{[i]}\right)_\tau \otimes E\right)$$

is an isomorphism for any $[i] \in \Lambda$. But since $\operatorname{\mathsf{char}} k = 0$, the cyclic groups have no homology, so that the right-hand side is isomorphic to

$$H_\bullet(\Lambda, \overline{k}_{[i]} \otimes E)_\tau.$$

And by Lemma 3.5, τ on $H_\bullet(\Lambda, \overline{k}_{[i]} \otimes E)$ is the identity map. □

Assume now that $E = A_\#$ for some associative unital A-algebra A. Then the bicomplex (3.5) is similar to the second bicomplex (2.1) for cyclic homology in the following sense: for any $i \geq 0$, the column $K_{i,\bullet}(A_\#)$ of (3.5) computes the Hochschild homology $HH_\bullet(A)$, with the same degree shift as in (2.1).

What happens is the following. Recall that to obtain the Hochschild homology complex, one uses the bar resolution $C_\bullet(A)$. However, to compute the Hochschild homology $HH_\bullet(A)$, any other resolution would do. In particular, we can take any integer $n \geq 2$, and consider the n-fold tensor product

$$C^n_\bullet(A) = C_\bullet(A) \otimes_A C_\bullet(A) \otimes_A \cdots \otimes_A C_\bullet(A).$$

This is obviously a complex of free A-bimodules, and it is quasiisomorphic to $A \otimes_A A \otimes_A \cdots \otimes_A A \cong A$, so that it is a good resolution. Using this resolution to compute $HH_\bullet(A)$, we obtain a complex $CH^n_\bullet(A)$ whose l-th term $CH^n_l(A)$ is the sum of several copies of $A^{\otimes(n+l)}$, and these copies are numbered by elements in the set

$$M^n_l = \Lambda_{inj}([n], [l+n])/\tau$$

of injective maps $[n] \to [l+n]$ considered modulo the action of the cyclic permutation $\tau : [l+n] \to [l+n]$. In other words, the terms of the complex $CH^n_\bullet(A)$ are numbered by wheels $[n+m]$, $m \geq 0$, with n marked points considered modulo cyclic permutation. These n points cut the wheel into n intervals of lengths $l_1, l_2, \dots, l_n$ with $l_1 + l_2 + \cdots + l_n = m + n$, and the corresponding term in $CH^n_l(A)$ computes the summand

$$A \otimes_{A^{opp} \otimes A} (C_{l_1-1}(A) \otimes_A C_{l_2-1}(A) \otimes_A \cdots \otimes_A C_{l_1-1}(A))$$

in

$$CH^n_\bullet(A) = A \otimes_{A^{opp} \otimes A} (C_\bullet(A) \otimes_A \cdots \otimes_A C_\bullet(A)).$$

The differential $\widetilde{b} : CH^n_{\bullet+1}(A) \to CH^n_\bullet(A)$ restricted to the term which corresponds to some injective $f : [n] \to [n+l+1]$ is the alternating sum of the maps m_i corresponding to surjective maps $[n+l+1] \to [n+l]$ such that the composition $[n] \to [n+l+1] \to [n+l]$ is still injective – in other words, we allow to contract edges of the marked wheel $[n+l+1]$ *unless an edge connects two marked points*. Of course, $CH^1_\bullet(A)$ is the usual Hochschild homology complex, and $\widetilde{b} = b$ is the usual Hochschild differential (since there is only one marked point, every edge can be contracted).

We leave it to the reader to check that the complex $CH^n_\bullet(A)$ is precisely isomorphic to the complex $K_{n,\bullet+n}(A_\#)$.

One can also show that the periodicity in $HC_\bullet(A)$ corresponds to shifting the bicomplex (3.5) by one column to the left, just as in (2.1), so that the Hodge filtration on $HC_\bullet(A)$ is also induced by the stupid filtration on (3.5) in the horizontal direction. Thus *a postriori*, (3.5) and (2.1) are even quasiisomorphic as bicomplexes, and the horizontal differential $\widetilde{B}$ in (3.5) can be identified with the Connes-Tsygan differential B. However, this is not at all easy to see by a direct computation.

3.5 Cyclic homology computed by arbitrary resolution.

To show why (3.5) is useful, let me show how it can be modified so that the bar resolution $C_\bullet(A)$ is replaced with an arbitrary projective resolution $P_\bullet$ of the diagonal bimodule S (I follow the exposition in my paper *Cyclic homology with coefficients*, `math.KT/0702068`, which is based on ideas of B. Tsygan).

For simplicity, I will only explain how to do this for the first two columns of (3.5). This gives a resolution-independent description of the Connes-Tsygan differential $B = \widetilde{B}$, but says nothing about possible higher differentials in the Hodge-to-de Rham spectral sequence.

Fix a projective resolution $P_\bullet$ with the augmentation map $r : P_\bullet \to A$. Consider the resolution $P^2_\bullet = P_\bullet \otimes_A P_\bullet$ of the same diagonal bimodule A. Note that the augmentation map r induces *two* quasiisomorphisms $r_0, r_1 : P^2_\bullet \to P_\bullet$ given by

$$r_0 = r \otimes_A \mathsf{id}, \qquad r_1 = \mathsf{id} \otimes_A r.$$

In general, there is no reason why these two maps should be equal. However, being two maps of projective resolutions of A which induce the same identity map on A itself, they should be chain-homotopic. Choose a chain homotopy $\iota : P^2_\bullet \to P_{\bullet+1}$.

Now consider the complexes

$$\overline{P}_\bullet = A \otimes_{A^{opp} \otimes A} P_\bullet, \quad \overline{P}^2_\bullet = A \otimes_{A^{opp} \otimes A} P^2_\bullet$$

which compute $HH_\bullet(A)$, and the maps $\overline{r}_0$, $\overline{r}_0$, $\overline{\iota}$ between them induces by r_0, r_1 and ι. Notice that the complex $\overline{P}^2_\bullet$ has another description: we have

$$\overline{P}^2_\bullet = \bigoplus_{l,\bullet-l} A \otimes_{A^{opp} \otimes A} P_l \otimes_A P_{\bullet-l},$$

and for any two A-bimodules M, N, we have

$$A \otimes_{A^{opp} \otimes A} (M \otimes_A N) = M \otimes N / \{ma \otimes n - m \otimes an, am \otimes n - m \otimes na \mid a \in A, m \in M, n \in N\},$$

which is manifestly symmetric in m and n. Thus we have a natural involution $\tau : \overline{P}^2_\bullet \to \overline{P}^2_\bullet$. This involution obviously interchanges $\overline{r}_0$ and $\overline{r}_1$, but there is no reason why it should be in any way compatible with the map $\bar{\iota}$ – all we can say is that $\tau \circ \bar{\iota}$ is another chain homotopy between $\bar{r}_0$ and $\bar{r}_1$. Thus the map

$$\widetilde{B} = \bar{\iota} - \tau \circ \bar{\iota} : \overline{P}_{\bullet}^2 \to \overline{P}_{\bullet+1}$$

commutes with the differentials.

Lemma 3.7. *The map $\widetilde{B}$ induces the same map on the Hochschild homology $HH_{\bullet}(A)$ as the Connes-Tsygan differential B.*

Sketch of a proof. One checks that the map we need to describe does not depend on choices: neither of a projective resolution $P_{\bullet}$, since any two such resolutions are chain-homotopy equivalent, nor of the map ι, since any two such are chain-homotopic to each other. Thus to compute it, we can take any $P_{\bullet}$ and any ι. If we take $P_{\bullet} = C_{\bullet}(A)$, the bar-resolution, and let ι be the sum of tautological maps $A^{\otimes l} \otimes A^{\otimes l'} \to A^{\otimes l+l'}$, then $\widetilde{B}$ is precisely the same as in the bicomplex (3.5). □

Remark 3.8. In the assumptions of the Hochschild-Kostant-Rosenberg Theorem, it would be very interesting to try to work out explicitly the map $\widetilde{B}$ for the Koszul resolution.