Lecture 4.

Combinatorics of the category Λ: cohomology of Λ and $\Lambda_{\leq n}$, periodicity, classifying spaces. Fibrations and cofibrations of small categories; Λ_∞ as a fibered category over Λ.

4.1 Cohomology of the category Λ and periodicity.

In the last lecture, we have shown that the homology $H_\bullet(\Lambda, E)$ with coefficients in some cyclic vector space $E \in \operatorname{Fun}(\Lambda, E)$ can be computed by the standard complex (3.2); in particular, we have the periodicity map $u : H_{\bullet+2}(\Lambda, E) \to H_\bullet(\Lambda, E)$ and the Connes' exact triangle

$$H_\bullet(\Delta^{opp}, j^*E) \longrightarrow H_\bullet(\Lambda, E) \xrightarrow{u} H_{\bullet-2}(\Lambda, E) \longrightarrow ,$$

where $j : \Delta^{opp} \to \Lambda$ is the embedding defined in the last lecture. Today, we want to give a more invariant description of the periodicity map. That such a description should exist is more-or-less clear. Indeed, homology $H_\bullet(\Lambda, -)$ — or rather, hyperhomology $\mathbb{H}(\Lambda, -)$ — is a functor from the derived category $\mathcal{D}(\Lambda, k)$ of the abelian category $\operatorname{Fun}(\Lambda, k)$ to the derived category $\mathcal{D}(k\text{-Vect})$. By definition, this functor is adjoint to the tautological embedding $\mathcal{D}(k\text{-Vect}) \to \mathcal{D}(\Lambda, k)$, $k \mapsto k^\Lambda$, so that by Yoneda Lemma, every natural transformation $H_{\bullet+2}(\Lambda, -) \to H_\bullet(\Lambda, -)$ should be induced by an element in

$$\operatorname{Ext}^2(k^\Lambda, k^\Lambda) = H^2(\Lambda, k).$$

Thus to describe periodicity, we have to compute the cohomology $H^\bullet(\Lambda, k)$ of the category Λ with constant coefficients $k = k^\Lambda \in \operatorname{Fun}(\Lambda, k)$.

The computation itself is not difficult: since the category Λ is self-dual, the complex (3.2) has an obvious dualization, and exactly the same argument as in the proof of Lemma 3.2 shows that dualized complex computes $H^\bullet(\Lambda, E)$ for any $E \in \operatorname{Fun}(\Lambda, k)$. For the constant functor k, this gives

$$H^\bullet(\Lambda, k) \cong k[u], \tag{4.1}$$

where, as before, $k[u]$ means "the space of polynomials in one formal variable u of degree 2". It is only slightly more difficult to see that the isomorphism (4.1) is an algebra isomorphism, and the action of the generator $u \in H^2(\Lambda, k)$ on homology $H_\bullet(\Lambda, -)$ is the periodicity map. One can argue, for instance, as follows. The same operation of "shifting the bicomplex by two columns" induces a periodicity map $H^\bullet(\Lambda, E) \to H^{\bullet+2}(\Lambda, E)$; this map is functorial, thus (1) induced by an element in $H^2(\Lambda, k)$, and obviously the same one, and (2) compatible with the algebra structure on

$$H^\bullet(\Lambda, k) = \operatorname{Ext}^\bullet(k^\Lambda, k^\Lambda),$$

so that $H^\bullet(\Lambda, k)$ must be a unital algebra over the polynomial algebra $k[u]$ generated by the periodicity map. Since by (4.1), it is isomorphic to $k[u]$ as a $k[u]$-module, it must also be isomorphic to $k[u]$ as an algebra.

However, it will be useful to have a more explicit description of the generator $u \in H^2(\Lambda, k)$.

To obtain such a description, we use the topological interpretation of the category Λ — in other words, we treat $[n] \in \Lambda$ as a wheel formed by marking n points on the circle S^1. Note that this defines a cellular decomposition of the circle: its 0-cells are vertices $v \in V([n])$, and its 1-cells are edges $e \in E([n])$. Denote by $C_\bullet([n])$ the corresponding complex of length 2 which computes the homology $H_\bullet(S^1, k)$. Any map $f \in \Lambda([n], [m])$ induces a cellular map $S^1 \to S^1$, or at any rate, a map which sends 0-skeleton into 0-skeleton, and thus induces a map $C_\bullet([n]) \to C_\bullet([m])$. In this way, $C_\bullet([n])$ becomes a length-2 complex of cyclic vector spaces. Since the homology of the circle $H_i(S^1, k)$ is equal to k for $i = 0, 1$ and 0 otherwise, and does not depend on the cellular decomposition, the homology of the complex $C_{\bullet} \in \operatorname{Fun}(\Lambda, k)$ is k^{Λ} in degree 0 and 1, and 0 in other degrees. Thus we have an exact sequence

$$0 \longrightarrow k^{\Lambda} \longrightarrow C_1 \longrightarrow C_0 \longrightarrow k \longrightarrow 0 \tag{4.2}$$

of cyclic vector spaces. Explicitly, $V([n]) \cong \Lambda([1],[n])$, so that $C_0([n]) = k[V([n])] = k[\Lambda([1],[n])]$, and C_0 is canonically isomorphic to the representable functor $k_{[1]}$. As for C_1, we have by definition

$$C_1([n]) = k[E([n])] = k[\Lambda([n],[1])],$$

and the map $C_1(f) : C_1([n]) \to C_1([m])$ corresponding to a map $f : [n] \to [m]$ is given by

$$C_1(f)(e) = \sum_{e' \in f^{o-1}(e)} e' \in k[E([m])] \tag{4.3}$$

for any edge $e \in E([n])$, so that C_1 is canonically identified with the corepresentable functor $k^{[1]}$. All in all, the exact sequence (4.2) can be rewritten as

$$0 \longrightarrow k^{\Lambda} \longrightarrow k^{[1]} \longrightarrow k_{[1]} \longrightarrow k^{\Lambda} \longrightarrow 0. \tag{4.4}$$

This represents by Yoneda a certain class in $H^2(\Lambda, k) = \operatorname{Ext}^2(k^{\Lambda}, k^{\Lambda})$.

Lemma 4.1. *The class $u' \in H^2(\Lambda, k)$ represented by* (4.4) *is equal to the periodicity generator u.*

Proof. Let us first prove the equality up to an invertible constant. To do this, it suffices to prove that the cone of the map $H_{\bullet+2}(\Lambda, k) \to H_{\bullet}(\Lambda, k)$ induced by u' is isomorphic to k in degree 0 and trivial in other degrees. This cone is the hyperhomology $\mathbb{H}(\Lambda, C_{\bullet})$. Since $C_0 = k_{[1]}$ is representable, it already has all the homology we want from the cone, so that we have to prove that

$$H_{\bullet}(\Lambda, C_1) = H_{\bullet}(\Lambda, k^{[1]}) = 0$$

(in all degrees). Denote by M the kernel of the natural map $k_{[1]} \to k^{\Lambda}$, so that we have short exact sequences

$$0 \longrightarrow M \longrightarrow k_{[1]} \longrightarrow k^{\Lambda} \longrightarrow 0,$$
$$0 \longrightarrow k^{\Lambda} \longrightarrow k^{[1]} \longrightarrow M \longrightarrow 0.$$

Computing the homology long exact sequence for the first of these exact sequences, we see that the boundary differential $\delta_1 : H_i(\Lambda, M) \to H_{i+1}(\Lambda, k)$ is non-trivial, so that the first short exact sequence is not split, and that in fact δ_1 is an isomorphism for all $i \geq 0$. To prove the claim, it suffices to check that the boundary differential $\delta_2 : H_{i+1}(\Lambda, M) \to H_i(\Lambda, k)$ in the second long exact sequence also is an isomorphism for all i. Since everything is compatible with with $k[u']$-action, it suffices to prove it for $i = 0$ – in other words, we have to prove that the generator of $H_0(\Lambda, k) = k$ goes to 0 under the map $k^{\Lambda} \to k^{[1]}$. But if not, this means by definition that the second short exact sequence is split. This is not possible: the duality $\Lambda \cong \Lambda^{opp}$ together with the usual duality $k\text{-Vect}^{opp} \to k\text{-Vect}$, $V \mapsto V^*$ induce a fully faithfull duality functor $\operatorname{Fun}(\Lambda, k)^o \to \operatorname{Fun}(\Lambda, k)$, and this functor sends our short exact sequences into each other.

As for the constant, we note that it obviously must be universal, thus invertible in any field, thus either 1 or -1. On the other hand, in the definition of (4.4) there is a choice: we have to choose an orientation of the cirle S^1. Switching the orientation changes the sign of u', so that we can always achieve $u = u'$. We leave it at that. □

4.2 Canonical resolution.

We can extend the exact sequence (4.4) to a resolution of the constant functor k^Λ by iterating it – the result is a complex of the form

$$\dots \longrightarrow k^{[1]} \longrightarrow k_{[1]} \longrightarrow k^{[1]} \longrightarrow k_{[1]},$$

where the maps $k^{[1]} \to k_{[1]}$ are as in (4.4), and the maps $k_{[1]} \to k^{[1]}$ are the composition maps $k_{[1]} \to k^\Lambda \to k^{[1]}$. Moreover, for any cyclic vector space $E \in \mathrm{Fun}(\Lambda, k)$, we have a canonical resolution

$$\dots \longrightarrow k^{[1]} \otimes E \longrightarrow k_{[1]} \otimes E \longrightarrow k^{[1]} \otimes E \longrightarrow k_{[1]} \otimes E. \tag{4.5}$$

The periodicity map for E is induced by $\mathsf{id} \otimes u \in \mathrm{Ext}^2(E, E)$, and it can be represented explicitly by the obvious periodicity endomorphism of (4.5) which shift everything to the left by two terms.

It is instructive to see what happens if compute $H_\bullet(\Lambda, E)$ by replacing E with (4.5), as in Lemma 3.6 in the last Lecture. Both $k_{[1]} \otimes E$ and $k^{[1]} \otimes E$ are clean in the sense of Definition 3.3, so that we can compute $H_\bullet(\Lambda, -)$ by the complex (3.3). Applying it to (4.5) gives a double complex $M_{i,j}(E)$ with terms

$$M_{i,j}(E) = \begin{cases} (k_{[1]}([j+1]) \otimes E([j+1]))_\tau, & i \text{ even}, \\ (k^{[1]}([j+1]) \otimes E([j+1]))_\tau, & i \text{ odd}. \end{cases}$$

To identify further $M_{0,j}(E) = E([j+1])$, we need to choose a vertex $v \in V([j+1])$ (for instance, we may fix the embedding $j : \Delta^{opp} \to \Lambda$), and to identify $M_{1,j}(E) = E([j+1])$, we need to to choose an edge $e \in E([j+1])$ (for instance, since choosing $v \in V([j+1])$ cuts the wheel and defines a total order on $E([j+1])$, we can take the last edge with respect to this order). To compute the differential $b : M_{i,j}(E) \to M_{i,j-1}(E)$, we note that for any contraction $[j+1] \to [j]$ of an edge $e' \in E([j+1])$, the corresponding face map $m_e : k_{[1]}([j+1]) \to k_{[1]}([j])$ sends the chosen vertex $v \in k[V([j+1])] = k_{[1]}([j+1])$ to the chosen vertex $v \in k[V([j])]$. On the other hand, it immediately follows from (4.3) that the face map $m'_{e'} : k^{[1]}([j+1]) \to k^{[1]}([j])$ sends the chosen last edge $e \in k[E([j+1])]$ to $e \in k[E([j])]$ if $e \neq e'$, and to 0 otherwise. Thus the diferential $b : M_{i,j}(E) \to M_{i,j-1}(E)$ is given by

$$b = \sum_{0 \leq l \leq j} (-1)^j r_l m_l,$$

where $r_l = 0$ if i is odd and $l = j$, and $r_l = 1$ otherwise. Thus $M_{\bullet,\bullet}(E)$ becomes exactly isomorphic to the original bicomplex (3.2) for the cyclic vector space E. We also have $H_\bullet(\Lambda, E \otimes k^{[1]}) = 0$, and $H_\bullet(\Lambda, E \otimes k_{[1]}) = H_\bullet(\Delta^{opp}, j^*E)$.

4.3 Nerves and geometric realizations.

To anyone who studied algebraic topology, the cohomology algebra $H^\bullet(\Lambda, k) = k[u]$ of the category Λ will seem familiar: the same algebra appears as the cohomology algebra $H^\bullet(\mathbb{C}P^\infty, k)$ of the infinite-dimensional complex projective space $\mathbb{C}P^\infty$, the classifying space $BU(1)$ for the unit circle group $U(1) = S^1$. This is not a simple coincidence. The relation between Λ and $\mathbb{C}P^\infty$ has been one of the recurring themes of the whole theory of cyclic homology from its very beginning.

The relation occurs at various levels, and while the most advanced ones are not properly understood even today, we do understand the picture up to a certain point. The next level after the cohomology isomorphism is that of the so-called *geometric realizations*.

Unfortunately, we do not have time to present the notion of the geometric realization in full detail (it is easily available in the literature; my personal favourite is the exposition in Chapter I of Gelfand-Manin's book, also Quillen has a nice and concise exposition in his paper on higher K-theory in Lecture Notes in Math., vol. 341). Let us just briefly remind the reader that to any small category Γ, one associated a simplicial set $N(\Gamma)$ called *the nerve* of the category Γ. By definition, 0-simplices in $N(\Gamma)$ are objects of Γ, 1-simplices are morphisms, 2-simplices are pairs of composable morphisms $[a_1] \to [a_2] \to [a_3]$, and so on – n-simplices in $N(\Gamma)$ are functors to Γ from the totally ordered set $[n+1]$ considered as a category in the usual way. Given a simplicial set $X \in \operatorname{Fun}(\Delta^{opp}, \mathrm{Sets})$, one forms a topological space $|X|$ called the *geometric realization* of X by gluing together the elementary simplices Δ^n, one for each n-simplex in $X([n+1])$. Given a small category Γ, we will call $|N(\Gamma)|$ its geometric realization, and we will denote it simply by $|\Gamma|$.

Here are some simple properties of the geometric realization.

(i) We have $|\Gamma| \cong |\Gamma^{opp}|$.

(ii) A functor $\gamma : \Gamma \to \Gamma'$ induces a map $|\gamma| : |\Gamma| \to |\Gamma'|$, and a map $\gamma_1 \to \gamma_2$ between functors γ_1, γ_2 induces a homotopy between $|\gamma_1|$ and $|\gamma_2|$.

(iii) Consequently, if a functor $\gamma : \Gamma \to \Gamma'$ has an adjoint, then $|\gamma|$ is a homotopy equivalence. In particular, if Γ has a final, or an initial object, then $|\Gamma|$ is contractible.

(iv) If Γ is a connected groupoid, and an object $[a] \in \Gamma$ has automorphism group is G, then up to homotopy, $|\Gamma|$ is the classifying space BG.

To any functor $E \in \operatorname{Fun}(\Gamma, k)$, one associates a constructible sheaf $\mathcal{E}$ of k-vector spaces on $|\Gamma|$ by the following rule: for any n-simplex $[a_0] \to \cdots \to [a_n]$ of $N(\Gamma)$, the restriction of $\mathcal{E}$ to the corresponding simplex $\Delta^n \subset |\Gamma|$ is the constant sheaf with fiber $E([a_0])$, and the gluing maps are either identical or induced by the action of morphisms in Γ. The gives an exact comparison functor $\operatorname{Fun}(\Gamma, k) \to \operatorname{Shv}(|\Gamma|, k)$. This functor is fully faithful, and it is even fully faithful on the level of derived categories: for any $E, E' \in \operatorname{Fun}(\Gamma, k)$ with corresponding sheaves $\mathcal{E}, \mathcal{E}' \in \operatorname{Shv}(|\Gamma|, k)$, the natural map

$$\operatorname{Ext}^{\bullet}(E, E') \to \operatorname{Ext}^{\bullet}(\mathcal{E}, \mathcal{E}')$$

is an isomorphism in all degrees (to prove it, one can, for instanse, use the Godement resolution of $E \in \operatorname{Fun}(\Gamma, k)$ by representable sheaves, as in Lecture 2). Of course, the comparison functor is not an equivalence: in general, the category $\operatorname{Shv}(|\Gamma|, k)$ is much larger. However, we have the following obvious fact.

Definition 4.2. A functor $E \in \operatorname{Fun}(\Gamma, k)$ is *locally constant* if for any morphism $f : [a] \to [a']$ in Γ, the corresponding map $E([a]) \to E([a'])$ is invertible.

Lemma 4.3. *The comparison functor induces an equivalence between the derived category $\mathcal{D}_{lc}(\Gamma, k)$ of complexes with locally constant homology and the derived category $\operatorname{Shv}_{lc}(|\Gamma|, k)$ of complexes of sheaves on $|\Gamma|$ whose homology sheaves are locally constant.* □

Corollary 4.4. *Assume that for any field k and for any locally constant $E \in \operatorname{Fun}(\Gamma, k)$, we have $H_{\bullet}(\Gamma, E) = E([a])$, where $[a] \in \Gamma$ is s fixed object. Then $|\Gamma|$ is contractible.*

Proof. By the well-known Whitehead Theorem, a map $f : X \to Y$ of CW-complexes is a homotopy equivalence if for any local systems A on Y, B on X, the induced maps $H_{\bullet}(X, f^*A) \to H_{\bullet}(Y, A)$, $H_{\bullet}(X, B) \to H_{\bullet}(Y, f_*B)$ are isomorphisms. □

Going back to the cyclic category Λ: our goal is to prove that $|\Lambda|$ is homotopically equivalent to $\mathbb{C}P^\infty$. We will do it indirectly, in two steps: first, we prove that the realization $|\Lambda_\infty|$ of the category Λ_∞ is contractible, then we prove that the projection functor $\Lambda_\infty \to \Lambda$ induces a fibration $|\Lambda_\infty| \to |\Lambda|$ whose fiber is the circle $S^1 = U(1)$ – thus $|\Lambda_\infty|$ can be taken as the contractible space $EU(1)$, and $|\Lambda|$ is homotopy equivalent to the classifying space $EU(1)/U(1) = BU(1) \cong \mathbb{C}P^\infty$. For the first step, we only need Corollary 4.4, but for the second step, we need to develop some machinery of fibrations for small categories.

4.4 Fibrations and cofibration of small categories.

The notion of a fibered and cofibered category was introduced by Grothendieck in SGA1, Ch.VI, which is perhaps still the best reference for those who can read French; nowadays, this machinery is usually called *Grothendieck construction*. Let me give the basic definitions.

Assume given a functor $\gamma : \Gamma' \to \Gamma$ between small categories Γ, Γ'. By the *fiber* $\Gamma'_{[a]}$ over an object $[a] \in \Gamma$ we understand the subcategory $\Gamma'_{[a]} \to \Gamma'$ of objects $[a'] \subset \Gamma'$ such that $\gamma([a']) = [a]$, and morphisms f such that $\gamma(f) = \mathsf{id}$. A morphism $f : [a] \to [b]$ in Γ' is called *Cartesian* if it has the following universal property:

- any morphism $f' : [a'] \to [b]$ such that $\gamma(f') = \gamma(f)$ factors through f by means of a unique map $[a'] \to [a]$ in $\Gamma'_{\gamma([a])}$.

Definition 4.5. A functor $\gamma : \Gamma' \to \Gamma$ is called a *fibration* if

(i) for any $f : [a] \to [b]$ in Γ, and any $b' \in \Gamma'_{[b]}$, there exists a Cartesian morphism $f' : [a'] \to [b']$ such that $\gamma(f') = f$, and

(ii) the composition of two Cartesian morphisms is Cartesian.

Condition (i) here mimics the "covering homotopy" condition in the definition of a fibration in algebraic topology, but it is in fact much more precise — indeed, the Cartesian covering morphism f', having the universal property, is uniquely defined. Grothendieck also introduced "cofibrations" as functors $\gamma : \Gamma' \to \Gamma$ such that $\gamma^{opp} : \Gamma'^{opp} \to \Gamma^{opp}$ is a fibration. This terminology is slightly unfortunate because the topological analogy is still a fibration – "cofibration" in topology means something completely different. For this reason, now the term "op-fibration" is sometimes used. However, we will stick to Grothendieck's original terminology.

Assume given a fibration $\gamma : \Gamma' \to \Gamma$ and a morphism $f : [a] \to [b]$ in Γ. Then for any $[b'] \in \Gamma'_{[b]}$, we by definition have a Cartesian morphism $f' : [a'] \to [b']$, and using the universal property of the Cartesian morphism, one checks that the correspondence $[b'] \mapsto [a']$ is functorial: we have a functor $f^* : \Gamma'_{[b]} \to \Gamma'_{[a]}$, $[b'] \mapsto [a']$. Using condition (ii) of Definition 4.5, one checks that for any composable pair of maps f, g, we have a natural isomorphism $(f \circ g)^* \cong g^* \circ f^*$, and there is a compatibility constraint for these isomorphisms when we are given a composable triple f, g, h. All in all, the correspondence $[a] \mapsto \Gamma'_{[a]}$, $f \mapsto f^*$ defines a contravriant "weak functor" from Γ to the category of small categories. Conversely, every such "weak functor", appropriately defined, arises in this way. This was the main reason for Grothendieck's definition of a fibration – it gives a nice and short replacement for the cumbersome notion of a weak functor, with all its higher isomorphisms and compatibility constraints.

Today, we will only need one basic fact about fibrations, and we will use it without a proof.

Definition 4.6. A fibration $\gamma : \Gamma' \to \Gamma$ is *locally constant* if for any f in Γ, the functor f^* is an equivalence.

Proposition 4.7. *Assume given a connected small category Γ and a locally constant fibration $\gamma : \Gamma' \to \Gamma$. Then the homotopy fiber of the induced map $|\gamma| : |\Gamma'| \to |\Gamma|$ is naturally homotopy equivalent to the realization $|\Gamma'_{[a]}|$ of the fiber over any object $[a] \in \Gamma$.* □

4.5 Computation of $|\Lambda|$.

We can now prove that the realization $|\Lambda|$ is equivalent to $\mathbb{C}P^\infty$. We start with the following.

Lemma 4.8. *The realization $|\Lambda_\infty|$ is contractible.*

Proof. By Corollary 4.4, it suffices to prove that $H_\bullet(\Lambda_\infty, E) \cong E([1])$ for any locally constant $E \in \mathrm{Fun}(\Lambda_\infty, k)$. The homology of the category Λ_∞ can be computed by a complex similar to (3.2): we take (3.2) and remove everything except for the two right-most columns. We leave it to the reader to check that this indeed computes $H_\bullet(\Lambda_\infty, E)$ (while the rows of the complex now have only length 2, they still compute the homology of the infinite cyclic group $\mathbb{Z} = \mathsf{Aut}([n])$, and the same proof as in Lemma 3.2 works). Since we now only have two columns, and one of them is contractible, the Connes' exact sequence reduces to an isomorphism

$$H_\bullet(\Delta^{opp}, j^*E) \cong H_\bullet(\Lambda_\infty, E).$$

Since j^*E is obviously locally constant, it suffices to check that the realization $|\Delta^{opp}|$ of the category Δ^{opp} is contractible. This is clear — Δ^{opp} has an initial object. □

Lemma 4.9. *The natural functor $\Lambda_\infty \to \Lambda$ is a locally constant fibration whose fiber is the groupoid $\mathsf{pt}_{\mathbb{Z}}$ with one object whose automorphisms group is $\mathbb{Z}$.*

Proof. We use the combinatorial description of the category Λ. Then for any $[n], [m] \in \Lambda$, the map $\Lambda_\infty([n], [m]) \to \Lambda([n], [m])$ is surjective by definition, and one checks easily that *any* map $f \in \Lambda_\infty([n], [m])$ is Cartesian. The fiber, again by definition, has one object, and its automorphism group is freely generated by the automorphism σ. □

Proposition 4.10. *We have a homotopy equivalence $|\Lambda| \cong \mathbb{C}P^\infty \cong BU(1)$.*

Proof. By Proposition 4.7 and Lemma 4.9, the homotopy fiber of the map $|\Lambda_\infty| \to |\Lambda|$ is homotopy equivalent to $|\mathsf{pt}_{\mathbb{Z}}|$, and since $|\Lambda_\infty|$ is contractible, this implies that $|\mathsf{pt}_{\mathbb{Z}}|$ is homotopy equivalent to the loop space of $|\Lambda|$. But $\mathsf{pt}_{\mathbb{Z}}$ is a groupoid, so that $|\mathsf{pt}_{\mathbb{Z}}|$ is equivalent to the classifying space $B\mathbb{Z} \cong S^1$. This means that $|\Lambda|$ has only one non-trivial homotopy group, namely $\pi_2(|\Lambda|) = \mathbb{Z}$, so that it must be the Eilenberg-MacLane space $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. □

As a corollary, we see that the derived category $\mathcal{D}_{lc}(\Lambda, k)$ of complexes of cyclic objects with locally constant homology objects is equivalent to the derived category of complexes of sheaves of $\mathbb{C}P^\infty$ with locally constant homology sheaves. The objects in this latter category are also know as $U(1)$-equivariant constructible sheaves on the point pt.