Lecture 5.

The structure of $\mathcal{D}_{lc}(\Lambda, k)$ and $\mathrm{Fun}(\Lambda, k)$; Dold-Kan equivalence, mixed complexes. Cyclic bimodules. Cyclic homology as a derived functor.

5.1 The structure of the category $\mathrm{Fun}(\Lambda, k)$.

In the last lecture, we have proved that the geometric realization $|\Lambda|$ of the Connes' cyclic category is homotopy equivalent to the infinite projective space $\mathbb{C}P^\infty$. In particular, we have an equivalence

$$\mathcal{D}_{lc}(\Lambda, k) \cong \mathcal{D}_{lc}(\mathrm{Shv}(\mathbb{C}P^\infty, k)),$$

where $\mathcal{D}_{lc}$ means "the full subcategory in the derived category $\mathcal{D}(\Lambda, k)$ spanned by complexes with locally constant homology", and similarly in the right-hand side. The category in the right-hand side is also equivalent to the derived category of S^1-equivariant sheaves on a point. Besides these topological descriptions, there is also the following very simple combinatorial description.

Let $\mathcal{D}^{per}(k\text{-Vect})$ be the periodic derived category of the category k-Vect – namely, $\mathcal{D}^{per}(k)$ is the triangulated category obtained by considering the category of quadruples $\langle V_+, V_-, d_+, d_-\rangle$ of two vector spaces V_+, V_- and two maps $d_+ : V_+ \to V_-$, $d_- : V_- \to V_+$ such that $d_+ \circ d_- = d_- \circ d_+ = 0$, and inverting quasiisomorphisms. Equivalently, $\mathcal{D}^{per}(k)$ is the homotopy category of 2-periodic complexes $V_\bullet$ of k-vector spaces (with $V_+ = V_{2\bullet}$, $V_- = V_{2\bullet+1}$, and d_+, d_- being the components of the differential). Just as the usual derived category $\mathcal{D}(k\text{-Vect})$ has filtered version $\mathcal{D}\mathsf{F}(k\text{-Vect})$, we define the filtered periodic category $\mathcal{D}\mathsf{F}^{per}(k\text{-Vect})$ by considering 2-periodic filtered complexes $F^\bullet V_\bullet$ such that $F^\bullet V_\bullet \cong F^{\bullet+1}V_{\bullet+2}$ – note the shift in the filtration! Then for any cyclic vector space $E \in \mathrm{Fun}(\Lambda, k)$, the periodic cyclic homology $HP_\bullet(E)$ equipped with the Hodge filtration is an object in $\mathcal{D}\mathsf{F}^{per}(k\text{-Vect})$, so that we have a natural functor

$$HP_\bullet(-) : \mathcal{D}(\Lambda, k) \to \mathcal{D}\mathsf{F}^{per}(k\text{-Vect}).$$

Exercise 5.1. *Show that the induced functor $\mathcal{D}_{lc}(\Lambda, k) \to \mathcal{D}\mathsf{F}^{per}(k\text{-Vect})$ is an equivalence of categories. Hint: both categories are generated by k, so that it suffices to compare $\mathrm{Ext}^\bullet(k, k)$.*

Thus an object $\mathcal{D}_{lc}(\Lambda, k)$, when compared to its periodic cyclic homology equipped with the Hodge filtration, contains exactly the same amount of information, we lose nothing by taking $HP_\bullet(-)$. What can be said about non-constant cyclic vector spaces — in other words, how complicated is the category $\mathrm{Fun}(\Lambda, k)$? Unfortunately, the answer is "very complicated".

This might not seem surprising, because the category Λ contains so many maps. However, so does the category Δ. Nevertheless, there is the following surprising fact, discovered about 50 years ago independently by A. Dold and D. Kan.

Theorem 5.1 (Dold,Kan). *The abelian category $\mathrm{Fun}(\Delta^{opp}, k)$ of simplicial k-vector spaces is equivalent to the category $C^{\leq 0}(k)$ of complexes of k-vector spaces cocentrated in non-positive degrees.*

Proof. There are many proofs, but they all involve either non-trivial computations, or non-trivial combinatorics. We will not give any of them, but we will indicate what the equivalence is. Given a simplicial vector space $E \in \mathrm{Fun}(\Delta^{opp}, k)$, we take its stadrd complex $E_\bullet$, and we replace it with its *normalized quotient* $N(E)_\bullet$ given by

$$N(E)_i = E_i / \sum \mathsf{Im}\, s_j,$$

where $s_j : E_{i-1} \to E_i$ are the degeneration maps (induced by surjective maps $[i] \to [i-1]$). $\square$

There exist also various generalizations of the Dold-Kan equivalence. First, the category $\operatorname{Fun}(\Delta, k)$ of co-simplicial vector spaces is equivalent to the category $C^{\geq 0}(k)$ of complexes concentrated in non-negative degrees (this is not surprising, since $\operatorname{Fun}(\Delta, k)$ and $\operatorname{Fun}(\Delta^{opp}, k)$ are more-or-less dual to each other). One can also consider the subcategory $\Delta_+ \subset \Delta$ with the same objects, and only those maps $[n] \to [m]$ which send the first element to the first element. Then $\operatorname{Fun}(\Delta_+^{opp}, k)$ is equivalent to the category of k-vector spaces graded by non-positive integers (restriction to $\Delta_+^{opp} \subset \Delta^{opp}$ corresponds to forgetting the differential in the complex). Finally, if one "truncates" Δ and considers the full subcategory $\Delta_{\leq n} \subset \Delta$ spanned by objects $[1], \dots, [n]$, then $\operatorname{Fun}(\Delta_{\leq n}^{opp}, k)$ is equivalent to the category $C^{[1-n,0]}(k)$ of complexes concentrated in degrees from $1-n$ to 0, and similarly for $\operatorname{Fun}(\Delta_{\leq n}, k)$ and for Δ_+.

Now, we have a natural embedding $\Delta^{opp} \subset \Lambda$, so that we have a flag of subcategories $\Delta_+^{opp} \subset \Delta^{opp} \subset \Lambda$. We know that Λ is self-dual, $\Lambda \cong \Lambda^{opp}$. One checks easily that Δ_+^{opp} is preserved by this self-duality — we have $\Delta_+^{opp} \cong \Delta_+$. The intermediate category $\Delta^{opp} \subset \Lambda$ is not preserved, so that by duality, we get an embedding $\Delta \subset \Lambda$. All in all, we have the following diagram.

$$
\begin{array}{ccc}
\Delta_+ \cong \Delta_+^{opp} & \longrightarrow & \Delta \\
\big\downarrow & & \big\downarrow \\
\Delta^{opp} & \longrightarrow & \Lambda \cong \Lambda^{opp}.
\end{array}
$$

Applying restrictions and the Dold-Kan equivalence, we associate to any cyclic vector space $E \in \operatorname{Fun}(\Lambda, k)$ a complex $E_\bullet \in C^{\leq 0}(k)$ and a complex $E^\bullet \in C^{\geq 0}(k)$, and since the diagram of categories commutes, we also have natural identifications $E_i \cong E^i$ as k-vector spaces. In other words, we have a collection E_i, $i \geq 0$ of k-vector spaces and *two* differentials $b : E_i \to E_{i-1}$, $B : E_i \to E_{i+1}$. One can check that these differentials anti-commute, $bB + Bb = 0$. The result is what is known in the literature as a *mixed complex*.

Definition 5.2. A *mixed complex* $E_\bullet$ is a collection E_i, $i \geq 0$ of k-vector spaces and two maps $b : E_i \to E_{i-1}$, $B : E_i \to E_{i+1}$ such that $b^2 = B^2 = bB + Bb = 0$.

Mixed complexes form a nice abelian category $M^{\leq 0}(k)$ which is not much more complicated than the category of complexes $C^{\leq 0}(k)$, and we have a comparison functor $\operatorname{Fun}(\Lambda, k) \to M(k)$. But the obvious analog of the Dold-Kan Theorem is *wrong* — the comparison functor is not an equivalence.

The only fact which is true is the following: define the derived category $\mathcal{D}(M(k))$ of mixed complexes by inverting the maps which are quasiisomorphisms with respect to the differential b. Then the comparison functor $\mathcal{D}_{lc}(\Lambda, k) \to \mathcal{D}(M(k))$ is an equivalence (and $\mathcal{D}(M(k))$ is equivalent to $\mathcal{D}\mathsf{F}^{per}(k)$ – this is an instance of the so-called *Koszul*, or $S-\Lambda$ *duality*). However, even when we pass to the restricted categories $\operatorname{Fun}(\Lambda_{\leq n}, k)$, $M^{[1-n,0]}(k)$, with the obvious notation, the comparison functor probably is an equivalence only if k has characteristic 0. And for non-locally constant functors, things only get worse.

To sum up: while in the literature on cyclic homology people often use mixed complexes as a basic object, especially in characteristic 0, in reality, cyclic vector spaces contain strictly more information. And we will see later at least one example where this extra information is crucially important.

5.2 Projecting to $\mathcal{D}_{lc}(\Lambda, k)$.

One moral of the above story is that it is much preferable to work with the locally constant subcategory $\mathcal{D}_{lc}(\Lambda, k)$ rather then with the whole category $\mathcal{D}(\Lambda, k)$. An immediate problem is that the cyclic vector space $A_\# \in \operatorname{Fun}(\Lambda, k)$ defined for an associative unital k-algebra A is not locally constant unless $A = k$. However, we can force it to be locally constant. Namely, the embedding $\mathcal{D}_{lc}(\Lambda, k) \subset \mathcal{D}(\Lambda, k)$ admits a left-adjoint functor $\mathsf{lc} : \mathcal{D}(\Lambda, k) \to \mathcal{D}_{lc}(\Lambda, k)$. Since $\mathcal{D}_{lc}(\Lambda, k) \subset \mathcal{D}(\Lambda, k)$ is a full subcategory, lc is identical on $\mathcal{D}(\Lambda, k)$, and since it contains the constant cyclic vector space k^Λ which corepresents the homology functor, the homology functor factors through lc, so that for any $E \in \mathcal{D}(\Lambda, k)$, we have a canonical isomorphism $H_\bullet(\Lambda, E) \cong H_\bullet(\Lambda, \mathsf{lc}(E))$.

The existence of the adjoint functor lc is easy to prove by general nonsense, but it is perhaps more interesting to use the following explicit construction.

Consider the category $\Lambda^{opp} \times \Lambda$, and consider the functor $\mathsf{I} \in \mathrm{Fun}(\Lambda^{opp} \times \Lambda, k)$ spanned by the Hom-functor: we set

$$\mathsf{I}([n] \times [m]) = k\left[\Lambda([m],[n])\right].$$

Denote by π, π^o the natural projections $\pi : \Lambda^{opp} \times \Lambda \to \Lambda$, $\pi : \Lambda^{opp} \times \Lambda \to \Lambda^{opp}$. We claim that for any $E \in \mathrm{Fun}(\Lambda, k)$, we have a natural isomorphism

$$H_\bullet(\Lambda^{opp} \times \Lambda, \mathsf{I} \otimes \pi^* E) \cong H_\bullet(\Lambda, k). \tag{5.1}$$

Indeed, by an obvious version of the Künneth formula, we can compute the homology in the left-hand side first along Λ^{opp}, and then along Λ. Then it suffices to show that for any $[n] \in \Lambda$, we have a functorial isomorphism

$$H_\bullet(\Lambda^{opp}, E([n]) \otimes \mathsf{I}|_{\Lambda^{opp} \times [n]}) \cong E([n]).$$

But here we can take $E([n])$ out of the brackets, so that it suffices to consider the case $E([n]) = k$, and the restriction $\mathsf{I}_{\Lambda^{opp} \times [n]}$ is nothing but the representable functor $k_{[n]}^{\Lambda^{opp}}$, so that its homology is indeed isomorphic to k concentrated in degree 0.

But on the other hand, we can compute the left-hand side of (5.1) by first using the projection π^o. By general nonsence, we have

$$H_\bullet(\Lambda^{opp} \times \Lambda, \mathsf{I} \otimes \pi^* E) \cong H_\bullet(\Lambda^{opp}, L^\bullet \pi^o_!(\mathsf{I} \otimes \pi^* E)),$$

and since $\Lambda \cong \Lambda^{opp}$, we can define $\mathsf{lc}(E) = L^\bullet \pi^o_!(\mathsf{I} \otimes \pi^* E))$. All we have to do is to prove that it is locally constant. Indeed, by the Künneth formula, for any $[n] \in \Lambda^{opp} \cong \Lambda$ we have

$$\mathsf{lc}(E)([n]) = H_\bullet(\Lambda, k_{[n]} \otimes E),$$

where the representable functor $k_{[n]}$ is the restriction of I to $[n] \times \Lambda \subset \Lambda^{opp} \times \Lambda$. But $k_{[n]}$ is clean in the sense of Definition 3.3, so that

$$H_\bullet(\Lambda, k_{[n]} \otimes E) \cong H_\bullet(\Delta^{opp}, k_{[n]}^{\Delta^{opp}} \otimes j^* E).$$

By the well-known Künneth formula for simplicial vector spaces, the right-hand side is canonically isomorphic to

$$H_\bullet(\Delta^{opp}, k_{[n]}^{\Delta^{opp}}) \cong H_\bullet(\Delta^{opp}, E) \otimes H_\bullet(\Delta^{opp}, E),$$

which is manifestly independent of $[n]$.

5.3 Cyclic bimodules.

If one writes down explicitly $\mathsf{lc}(E)$ by using the complex (3.3), the result is very similar to the "third bicomplex" (3.5) for cyclic homology which we defined in Lecture 3. One can also clearly see why that construction only worked in characteristic 0. The columns in (3.5) are naturally assembled into a cyclic object, not in a simplicial one; when we simply imposed the differential $\widetilde{B}$ on them, we in effect forgot the cyclic structure and only considered the underlying simplicial structure. In $\mathsf{char}\, 0$, this did not matter – the cyclic group action on each column is actually trivial, so that we we compute $H_\bullet(\Lambda, \mathsf{lc}(E))$ by (3.3), taking coinvariants with respect to τ can be omitted. In the general case, we do need to compute honestly the cyclic homology $H_\bullet(\Lambda, \mathsf{lc}(E))$.

However, the bicomplex (3.5), although it only worked in $\operatorname{char} 0$, was very interesting for the computation of the cyclic homology $HC_\bullet(A)$ of an associative algebra A, because it had a version where the bar resolution $C_\bullet(A)$ of the diagonal A-bimodule could be replaced with an arbitrary resolution $P_\bullet$ (at least for the two rightmost columns). Now that we know the full truth, can we perhaps give a version of that construction which is valid in any characteristic and for all columns, not only the two rightmost ones?

It turns out that we can do even better — it is possible to obtain the whole $\mathsf{lc}_\bullet(A_\#)$ as an object of $\mathcal{D}_{lc}(\Lambda, k)$ completely canonically, without any explicit choice at all, neither of a resolution $P_\bullet$, nor of the homotopy ι, as in Lecture 3, part 3.5. Or rather, the choices do occur, but they are all packed into a single choice of a projective resolution in some appropriate abelian category, and cyclic homology is obtained as a derived functor on this abelian category (just as Hochschild homology is the derived functor on the abelian category A-bimod of A-bimodules).

To construct this new category, which we call the category of *cyclic A-bimodules*, we use the technique of fibered and cofibered categories explained in Lecture 4.

Assume given a small category Γ and a category $\mathcal{C}$ equipped with a cofibration $\pi : \mathcal{C} \to \Gamma$. Thus for any $[a] \in \Gamma$, we have the fiber $\mathcal{C}_{[a]}$, and for any map $f : [a] \to [b]$, we have a transition functor $f_! : \mathcal{C}_{[a]} \to \mathcal{C}_{[b]}$. Denote by $\mathrm{Sec}(\mathcal{C})$ the category of sections $\Gamma \to \mathcal{C}$ of the projection $\pi : \mathcal{C} \to \Gamma$. Explicitly, an object $M \in \mathrm{Sec}(\mathcal{C})$ is given by a collection of objects $M_{[a]} \in \mathcal{C}_{[a]}$ for all $[a] \in \Gamma$, and of transtion maps $\iota_f : f_! M_{[a]} \to M_{[b]}$ for all $f : [a] \to [b]$, subject ot natural compatibilities.

Proposition 5.3. *Assume that all the fibers $\mathcal{C}_{[a]}$ of the cofibration $\pi : \mathcal{C} \to \Gamma$ are abelian, and all the transition functors $f_! : \mathcal{C}_{[a]} \to \mathcal{C}_{[b]}$ are left-exact. Then the category $\mathrm{Sec}(\mathcal{C})$ is abelian.*

Sketch of a proof. To prove that an additive category is abelian, one has to show that it has kernels and cokernels, and they satisfy some additional conditions (such as “the cokernel of the kernel is isomorphic to the kernel of the cokernel). The kernel and cokernel of a map $\varphi : M \to N$ in $\mathrm{Sec}(\mathcal{C})$ are taken fiberwise, $\mathsf{Coker}\,\varphi_{[a]} = \mathsf{Coker}\,\varphi_{[a]}$, $\mathsf{Ker}\,\varphi_{[a]} = \mathsf{Ker}\,\varphi_{[a]}$. The transition maps of the kernel are induced from those of M, and to construct the transtion maps for the cokernel, one uses the fact that the transition functors $f_!$ are right-exact. All the extra conditions can be checked fiberwise, where they follow from the assumption that all fibers are abelian. □

As we can see from its explicit description, the category $\mathrm{Sec}(\mathcal{C})$ is rather large. One can define a smaller subcategory by only considering those sections that are *Cartesian* – that is, any $f : [a] \to [b]$ goes to a Cartesian map in $\mathcal{C}$. Equivalently, in the explicit description above, all the transition maps $\iota_f : f_! M_{[a]} \to M_{[b]}$ must be isomorphisms. This is often a much smaller category, but it need not be abelian (unless all the transition functors are exact, not just right-exact, which rarely happens in practice). A reasonable thing to do is to consider the derived category $\mathcal{D}(\mathrm{Sec}(\mathcal{C}))$ and the full subcategory $\mathcal{D}_{cart}(\mathrm{Sec}(\mathcal{C})) \subset \mathcal{D}(\mathrm{Sec}(\mathcal{C}))$ of complexes with Cartesian cohomology.

Assume now given an associative unital algebra A, and consider the category A-bimod of A-bimodules. This is a unital tensor category: we have the (non-symmetric) associative tensor product functor $m : A\text{-}\mathsf{bimod} \times A\text{-}\mathsf{bimod} \to A\text{-}\mathsf{bimod}$, $M_1 \times M_2 \mapsto M_1 \otimes_A M_2$. Moreover, we can also consider the category $A^{\otimes 2}$-bimod of $A^{\otimes 2}$-bimodules, and the exterior product functor $A\text{-}\mathsf{bimod} \times \mathcal{A}\text{-}\mathsf{bimod} \to A^{\otimes 2}\text{-}\mathsf{bimod}$, $M_1 \otimes M_2 \mapsto M_1 \boxtimes M_2$ is a fully faithful embedding. The tensor product functor then obviously extends to a right-exact functor $m : A^{\otimes 2}\text{-}\mathsf{bimod} \to A\text{-}\mathsf{bimod}$. Since the tensor product on A-bimod is associative, we can iterate this and obtain the right-exact tensor product functors $m_n : \mathcal{A}^{\otimes n}\text{-}\mathsf{bimod} \to A\text{-}\mathsf{bimod}$ for any $n \geq 1$. For $n = 0$, we take $A^{\otimes 0}$ to be k, and $m_0 : k\text{-Vect} \to A\text{-}\mathsf{bimod}$ is the functor which sends k to the unit object of A-bimod.

What we want to do now is to take the construction of the cyclic vector space $A_{\#}$, and replace the unital associative algebra A with the unital associative tensor category A-bimod. The result is a category cofibered over Λ which we denote by A-$\mathsf{bimod}_{\#}$. The fibers are given by

$$A\text{-}\mathsf{bimod}_{\#}([n]) = A^{\otimes n}\text{-}\mathsf{bimod},$$

and the transition functors $f_!$ are induced by the multiplication functors m_n by the same formula (3.4) as in the definition of the cyclic vector space $A_{\#}$.

Definition 5.4. A *cyclic A-bimodule M* is a Cartesian section of the cofibration A-$\mathsf{bimod}_{\#} \to \Lambda$.

Explicitly, a cyclic A-bimodule M is given a collection of $M_{[n]} \in A^{\otimes n}$-$\mathsf{bimod}$, $n \geq 1$, and transition maps between them. However, because all transition maps are isomorphisms, the bimodules $M_{[n]}$, $n \geq 2$ can be computed from the first bimodule $M_1 = M_{[1]}$ — it suffices to apply the transition functor $f_!$ for some map $f : [1] \to [n]$. Since such a map is not unique, extending a given $M_1 \in A$-bimod to a cyclic bimodule requires extra data. It is enough, for instance, to specify an $A^{\otimes 2}$-bimodule isomorphism $\tau : A \boxtimes M \to M \boxtimes A$ such that the induced maps $\tau_{23} : A \boxtimes A \boxtimes M \to A \boxtimes M \boxtimes A$, $\tau_{12} : A \boxtimes M \boxtimes A \to M \boxtimes A \boxtimes A$, $\tau_{23} : M \boxtimes A \boxtimes A \to A \boxtimes A \boxtimes M$ satisfy

$$\tau_{31} \circ \tau_{12} \circ \tau_{23} = \mathsf{id}\,. \tag{5.2}$$

The category of cyclic A-bimodules is abelian, but this is an accident: the category that must be abelian for general reasons is the category $\mathrm{Sec}(A\text{-}\mathsf{bimod}_{\#})$ of all sections of the cofibration A-$\mathsf{bimod}_{\#} \to A$. Thus we consider the derived category $\mathcal{D}\Lambda(A\text{-}\mathsf{bimod}) = \mathcal{D}(\mathrm{Sec}(A\text{-}\mathsf{bimod}_{\#}))$, and we define the *derived category of cyclic bimodules* $\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod})$ as the full subcategory

$$\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod}) = \mathcal{D}_{cart}(\mathrm{Sec}(A\text{-}\mathsf{bimod}_{\#})) \subset \mathcal{D}(\mathrm{Sec}(A\text{-}\mathsf{bimod}_{\#})) = \mathcal{D}\Lambda(A\text{-}\mathsf{bimod})$$

of complexes with Cartesian cohomology.

We note that even though the category $\mathrm{Sec}_{cart}(A\text{-}\mathsf{bimod}_{\#})$ of cyclic bimodules *per se* happens to be abelian, its derived category is smaller than $\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod})$. For instance, if $A = k$, so that $A^{\otimes n} = k$ for any $n \geq 0$, with identical transition funcotrs, then $\mathrm{Sec}(A\text{-}\mathsf{bimod})$ is exactly equivalent to $\mathrm{Fun}(\Lambda, k)$, and the Cartesian sections correspond to locally constant functors. But every locally constant cyclic vector space is constant, while $\mathcal{D}\Lambda_{lc}(k\text{-}\mathsf{bimod}) \cong \mathcal{D}_{lc}(\Lambda, k)$ is a non-trivial category.

5.4 Cyclic homology as a derived functor.

We now recall that the Hochschild homology $H_{\bullet}(A, M)$ with coefficients in an A-bimodule M is by definition the derived functor of the functor $M \mapsto A \otimes_{A^{opp} \otimes A} M$, which can be equivalently described as the following right-exact *trace functor*

$$\mathsf{tr}(M) = M/\{am - ma \mid a \in A, m \in M\}.$$

We prefer this description because it clearly has the following "trace property": for any two A-bimodules M, N, there exists a canonical isomorphism $\mathsf{tr}(M \otimes_A N) \cong \mathsf{tr}(N \otimes M)$. Even more generally, for any $A^{\otimes n}$-module M_n, we can define

$$\mathsf{tr}(M_n) = M/\{am - m\sigma(a) \mid a \in A^{\otimes n}, m \in M_n\},$$

where $\sigma : A^{\otimes n} \to A^{\otimes n}$ is the cyclic permutation. These trace functors obviously commute with the transition functors of the cofibered category A-$\mathsf{bimod}_{\#}$, so that tr extends to a functor $\mathsf{tr} : A\text{-}\mathsf{bimod}_{\#} \to k\text{-Vect}$ which sends every Cartesian map to an isomorphism of vector spaces.

We can now apply the trace functor tr fiberwise, to obtain a Cartesian functor $\mathsf{tr} : A\text{-}\mathsf{bimod}_\# \to k\text{-Vect}^\Lambda$, where $k\text{-Vect}^\Lambda = k\text{-Vect} \times \Lambda$ is the constant cofibration with fiber k-Vect. This induces a right-exact functor

$$\mathsf{tr} : \mathrm{Sec}(A\text{-}\mathsf{bimod}_\#) \to \mathrm{Fun}(\Lambda, k),$$

and since $\mathsf{tr} : A\text{-}\mathsf{bimod}_\# \to k\text{-Vect}^\Lambda$ is Cartesian, the derived functor $L^\bullet \mathsf{tr} : \mathcal{D}\Lambda(A\text{-}\mathsf{bimod}) \to \mathcal{D}(\Lambda, k)$ sends $\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod})$ into $\mathcal{D}_{lc}(\Lambda, k)$.

Definition 5.5. The *cyclic homology* $HC_\bullet(A, M)$ of the algebra A with coefficients in some $M \in \mathcal{D}\Lambda(A\text{-}\mathsf{bimod})$ is given by

$$HC_\bullet(A, M) = H_\bullet(\Lambda, L^\bullet \mathsf{tr}(M)).$$

In general, it is not easy to construct cyclic bimodules. However, one cyclic bimodule manifestly exists for any algebra A — this is $A_\#$, with the diagonal $A^{\otimes n}$-bimodule structure on every $A_\#([n]) = A^{\otimes n}$.

Proposition 5.6. *For any algebra A, we have $HC_\bullet(A, A_\#) \cong HC_\bullet(A)$.*

Proof. Notice that we can define a simpler notion of cyclic homology with coefficients in some $M \in \mathcal{D}\Lambda(A\text{-}\mathsf{bimod})$ – we can forget the $A^{\otimes n}$-bimodule structure on $M([n])$, and treat M simply as a complex of cyclic vector spaces. Denote $H_\bullet(\Lambda, M)$ by $HC'_\bullet(A, M)$. We have obvious projection maps $M([n]) \to \mathsf{tr}(M([n]))$ which induce a functorial map

$$HC'_\bullet(A, M) \to HC_\bullet(A, M). \tag{5.3}$$

We have to show that this map is an isomorphism for $M = A_\#$. It suffices to prove that it is an isomorphism for any $M \in \mathcal{D}\Lambda(A\text{-}\mathsf{bimod})$, or even for any $M \in \mathrm{Sec}(A\text{-}\mathsf{bimod}_\#)$. We note that the evaluation at $[n] \in \Lambda$ induces a functor $\mathrm{Sec}(A\text{-}\mathsf{bimod}) \to A^{\otimes n}\text{-}\mathsf{bimod}$, which has a left-adjoint $i_!^{[n]} : A^{\otimes n} \to \mathrm{Sec}(A\text{-}\mathsf{bimod})$. Explicitly, for any $A^{\otimes n}$-bimodule P, we have

$$i_!^{[n]} P([m]) = \bigoplus_{f:[n]\to[m]} f_! P. \tag{5.4}$$

If P is projective, then $i_!^{[n]} P$ is projective in $\mathrm{Sec}(A\text{-}\mathsf{bimod})$ by adjunction, and $\mathrm{Sec}(A\text{-}\mathsf{bimod})$ obviously has enough projectives of this type, so it is enough to prove that (5.3) is an isomorphism for $M = i_!^{[n]} P$. Even further, it is enough to consider objects $P^n \in \mathrm{Sec}(A\text{-}\mathsf{bimod})$ given by

$$P^n = i_!^{[n]} A^{\otimes n} \otimes A^{\otimes n},$$

where on the right-hand side we have the free $A^{\otimes n}$-bimodule with one generator.

Since P^n is projective, we have $L^i \mathsf{tr}(P^n) = 0$ for $i \geq 1$, and $\mathsf{tr}(P^n) \in \mathrm{Fun}(\Lambda, k)$ is isomorphic to $A^{\otimes n} \otimes k_{[n]}$; thus the right-hand side of (5.3) with $M = P^n$ is canonically isomorphic to $A^{\otimes n}$ in degree 0, and trivial otherwise. As for the left-hand side, we see from (5.4) that

$$P^n([m]) = \bigoplus_{f:[n]\to[m]} A^{\otimes(n+m)}.$$

In particular, it is clean, so that $H_\bullet(\Lambda, P^n)$ can be computed by the complex (3.3). We leave it to the reader to check that the resulting complex $P^n_\bullet$ can be described as follows: if we take the augmented bar resolution $C_\bullet(A)$, $C_i(A) = A^{\otimes i+1}$ and consider the n-fold tensor power $C^n_\bullet = C_\bullet(A)^{\otimes n}$, then

$$P^n_i = C^n_{i+1}$$

for any $i \geq 0$. Since the whole complex $C^n_\bullet$, being the n-fold tensor power of the acyclic complex $C_\bullet(A)$, is itself acyclic, the complex $P^n_\bullet$ is a resolution for the 0-th term C^n_0, which is again $A^{\otimes n}$. $\square$