Lecture 6.

Cyclic homology for general tensor categories. Morita-invariance. Example: cyclic homology of a group algebra. Regulator map.

6.1 Cyclic homology for general tensor categories.

In the last lecture, we have constructed the derived category $\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod})$ of cyclic bimodules over an associative algebra A, and we have re-defined cyclic homology by means of a trace functor $\mathsf{tr} : \mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod}) \to \mathcal{D}_{lc}(\Lambda, k)$. The algebra A itself essentially only appeared in the construction though the tensor category A-bimod of A-bimodules. A natural question is, can we do the same construction for a more general tensor category $\mathcal{C}$?

To start with, we need to construct a category $\mathcal{C}_\#$ cofibered over Λ. Here there is one problem: there is no well-defined tensor product for general abelian categories. Namely, we can introduce the following.

Definition 6.1. Assume given two abelian k-linear categories $\mathcal{C}_1$, $\mathcal{C}_2$. The *tensor product* $\mathcal{C}_1 \otimes \mathcal{C}_2$ is a k-linear abelian category equipped with a functor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2$ which is k-linear and right-exact in each variable, and has the following universal property:

- for any k-linear abelian category $\mathcal{C}'$, any functor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}'$ which is k-linear and right-exact in each variable factors through $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2$, and the facrorization is unique up to an isomorphism.

The problem is, while the tensor product in this sense is obviously unique up to an equivalence, it does not always exist. However, it does exist for categories of modules or bimodules: one can show that for any k-algebras A, B, we have $A\text{-}\mathsf{mod} \otimes B\text{-}\mathsf{mod} \cong (A \otimes B)\text{-}\mathsf{mod}$, $A\text{-}\mathsf{bimod} \otimes B\text{-}\mathsf{bimod} \cong (A \otimes B)\text{-}\mathsf{bimod}$ – thus the category $A^{\otimes n}\text{-}\mathsf{bimod}$ which we used in the last lecture is actually $A\text{-}\mathsf{bimod}^{\otimes n}$ in the sense of Definition 6.2. There are other interesting cases, too. Thus we simply impose this as an assumption.

Definition 6.2. A k-linear abelian tensor category $\mathcal{C}$ is *good* if it has arbitrary sums, the tensor product functor is right-exact in each variable, and for any n, there exists a tensor product $\mathcal{C}^{\otimes n}$ in the sense of Definition 6.2.

Remark 6.3. Sometimes in the representation-theoretic literature, "tensor category" means "symmetric tensor category" – that is, the tensor product is not only bilinear, but also symmetric – and tensor categories in the normal sense are called "monoidal". The reason for this is completely unclear to me, and this is bad terminology – in the standard language of category theory, "monoidal" does not imply that the tensor product is a bilinear functor.

Given a good k-linear unital tensor category $\mathcal{C}$, we can literally repeat the construction of the last lecture and obtain a category $\mathcal{C}_\#$ which is cofibered over Λ – the fiber $(\mathcal{C}_\#)_{[n]}$ is the category $\mathcal{C}^{\otimes n}$, and the transition functors $f_!$ are obtained from the tensor product functors $m_n : \mathcal{C}^{\otimes n} \to \mathcal{C}$ (for $n = 0$, we take $\mathcal{C}^{\otimes n} = k\text{-Vect}$, and $m_0 : k\text{-Vect} \to \mathcal{C}$ is the functor which sends k to the unit object in $\mathcal{C}$). Again, the category $\operatorname{Sec}(\mathcal{C}_\#)$ of sections of the cofibration $\mathcal{C}_\# \to \Lambda$ is abelian by Proposition 5.3, so that we can consider the derived category $\mathcal{D}\Lambda(\mathcal{C}) = \mathcal{D}(\operatorname{Sec}(\mathcal{C}_\#))$ and the full triangulated subcategory $\mathcal{D}\Lambda_{lc}(\mathcal{C}) = \mathcal{D}_{cart}(\operatorname{Sec}(\mathcal{C}_\#))$ spanned by Cartesian sections. We will call $\mathcal{D}\Lambda_{lc}(\mathcal{C}_\#)$ the *cyclic envelope* of $\mathcal{C}_\#$.

Cyclic envelope only depends on the tensor category $\mathcal{C}$. However, already to define Hochschild homology $HH_\bullet(\mathcal{C})$ of the category $\mathcal{C}$, we need an extra datum – a right-exact "trace functor".

Definition 6.4. Assume given a good k-linear tensor category $\mathcal{C}$. A *trace functor* on $\mathcal{C}$ is a functor $\mathsf{tr} : \mathcal{C} \to k\text{-Vect}$ which is extended to a functor $\mathsf{tr} : \mathcal{C}_\# \to k\text{-Vect}$ in such a way that $\mathsf{tr}(f)$ is invertible for any Cartesian map f in $\mathcal{C}_\#/\Lambda$.

Explicitly, a trace functor is given by a functor $\mathsf{tr} : \mathcal{C} \to k\text{-Vect}$ and an isomorphism

$$\tau : \mathsf{tr}(M \otimes N) \cong \mathsf{tr}(N \otimes M) \tag{6.1}$$

for any two objects $M, N \in \mathcal{C}$. The isomorphism τ should be functorial in both M and N, and satisfy the condition $\tau_{31} \circ \tau_{12} \circ \tau_{23} = \mathsf{id}$, as in (5.2). We leave it to the reader to check that such an isomorphism τ uniquely defines an extension of tr to the whole category $\mathcal{C}_\#$.

Given a good k-linear tensor category $\mathcal{C}$ equipped with a trace functor tr, we can repeat the construction of the last lecture: we extend tr to a functor $\mathsf{tr} : \mathrm{Sec}(\mathcal{C}_\#) \to \mathrm{Fun}(\Lambda, k)$, and consider the corresponding dervied functor $L^\bullet \mathsf{tr} : \mathcal{D}\Lambda(\mathcal{C}) \to \mathcal{D}(\Lambda, k)$. As before, it sends $\mathcal{D}\Lambda_{lc}(\mathcal{C}) \subset \mathcal{D}\Lambda(\mathcal{C})$ into $\mathcal{D}_{lc}(\Lambda, k)$.

Definition 6.5. *Hochschild homology* $HH_\bullet(\mathcal{C}, \mathsf{tr})$ *of the pair* $\langle \mathcal{C}, \mathsf{tr} \rangle$ *is given by*

$$HH_\bullet(\mathcal{C}, \mathsf{tr}) = L^\bullet \mathsf{tr}(\mathsf{I}),$$

where $\mathsf{I} \subset \mathcal{C}$ *is the unit object. Cyclic homology* $HC_\bullet(\mathcal{C}, \mathsf{tr})$ *of the pair* $\langle \mathcal{C}, \mathsf{tr} \rangle$ *is given by*

$$HC_\bullet(\mathcal{C}, \mathsf{tr}) = H_\bullet(\Lambda, L^\bullet \mathsf{tr}\, \mathsf{I}_\#),$$

where $\mathsf{I}_\# \in \mathrm{Sec}_{cart}(\mathcal{C}_\#)$ *is the Cartesian section of* $\mathcal{C}_\# \to \Lambda$ *which sends an object* $[n] \in \Lambda$ *to* $\mathsf{I}^{\boxtimes n} \in \mathcal{C}^{\otimes n}$*, the* n*-th power of the unit object* $\mathsf{I} \in \mathcal{C}$*.*

Of course, in the case $\mathcal{C} = A\text{-}\mathsf{bimod}$, tr as in the last lecture, we have $HC_\bullet(A\text{-}\mathsf{bimod}, \mathsf{tr}) = HC_\bullet(A, A_\#) = HC_\bullet(A)$ by virtue of Proposition 5.6.

6.2 Morita-invariance of cyclic homology.

As an application of the general formalism developed above, we prove that Hochschild and cyclic homology of an associative algebra A only depends on the category $A\text{-}\mathsf{mod}$ of left A-modules. This is known as *Morita invariance.*

A typical situation is the following. Assume given two k-algebras A, B, and a k-linear functor $F : A\text{-}\mathsf{mod} \to B\text{-}\mathsf{mod}$. Assume that F is right-exact and commutes with infinite direct sums. Consider the B-module $P = F(A)$. Since $\mathrm{End}_A(A) = A^{opp}$, P is not only a left B-module, but also a right A-module – in other words, an $A - B$-bimodule. By definition, we have $F(A) = A \otimes_A P$; since F is right-exact and commutes with arbitrary sums, the same is true for any $M \in A\text{-}\mathsf{mod}$ – the bimodule P represents the functor F in the sense that we have a functorial isomorphism

$$F(M) \cong M \otimes_A P.$$

If F is an equivalence of categories, then the inverse equivalence F^{-1} is of course also right-exact and commutes with sums; thus we have a $B - A$-bimodule P^o representing F^{-1}, and since $F \circ F^{-1} \cong \mathsf{Id}$, $F^{-1} \circ F \cong \mathsf{Id}$, we have isomorphisms

$$A \cong P \otimes_B P^o \in A\text{-}\mathsf{bimod}, \qquad B \cong P^o \otimes_A P \in B\text{-}\mathsf{bimod}. \tag{6.2}$$

Proposition 6.6. *Assume given two associative* k*-algebras* A*,* B*, and an equivalence* $A\text{-}\mathsf{mod} \cong B\text{-}\mathsf{mod}$*. Then there exist natural isomorphisms* $HH_\bullet(A) \cong HH_\bullet(B)$*,* $HC_\bullet(A) \cong HC_\bullet(B)$*.*

Proof. As we have already proved, every right-exact k-linear functor $G : A\text{-}\mathsf{mod} \to A\text{-}\mathsf{mod}$ which commutes with sums is represented by an A-bimodule Q. Conversely, every $Q \in A\text{-}\mathsf{bimod}$ represents such a functor. Tensor product of bimodules corresponds to the composition of functor. Therefore the k-linear tensor category $A\text{-}\mathsf{bimod}$ only depends on the k-linear abelian category $A\text{-}\mathsf{mod}$, and can be recovered as the category of endofunctors of $A\text{-}\mathsf{mod}$ of a certain kind (k-linear, right-exact, preserving sums). Thus in our situation, we have a natural equivalence $F : A\text{-}\mathsf{bimod} \cong B\text{-}\mathsf{bimod}$ of k-linear abelian tensor categories. It induces an equivalence of the corresponding categories of cyclic bimodules. To finish the proof, it suffices to prove that the equivalence $A\text{-}\mathsf{bimod} \cong B\text{-}\mathsf{bimod}$ is compatible with the natural trace functors on both side. This is obvious: for any $M \in A\text{-}\mathsf{bimod}$, we have

$$\mathsf{tr}(M) = A \otimes_{A^{opp}\otimes A} M \cong B \otimes_{B^{opp}\otimes B} (P \otimes P^o) \otimes_{A^{opp}\otimes A} M \cong B \otimes_{B^{opp}\otimes B} F(M) = \mathsf{tr}(F(M)),$$

where P and P^o are as in (6.2). □

6.3 Example: group algebras

Traditionally, in every exposition of cyclic homology, the authors devote some time to one very special case, that of a group algebra. I don't really know why — whether it's because this is needed to construct the regulator map from higher algebraic K-theory, or because there are interesting new things special for the group algebra case, or for some other reason. But let me follow the tradition. This will also give us an example where the general theory of cyclic homology for tensor categories is applied to a tensor category which is not a category of bimodules.

Assume given a group G, and consider the group algebra $k[G]$. This is an associative unital algebra, so it has Hochschild and cyclic homology, and the category of $k[G]$-bimodules is a tensor category. However, since G is a group, the category $G\text{-}\mathsf{mod} = k[G]\text{-}\mathsf{mod}$ of representation of G a.k.a. left $k[G]$-modules is a tensor category in its own right. Moreover, there is an obvious functor $\gamma : G\text{-}\mathsf{mod} \to k[G]\text{-}\mathsf{bimod}$ which sends a representation $V \in G\text{-}\mathsf{mod}$ to a functor $G\text{-}\mathsf{bimod} \to G\text{-}\mathsf{bimod}$ given by $M \mapsto M \otimes V$ (here we use the interpretation of $k[G]$-bimodules as endofunctors of the category $G\text{-}\mathsf{mod}$). This functor is obviously exact and obviously tensor. Explicitly, it is given by

$$\gamma(V) = V \otimes R,$$

where we denote $R = k[G]$, the left $k[G]$-action on $V \otimes R$ is through V and R, and the right action is through R: we have $g_1(v \otimes g)g_2 = g_1 v \otimes g_1 g g_2$. If we have two representations $V_1, V_2 \in G\text{-}\mathsf{mod}$, the natural isomorphism $\gamma(V_1 \otimes V_2) \cong \gamma(V_1) \otimes_{k[G]} \gamma(V_2)$ is given by the map

$$V_1 \otimes V_2 \otimes R \to (V_1 \otimes R) \otimes_{k[G]} (V_2 \otimes R) \tag{6.3}$$

which sends $v_1 \otimes v_2 \otimes g$ to $(v_1 \otimes 1) \otimes (v_2 \otimes g)$, where $1 \in G$ is the unity element.

Since the functor γ is tensor, the usual trace functor tr on $k[G]\text{-}\mathsf{bimod}$ gives by restriction a trace functor $\mathsf{tr}^R = \mathsf{tr} \circ \gamma$ on $G\text{-}\mathsf{mod}$. Explicitly, it is given by

$$\mathsf{tr}^R(V) = (V \otimes R)/\{g_1 v \otimes g_1 g - v \otimes g g_1 \mid g, g_1 \in G, v \in V\},$$

and since the quotient is over all g and all g_1, we might as well replace g with gg_1^{-1}. Then we have

$$\mathsf{tr}^R(V) = (V \otimes R)/\{g_1 v \otimes g_1 g g_1^{-1} - v \otimes g\} = (V \otimes R)_G,$$

the G-coinvariants in the G-representation $V \otimes R$, where R is equipped with the adjoint G-action. One can also check, and this is important, that the identification survives on the level of derived functors — the natural map

$$L^{\bullet}\,\mathsf{tr}(\gamma(V)) \to L^{\bullet}\,\mathsf{tr}^R(V) = H_{\bullet}(G, V \otimes R) \tag{6.4}$$

is an isomorphism in all degrees. For example, for the trivial representation $V = k$, we obtain an isomorphism $HH_\bullet(k[G]) \cong H_\bullet(G, R)$. The isomorphism $\tau^R : \mathsf{tr}^R(V_1 \otimes V_2) \cong \mathsf{tr}^R(V_2 \otimes V_1)$ of (6.1) is induced by the usual symmetry isomorphism $V_1 \otimes V_2 \to V_2 \otimes V_1$ and the isomorphism (6.3); explicitly, τ^R is the map on the spaces of coinvariants induced by the map

$$\widetilde{\tau}^R : V_1 \otimes V_2 \otimes R \to V_2 \otimes V_1 \otimes R, \qquad \widetilde{\tau}^R(v_1 \otimes v_2 \otimes g) = gv_2 \otimes v_1 \otimes g. \tag{6.5}$$

One easily checks that the map $\widetilde{\tau}^R$ defined in this way is actually a map of G-representations.

Applying the general theory of cyclic homology with coefficients, we extend this isomorphism to an isomorphism

$$HC_\bullet(k[G]) \cong HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^R).$$

We now note that the adjoint representation $R = k[G]$ canonically splits into a direct sum $R = \bigoplus_{\langle g \rangle} R^g$ over the conjugacy classes $\langle g \rangle \subset G$, $R^g = k[\langle g \rangle]$, and this induces a canonical direct sum decomposition

$$\mathsf{tr}^R = \bigoplus_{\langle g \rangle} \mathsf{tr}^g \tag{6.6}$$

of the trace functor tr^R: we set $\mathsf{tr}^g(V) = (V \otimes R^g)_G$, and since the isomorphism $\widetilde{\mathsf{tr}}^R$ of (6.5) obviously respects the direct sum decomposition, the isomorphism τ^R induces isomorphisms (6.1) for every component tr^g. Therefore we actually have a canonical direct sum decomposition of cyclic homology:

$$HC_\bullet(k[G]) = \bigoplus_{\langle g \rangle} HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^g), \tag{6.7}$$

and a corresponding decomposition for $HH_\bullet(k[G])$.

However, we can say more. Consider the component tr^1 in the decomposition (6.6) which corresponds to the unity element $1 \in G$. Then we have $\mathsf{tr}^1(V) = V_G$, the space of G-coinvariants, and

$$HH_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1) \cong H_\bullet(G, k).$$

What can we say about the cyclic homology $HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1)$? Looking at (6.5), we see that the isomorphism $\mathsf{tr}^1(V_1 \otimes V_2) \cong \mathsf{tr}^1(V_2 \otimes V_1)$ for the trace functor tr^1 is induced by the symmetry isomorphism $v_1 \otimes v_2 \to v_2 \otimes v_1$. We can rephrase this in the following way: since the tensor category G-mod is symmetric, *any* right-exact functor $F : G\text{-}\mathsf{mod} \to k\text{-Vect}$ canonically extends to a trace functor $F_\# : G\text{-}\mathsf{mod}_\# \to k\text{-Vect}$, and it is this trace functor structure that tr^1 has — we have $\mathsf{tr}^1 \cong \mathsf{Coinv}_\#$, where $\mathsf{Coinv} : G\text{-}\mathsf{mod} \to k\text{-Vect}$ is the coinvariants functor, $V \mapsto V_G$.

In other words, the identity functor $\mathsf{Id} : G\text{-}\mathsf{mod} \to G\text{-}\mathsf{mod}$ *can also be considered as a trace functor*, albeit with values in G-mod rather than k-Vect, so that we have a functor

$$L^\bullet \mathsf{Id} : \mathcal{D}\Lambda(G\text{-}\mathsf{mod}) \to \mathcal{D}(\Lambda, G\text{-}\mathsf{mod}) = \mathcal{D}(\Lambda \times \mathsf{pt}_G, k),$$

where pt_G is the category with one object with automorphism group G, and the trace functor $L^\bullet \mathsf{tr}^1$ factors through $L^\bullet \mathsf{Id}$, so that we have

$$HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1) = H_\bullet(\Lambda \times \mathsf{pt}_G, L^\bullet \mathsf{Id}(\mathsf{I}_\#)).$$

Moreover, Id is exact, so that there is no need to take its derived functor, and we simply have $L^\bullet \mathsf{Id}(\mathsf{I}_\#) = \mathsf{Id}(\mathsf{I}_\#) = k^{\Lambda \times \mathsf{pt}_G}$, the constant cyclic k-vector space with the trivial action of G. Thus by the Künneth formule, we have

$$HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1) \cong H_\bullet(\Lambda \times \mathsf{pt}_G), k) = H_\bullet(\Lambda, k) \otimes H_\bullet(\mathsf{pt}_G, k).$$

Since $H_\bullet(\mathsf{pt}_G, k) = H_\bullet(G, k) = HH_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1)$, we conclude that *the Hodge-to-de Rham spectral sequence for the cyclic homology $HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1)$ canonically degenerates*: we have a canonical isomorphism

$$HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1) \cong HH_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1)[u^{-1}] \tag{6.8}$$

for the unity component in the direct sum decomposition (6.7).

6.4 The regulator map

To finish today's lecture, let me give the standard application of the above computation of groups algebras: I will construct the *higher regulator* a.k.a. *higher Chern character map* from Quillen's higher K-theory to cyclic homology.

Recall that to define higher K-theory of an algebra A, one considers the group $GL_\infty(A) = \lim_{\to} GL_N(A)$ of infinite matrices over A and its classifying space $BGL_\infty(A)$. This is of course an Eilenberg-MacLane space of type $K(\pi, 1)$. However, Quillen defined a certain very non-trivial operation called *the plus-construction* with replaces a topological space X with another topological space X^+ so that the homology is the same, $H_\bullet(X, \mathbb{Z}) \cong H_\bullet(X^+, \mathbb{Z})$, but X^+ has an abelian fundamental group. Then by definition, higher K-groups of A are given by

$$K^\bullet(A) = \pi_\bullet(BGL_\infty(A)^+),$$

the homotopy groups of the plus-construction $BGL_\infty(A)$.

These groups are very hard to compute (not surprisingly, since homotopy groups in general are hard to compute). Fortunately, to construct the regulator, we do not need to do it. Namely, for any topological space X, there exists a canonical Hurewitz map $\pi_\bullet(X) \to H_\bullet(X)$. The regulator map factors through the Hurewitz map for BGL_∞^+, so that the source of the map we will construct is actually the homology $H_\bullet(BGL_\infty^+)$. At this point, we can also get rid of the plus-construction: by its very definition, $H_\bullet(X) = H_\bullet(X^+)$, so that $H_\bullet(BGL_\infty^+) = H_\bullet(BGL_\infty) = H_\bullet(GL_\infty, \mathbb{Z})$, the homology of the group $GL_\infty(A)$ with trivial coefficients. In fact, our map will further factor through $H_\bullet(GL_\infty(A), k)$.

What is the natural target of the regulator map? Comparison with the Chern character map in algebraic geometry suggests at first that this should the de Rham cohomology groups $H^\bullet_{DR}(-)$ — in our situation, these correspond to the periodic cyclic homology groups $HP_\bullet(A)$. However, it is known that the Chern character actually behaves nicely with respect to the Hodge filtration — the Chern character map $K_0(X) \to \bigoplus_i H^{2i}_{DR}(X)$ for an smooth algebraic variety X actually factors through $\bigoplus_i F^i H^{2i}_{DR}(X)$. In the non-commutative situation, this corresponds to taking the 0-th graded piece of the Hodge filtration on $HP_\bullet(A)$. This has its own name.

Definition 6.7. The *negative cyclic homology* $HC^-_\bullet(A)$ of an algebra A is the 0-th term $F^0 HP_\bullet(A)$ of the Hodge filtration on $HP_\bullet(A)$.

If we compute $HP_\bullet(A)$ by the standard periodic bicomplex, then computing $HC^-_\bullet(A)$ amount to removing all the columns *to the left* of the 0-th one — as opposed to the usual $HC_\bullet(A)$, where we remove everything to the right. This explains the adjective "negative".

So, what we want to do is to construct a map $H_\bullet(GL_\infty(A), k) \to HC^-_\bullet(A)$. This is done in three steps.

First, fix some integer N, and consider the group algebra $k[GL_N(A)]$. This has a natural map into the algebra $\mathrm{Mat}_N(A)$ of $N \times N$-matrices in A – an element $g \in GL_N(A)$ goes to itself considered as an element in $\mathrm{Mat}_N(A)$. The map of algebras induces a map of negative cyclic homology; passing to the limit, we obtain a map

$$\lim_{\to} HC^-_\bullet(k[GL_N(A)]) \to \lim_{\to} HC^-_\bullet(\mathrm{Mat}_N(A)).$$

Second, we observe that by Morita-invariance of cyclic homology, the directed system in the right-hand side is actually constant — we have $HC^-_\bullet(\mathrm{Mat}_N(A)) \cong HC^-_\bullet(A)$ for any N. Thus we have constructed a map

$$\lim_{\to} HC^-_\bullet(k[GL_N(A)]) \to HC^-_\bullet(A).$$

Finally, we use the direct sum decomposition (6.7) — we take the graded piece of (6.7) corresponding to the unity element $1 \in GL_N(A)$, and apply the canonical Hodge-to-de Rham degeneration (6.8). This gives a canonical map

$$H_\bullet(GL_\infty(A), k) = \lim_{\to} H_\bullet(GL_N(A), k) \to \lim_{\to} HC^-_\bullet(k[GL_N(A)]).$$

Composing the two maps above, and plugging in the Hurewitz map, we obtain the higher regulator map $K_\bullet(A) \to HC^-_\bullet(A)$.