Lecture 7.

Cartier isomorphism in the commutative case. The categories Λ_p. Frobenius and quasi-Frobenius maps. Non-commutative case: the Cartier isomorphism for algebras with a quasi-Frobenius map. Remarks on the general case.

7.1 Cartier isomorphism in the commutative case.

The goal of this lecture is to explain the construction of the so-called *Cartier isomorphism* for algebras over a finite field k. We start by recalling what happens in the commutative case.

Fix a finite field k of characteristic $p = \operatorname{char} k$, and consider a smooth affine variety $X = \operatorname{Spec} A$ over k. Assume that $p > \dim X$, and consider the de Rham complex $\Omega^{\bullet}_X$. This complex behaves very differently from what we have in characteristic 0. For instance, in characteristic 0, a function f is closed with respect to the de Rham differential if and only if it is locally constant. In our situation, however, the p-th power a^p of any $a \in A$ is closed: we have $df^p = pf^{p-1}df = 0$. About fifty years ago, P. Cartier has shown that this gives all the closed functions, and moreover, the situation in higher degrees is similar — for any $n \geq 0$, there exists a canonical *Cartier isomorphism*

$$C : H^n_{DR}(X) \cong \Omega^n_{X^{(1)}}$$

between the de Rham cohomology group $H^n_{DR}(X)$ and the space $\Omega^n_{X^{(1)}}$ of n-forms on the so-called "Frobenius twist" $X^{(1)} = \operatorname{Spec} A^{(1)}$ of the variety X — $A^{(1)}$ coincides with A as a ring, but the k-algebra structure is twisted by the Frobenius automorphism of the field k.

Let us briefly sketch the construction of the inverse isomorphism $C^{-1} : \Omega^n_{X^{(1)}} \to H^n_{DR}(X)$ (this is simpler). Consider the ring $W(k)$ of Witt vectors of the field k — that is, the unramified extension of $\mathbb{Z}_p$ whose residue field is k. Since X is an affine variety, we can lift it to a smooth variety $\widetilde{X}$ over $W(k)$ so that $X = \widetilde{X} \otimes_{W(k)} k$. Moreover, we can lift the Frobenius map $F : X \to X^{(1)}$ to a map $\widetilde{F} : \widetilde{X} \to \widetilde{X}^{(1)}$, where $\widetilde{X}^{(1)}$ means the twist with respect to the Frobenius automorphism of $W(k)$. For any 1-form $fdg \in \Omega^1_{\widetilde{X}}$, we have

$$\widetilde{F}^*(fdg) = f^p dg^p \mod p,$$

so that the pullback map $\widetilde{F}^* : \Omega^1_{\widetilde{X}^{(1)}} \to \Omega^1_{\widetilde{X}}$ is divisible by p, and cosequently, $\widetilde{F}^*$ on $\Omega^n_{\widetilde{X}^{(1)}}$ is divisible by p^n. Let us make this division and consider the map

$$\overline{F} : \widetilde{\Omega}^{\bullet}_{\widetilde{X}^{(1)}} \to \Omega^{\bullet}_{\widetilde{X}}$$

given by $\overline{F} = \frac{1}{p^n}\widetilde{F}^*$ in degree n, where $\widetilde{\Omega}^{\bullet}_{\widetilde{X}^{(1)}}$ is the de Rham complex of the variety $\widetilde{X}^{(1)}$ with differential multiplied by p. Then it is not difficult to check — for instance, by a computation in local coordinates — that the map $\overline{F}$ is a quasiisomorphism. Reducing it modulo p, we obtain a quasiisomorphism

$$\bigoplus_n \Omega^n_{X^{(1)}} \cong \Omega^{\bullet}_X,$$

where the differential in the left-hand side, being divisible by p, reduces to 0. One then checks that the components of this quasiisomorphisms in individual degrees do not depend on our choices — neither of the lifting $\widetilde{X}$, nor on the lifting $\widetilde{F}$. These are the inverse Cartier maps.

We note that the Cartier maps are not easy to write down by an explicit formula even when X is a curve, expect for one especially simple case — and contrary to the expectations, the simple case is not the affine line $X = \operatorname{Spec} k[t]$, but the multiplicative group $X = \operatorname{Spec} k[t, t^{-1}]$. In this case, we have

$$C^{-1}\left(f\frac{dt}{t}\right) = f^p\frac{dt}{t}$$

for any $f \in k[t, t^{-1}]$. Analogously, in dimension n, we have a similar explicit formula for the torus $X = T = \operatorname{Spec} k[L]$, the group algebra of a lattice $L = \mathbb{Z}^n$.

7.2 The categories Λ_p.

To generalize this construction to the non-commutative case, we need one piece of linear algebra which we now describe.

Recall that in the combinatorial description, the cyclic category Λ was obtained as a quotient of the category Λ_∞: for any $[m], [n] \in \Lambda$, we have $\Lambda([n], [m]) = \Lambda_\infty([n], [m])/\sigma$. For any positive integer l, we can define a category Λ_l by a similar procedure: Λ_l has the same objects as Λ, and we set

$$\Lambda_l([n], [m]) = \Lambda_\infty([n], [m])/\sigma^l$$

for any $[n], [m] \in \Lambda_l$. We have an obvious projection $\pi : \Lambda_l \to \Lambda$; just as the projection $\Lambda_\infty \to \Lambda$, this is a connected bifibration whose fiber is the groupoid pt_l with one object and $\mathbb{Z}/l\mathbb{Z}$ as its automorphism group. One the other hand, we also have an embedding $i : \Lambda_l \to \Lambda$ which sends $[n] \in \Lambda_l$ to $[nl] \in \Lambda$. Just as for Λ, the embedding $j : \Delta^{opp} \to \Lambda_\infty$ induces an embedding $j_l : \Delta^{opp} \to \Lambda_l$.

It turns out that most of the facts about the homology of the category Λ immediately generalize to Λ_l, with the same proofs. In particular, for any $E \in \operatorname{Fun}(\Lambda_l, k)$, the homology $H_\bullet(\Lambda_l, E)$ can be computed by a bicomplex

$$
\begin{array}{ccccccc}
\dots & \longrightarrow & E([1]) & \xrightarrow{\mathsf{id}} & E([1]) & \xrightarrow{\mathsf{id}-\tau} & E([1]) \\
 & & \uparrow b & & \uparrow b' & & \uparrow b \\
\dots & \longrightarrow & E([2]) & \xrightarrow{\mathsf{id}+\dots+\tau^{l-1}} & E([2]) & \xrightarrow{\mathsf{id}-\tau} & E([2]) \\
 & & \uparrow b & & \uparrow b' & & \uparrow b \\
\dots & & \dots & & \dots & & \dots \\
 & & \uparrow b & & \uparrow b' & & \uparrow b \\
\dots & \longrightarrow & E([n]) & \xrightarrow{\mathsf{id}+\tau+\dots+\tau^{ln-1}} & E([n]) & \xrightarrow{\mathsf{id}-\tau} & E([n]), \\
 & & \uparrow b & & \uparrow b' & & \uparrow b
\end{array}
\tag{7.1}
$$

we have a periodicity map $H_{\bullet+2}(\Lambda_l, E) \to H_\bullet(\Lambda_l, E)$ which fits into a Connes' exact sequence

$$H_\bullet(\Delta^{opp}, j_l^* E) \longrightarrow H_\bullet(\Lambda_l, E) \xrightarrow{u} H_{\bullet-2}(\Lambda_l, E) \longrightarrow ,$$

and the periodicity map u is induced by the action of the generator u of the cohomology algebra $H^\bullet(\Lambda_l, k) \cong k[u]$. As in Lecture 4, this generator admits an explicit Yoneda representation by a length-2 complex $j_{l*}^{opp} j_l^{opp*} k \to j_{l!} j_l^* k$. Moreover it is easy to check that this complex coincides with the pullback of the analogous complex in $\operatorname{Fun}(\Lambda, k)$ with respect to $i : \Lambda_l \to \Lambda$, so that i induces an isomorphism

$$i^* : H^\bullet(\Lambda, k) \to H^\bullet(\Lambda_l, k)$$

sending the periodicity generator to the periodicity generator. However, there is also one new and slightly surprising fact.

Lemma 7.1. *For any associative unital algebra A over k, the natural map*

$$M : H_\bullet(\Lambda_l, i^* A_\#) \to H_\bullet(\Lambda_l, A_\#)$$

is an isomorphism.

Proof. Since i^* is compatible with the periodicity maps, it suffices to prove that the natural map

$$H_\bullet(\Delta^{opp}, j_l^* i^* A_\#) \to H_\bullet(\Delta^{opp}, j^* A_\#)$$

on Hochschild homology is an isomorphism. By definition, we have

$$j^* A_\# \cong C_\bullet(A) \otimes_{A \otimes A^{opp}} A,$$

where $C_\bullet(A)$ is the bar resolution considered as a simplicial set. Writing down explicitly the definition of $i : \Lambda_l \to \Lambda$, one deduces that

$$j_l^* i^* A_\# \cong (C_\bullet(A) \otimes_A \cdots \otimes_A C_\bullet(A)) \otimes_{A \otimes A^{opp}} A,$$

with l factors $C_\bullet(A)$. But since $C_\bullet(A)$ is a resolution of A, so is the product in the right-hand side. We conclude that $H_\bullet(\Delta^{opp}, j_l^* i^* A_\#)$ is just the Hochschild homology $HH_\bullet(A)$ computed by a diffewrent resolution, and M is indeed an isomorphism. □

Exercise 7.1. *Prove that the map M is an isomorphism for any cyclic vector space $E \in$ Fun(Λ, k), not just for $A_\#$. Hint: use the acyclic models method, and show that* Fun(Λ, k) *has a generator of the form $A_\#$.*

7.3 Frobenius and quasi-Frobenius maps.

Assume now given an associative unital algebra A over k; motivated by comparison theorems of Lecture 2, we want to construct a Cartier isomorphism of the form

$$HH_\bullet(A^{(1)})((u)) \cong HP_\bullet(A). \tag{7.2}$$

Unfortunately, the procedure that we have used in the commutative case breaks down right away: there is no Frobenius map in the non-commutative case. The endomorphism $F : A \to A$ given by $a \mapsto a^p$ is neither additive nor multiplicative for a general non-commutative algebra A.

To analyze the difficulty, split F into the composition

$$A^{(1)} \xrightarrow{\ \varphi\ } A^{\otimes p} \xrightarrow{\ m\ } A$$

of the map φ given by $\varphi(a) = a^{\otimes p}$, and the multiplication map $m : A^{\otimes p} \to A$, $m(a_1 \otimes \cdots \otimes a_p) = a_1 \dots a_p$. The map φ is not additive, nor multiplicative, but this is always so, be A commutative or not. It is the map m that causes the problem: if A is commutative, it is an algebra map, and in general it is not.

This is where the p-cyclic category Λ helps. Although the map m is not an algebra map, so that no Frobenius map acts on A, we still can get an action of this nonexisting Frobenius on Hochschild and cyclic homology by extending m to the isomorphism

$$M : H_\bullet(\Lambda_p, i^* A_\#) \to H_\bullet(\Lambda, A_\#)$$

of Lemma 7.1. As for the map φ, which behaves very badly in all cases, it turns out that it can be replaced by a different map within a certain large class of them. Namely, the only important property of the map φ is the following.

Lemma 7.2. *For any vector space V over k, the map $\varphi : V^{(1)} \to V^{\otimes p}$ induces an isomorphism*

$$H_i(\mathbb{Z}/p\mathbb{Z}, V^{(1)}) \cong H_i(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$$

for any $i \geq 1$, where the cyclic group $\mathbb{Z}/p\mathbb{Z}$ acts trivially on $V^{(1)}$, and by the cyclic permutation on $V^{\otimes p}$.

Proof. The homology of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ with coefficients in some representation M can be computed by the standard periodic complex $M_\bullet$ with terms $M_i = M$, $i \geq 0$, and the differentials $d_- = 1 - \sigma$ in odd degrees and $d_+ = 1 + \sigma + \dots + \sigma^{p-1} = (1-\sigma)^{p-1}$ in even degrees, where σ is the generator of $\mathbb{Z}/p\mathbb{Z}$. For the trivial representation $V^{(1)}$, $d_+ = d_- = 0$. The map φ obviously sends $V^{(1)}$ into the σ-invariant subspace in $V^{\otimes p}$, thus into the kernel of both d_+ and d_-. We have to show that (1) φ becomes additive modulo the image of the corresponding differential d_-, d_+, and (2) it actually becomes an isomorphism. Choose a basis in V, so that $V = k[S]$ is the k-vector space generated by a set S. Then $V^{\otimes p} = k[S^p]$. Decompose $S^p = S \coprod (S \setminus S)$, where $S \subset S^p$ is embedded as the diagonal, and consider the corresponding decomposition $V^{\otimes p} = V \oplus V'$, where $V' = k[S^p \setminus S]$. This decomposition is $\mathbb{Z}/p\mathbb{Z}$-invariant, thus compatible with d_+ and d_-; morever, φ obviously becomes an additive isomorphism if we replace $V^{\otimes p}$ with its quotient $V^{\otimes p}/V' = V$. Thus it suffices to prove that the complex which computes $H_\bullet(\mathbb{Z}/p\mathbb{Z}, V')$ is acyclic in degrees ≥ 1. This is obvious — the $\mathbb{Z}/p\mathbb{Z}$-action on $S^p \setminus S$ is free. □

For a more natural formulation of Lemma 7.2, one can invert the periodicity endomorphism of the homology functor $H_\bullet(\mathbb{Z}/p\mathbb{Z}, -)$ to obtain the so-called *Tate homology* $\check{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, -)$ (this is the same procedure that we used in passing from $HC_\bullet(-)$ to $HP_\bullet(-)$). Then Lemma 7.2 claims that φ induces a canonical isomorphism

$$\check{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, V^{(1)}) \cong \check{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$$

in all degrees. We will call it *the standard isomorphism.*

Definition 7.3. A *quasi-Frobenius map* for an associative unital algebra A over k is a $\mathbb{Z}/p\mathbb{Z}$-equivariant algebra map $\Phi : A^{(1)} \to A^{\otimes p}$ which induces the standard isomorphism on Tate homology $\check{H}_\bullet(-)$.

Given an algebra A with a quasi-Frobenius map Φ, we can construct an inverse Cartier map (7.2) right away. Namely, comparing the bicomplex (7.1) with the usual cyclic bicomplex (3.2), we see that the only difference is that the differential $1 + \tau + \dots + \tau^{n-1}$ is replaced with

$$1 + \tau + \dots + \tau^{np-1} = (1 + \sigma + \circ + \sigma^{p-1}) \circ (1 + \tau + \dots + \tau^{n-1}),$$

where we have used the fact that $\sigma = \tau^n$. But for some $E' \in \operatorname{Fun}(\Lambda_p, k)$ of the form $E' = \pi^* E$, $E \in \operatorname{Fun}(\Lambda, p)$, we have $\sigma = 1$, so that $1 + \sigma + \circ + \sigma^{p-1} = p = 0$. Therefore we have a natural identification

$$HP_\bullet(\pi^* A^{(1)}_\#) \cong HH_\bullet(A^{(1)})((u)), \tag{7.3}$$

where on the left-hand side we have a periodic version of the homology $H_\bullet(\Lambda_p, \pi^* A^{(1)}_\#)$. On the other hand, the quasi-Frobenius map Φ induces a map $\Phi : \pi^* A_\# \to i^* A_\#$, which induces a map on periodic homology. We define the inverse Cartier map C^{-1} as the map

$$C^{-1} = M \circ \Phi : HH_\bullet(A^{(1)})((u)) \cong HP_\bullet(\pi^* A^{(1)}_\#) \to HP_\bullet(i^* A_\#) \to HP_\bullet(A). \tag{7.4}$$

We must say that this comparatively easy situation is quite rare — in fact, the only situation where I know that a quasi-Frobenius map exists is the case of a group algebra $A = k[G]$ of some group G (one can take, for instance, the map $\Phi : k[G] \to k[G^p] = k[G]^{\otimes p}$ given by $\Phi(g) = g^{\otimes p}$, $g \in G$). This is perhaps not surprising, since in the commutative case, the situation was also explicit and simple only for the torus $A = k[L]$. It remains to do three things.

(i) Prove that the map C^{-1} is an isomorphism.

(ii) Compare it to the usual inverse Cartier isomoprhism in the commutative case.

(iii) Explain what to do when no quasi-Frobenius map is available.

I will give a sketch of (i) next, under an additional assumption that the algebra A has finite homological dimension — it seems that this is a necessary assumption. I will leave (ii) as a not very difficult but tedious exersize. As for (iii), this is unfortunately quite involved, and I cannot really present the procedure here in any detail, however sketchy; let me just mention that the only new thing in the general case is a certain generalization of the notion of a quasi-Frobenius map, while everything that concerns cyclic homology *per se* remains more-or-less the same as in the simple case. I refer the reader to Section 5 of my paper `arXiv.math/0708.1574` for an introductory exposition, with the detailed proofs given in `arXiv.math/0611623`.

7.4 Cartier isomorphism for algebras with a quasi-Frobenius map.

We assume given an associative algebra A/k with a quasi-Frobenius map Φ, and we want to prove that the corresponding inverse Cartier map (7.4) is an isomorphism. We note that the map M induces an isomorphism by Lemma 7.1, so that what we have to prove is that Φ also induces an isomorphism on periodic cyclic homology.

We will need one technical notion. Note that the embedding $j : \Delta^{opp} \to \Lambda_p$ extends to an embedding $\widetilde{j} : \Delta^{opp} \times \mathsf{pt}_p \to \Lambda_p$. Thus every $E \in \operatorname{Fun}(\Lambda_p, k)$ gives by restriction a simplicial $\mathbb{Z}/p\mathbb{Z}$-representation $\widetilde{j}^*E \in \operatorname{Fun}(\Delta^{opp} \times \mathsf{pt}_p, k) \cong \operatorname{Fun}(\Delta^{opp}, \mathbb{Z}/p\mathbb{Z}\text{-}\mathsf{mod})$. By the Dold-Kan equivalence, $\widetilde{j}^*E$ can treated as a complex of $\mathbb{Z}/p\mathbb{Z}$-representations.

Definition 7.4. An object $E \in \operatorname{Fun}(\Lambda_p, k)$ is *small* if $\widetilde{j}^*E$ is chain-homotopic to a complex of $\mathbb{Z}/p\mathbb{Z}$-modules which is of finite length.

Lemma 7.5. *Assume given a small $E \in \operatorname{Fun}(\Lambda_p, k)$ such that $E([n])$ is a free $\mathbb{Z}/p\mathbb{Z}$-module for any $[n] \in \Lambda_p$ (the action of $\mathbb{Z}/p\mathbb{Z}$ is generated by $\sigma = \tau^n$). Then we have*

$$HP_\bullet(E) = 0.$$

Sktech of a proof. We have $H_\bullet(\Lambda_p, E) = H_\bullet(\Lambda, L^\bullet\pi_!E)$, and the Connes' exact sequence for $L^\bullet\pi_!E$ gives an exact triangle

$$H_{\bullet-1}(\Lambda_p, E) \longrightarrow H_\bullet(\Delta^{opp}, \widetilde{j}^*E) \longrightarrow H_\bullet(\Lambda_p, E) \xrightarrow{\pi^*u} ,$$

where the connecting map is induced by the pullback $\pi^*u \in H^2(\Lambda, k)$ of the generator $u \in H^2(\Lambda, k)$. Computing $H^2(\Lambda_p, k)$ by a cohomological version of the bicomplex (7.1), as in Lecture 4, we find that $\pi^*u = 0$ (this is the same computation as in (7.3)). Therefore, to prove that

$$HP_\bullet(E) = \lim_{\overleftarrow{u}} H_\bullet(\Lambda_p, E)$$

vanishes, it suffices to prove the vanishing of

$$\lim_{\overleftarrow{u}} H_\bullet(\Delta^{opp} \times \mathsf{pt}_p, \widetilde{j}^*E),$$

where $\widetilde{j}^*u \in H^2(\Delta^{opp} \times \mathsf{pt}_p, k)$ is the restriction of the periodicity generator $u \in H^2(\Lambda_p, k)$. Using the Yoneda representation of u, we see that with respect to the Künneth isomorphism $H^2(\Delta^{opp} \times$

$\mathsf{pt}_p, k) \cong H^{\bullet}(\mathbb{Z}/p\mathbb{Z}, k)$, the class $\widetilde{j}^*u$ corresponds to the periodicity generator of $H^2(\mathbb{Z}/p\mathbb{Z}, k)$. Therefore in the spectral sequence

$$H_{\bullet}(\Delta^{opp}, \lim_{\overleftarrow{u}} H_{\bullet}(\mathbb{Z}/p\mathbb{Z}, \widetilde{j}^*E)) \Rightarrow \lim_{\overleftarrow{u}} H_{\bullet}(\Delta^{opp} \times \mathsf{pt}_p, \widetilde{j}^*E),$$

the limit in the left-hand side is the Tate homology $\check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, \widetilde{j}^*E)$. Since E is small, the spectral sequence converges, and since $E([n])$ is a free $\mathbb{Z}/p\mathbb{Z}$-representation for every $[n]$, the Tate homology in question is equal to 0. $\square$

Proposition 7.6. *Assume given an associative algebra A equipped with a quasi-Frobenius map $\Phi : A^{(1)} \to A^{\otimes p}$, and assume that the category A-bimod of A-bimodules has finite homological dimension. Then the Cartier map* (7.4) *for the algebra A is an isomorphism.*

Proof. We first note that the object $i^*A_\# \in \operatorname{Fun}(\Lambda_p, k)$ is small in the sense of Definition 7.4. Indeed, by assumption, the diagonal A-bimodule A admits a finite projective resolution $P_{\bullet}$. Therefore the bar resolution $C_{\bullet}(A)$ is chain-homotopic to a finite complex $P_{\bullet}$, its p-th power $C_{\bullet}(A) \otimes_A \cdots \otimes_A C_{\bullet}(A)$ is chain-homotopic to the finite complex $P_{\bullet} \otimes_A \cdots \otimes_A P_{\bullet}$, and the induced chain homotopy equivalence between $i^*A_\#$ and the finite complex

$$(P_{\bullet} \otimes_A \cdots \otimes_A P_{\bullet}) \otimes_{A^{opp} \otimes A} A$$

is obviously compatible with the $\mathbb{Z}/p\mathbb{Z}$-action. Moreover, $\pi^*A_\#$ is also small. It remains to notice that any quasi-Frobenius map Φ must be injective (otherwise it sends some element $a \in A^{(1)} = \check{H}_0(\mathbb{Z}/p\mathbb{Z}, A^{(1)})$ to 0), and its cokernel $A^{\otimes p}/\Phi(A^{\otimes p})$ by definition has no Tate homology.

Exercise 7.2. *Prove that for some $k[\mathbb{Z}/p\mathbb{Z}]$-module V, $\check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, V) = 0$ if and only if V is free. Hint: identifying $k[\mathbb{Z}/p\mathbb{Z}] = k[t]/t^p$, $\sigma \mapsto 1 + t$, show that V decomposes into a direct sum of modules of the form $k[t]/t^l$, $0 < l \leq p$, and check the statement for all l.*

We conclude that $A^{\otimes p}/\Phi(A^{\otimes p})$ is a free $k[\mathbb{Z}/p\mathbb{Z}]$-module. Therefore for every n, the module $A^{\otimes pn}/\Phi^{\otimes n}(A^{(1)\otimes n})$ is free, and has no Tate homology. This means that the cokernel $i^*A_\#/\Phi(\pi^*A_\#)$ satisfies the assumptions of Lemma 7.5, and Φ indeed induces an isomorphism between $HP_{\bullet}(\pi^*A_\#)$ and $HP_{\bullet}(i^*A_\#)$. $\square$

Remark 7.7. In the smooth commutative case, the assumption that A-bimod has finite homological dimension just means that A is of finite type over k. In the general case of the theorem, when no quasi-Frobenius map is available, one actually needs to assume that the homological dimension is less than $2p - 1$. In the commutative case, this reduces to $p > \mathsf{dim}\operatorname{Spec} A$.