Lecture 9.

The language of operads. Poisson and associative operad. Gerstenhaber operad and small discs. Braided algebras. Deligne Conjecture.

9.1 The language of operads.

These days, it has become common practice to use the language of the so-called *operads* to describe various non-trivial algebraic structures such as that of a Gerstenhabe algebra. It must be mentioned that the notion of an operad has been introduced 35 years ago by P. May essentially as a quick hack; it is not very natural, and in many cases it is not quite what one needs, so that descriptions using operads tend to be somewhat ugly and somewhat artificial. But at least, from the formal point of view, everything is well-defined. We will only sketch most proofs. For a complete exposition which covers much if not all the material in this lecture, I refer the reader, for instance, to the paper `arXiv:0709.1228` by V. Ginzburg and M. Kapranov which is now considered one of the standard references on the subject (the paper was published in 1994, and I am grateful to V. Ginzburg who finally put it on arxiv in 2007). Another reference is the foundational paper `arXiv:hep-th/9403055` by E. Getzler and J.D.S. Jones, but this has to be used with care, since some advanced parts of it were later found to be wrong.

To define an operad, let Γ be the category of finite sets, and let $\Gamma^{[2]}$ be the category of arrows in Γ (objects are morphisms $f : S' \to S$ between $S', S \in \Gamma$, morphisms are commutative squares). Then Γ has a natural embedding into $\Gamma^{[2]}$: every finite set S has a canonical morphism $p^S : S \to \mathsf{pt}$ into the finite set $\mathsf{pt} \in \Gamma$ with a single element. We note that every $f \in \Gamma^{[2]}$, $f : S' \to S$ canonically decomposes into a coproduct

$$f = \coprod_{s \in S} f^s, \tag{9.1}$$

where $f^s \in \Gamma^{[2]}$ is the canonical map $p^{f^{-1}(s)} f^{-1}(s) \to \mathsf{pt}$ corresponding to the preimage $f^{-1}(s) \subset S'$.

Definition 9.1. An *operad* $O_{\bullet}$ of k-vector spaces is a rule which assigns a vector space O_f to any $f \in \Gamma^{[2]}$ together with the following operations:

(i) for any pair $f : S' \to S$, $g : S'' \to S'$ of composable maps, a map $\mu_{f,g} : O_f \otimes O_g \to O_{f \circ g}$,

(ii) for any $f \in \Gamma^{[2]}$, $f : S' \to S$, an isomorphism

$$O_f \cong \bigotimes_s O_{f^s},$$

where $f^s = p^{f^{-1}(s)}$ are as in (9.1).

Moreover, the assignment $f \mapsto O_f$ should be functorial with respect to isomorphisms in $\Gamma^{[2]}$, the maps in (i) and (ii) should be functorial maps, and for any triple $f, g, h \in \Gamma^{[2]}$ of composable maps, the square

$$\begin{array}{ccc} O_f \otimes O_g \otimes O_h & \longrightarrow & O_{f\circ g} \otimes O_h \\ \downarrow & & \downarrow \\ O_f \otimes O_{g\circ h} & \longrightarrow & O_{f\circ g\circ h} \end{array}$$

should be commutative.

It is useful to require also that $O_{\mathsf{id}} \cong k$ for an identity map $\mathsf{id} : S \to S$, and we shall do so. We note that by virtue of (ii), it is sufficient to specify only the vector spaces O_{p^S} for the canonical maps $p_S : S \to \mathsf{pt}$ (these are usually denoted O_S, or simply O_n, where n is the cardinality of S). However, the way we have formulated the definition makes it slightly more natural, and slightly easier to generalize.

Definition 9.2. An *algebra* A over an operad $O_{\bullet}$ of k-vector spaces is a k-vector space A, together with an action map

$$a_f : O_f \otimes A^{\otimes S_1} \to A^{\otimes S_2}$$

for any $f \in \Gamma^{[2]}$, $f : S_1 \to S_2$, where for any finite set $S \in \Gamma$, we denote by $A^{\otimes S}$ the tensor product of copies of A numbered by elements $s \in S$. The maps a_f should be functorial with respect to isomoprhisms in $\Gamma^{[2]}$ and satisfy the following rules:

(i) For a pair $f, g \in \Gamma^{[2]}$ of composable maps, we should have $a_f \circ a_g = a_{f \circ g} \circ \mu_{f,g}$.

(ii) For any $f \in \Gamma^{[2]}$, $f : S' \to S$, we should have

$$a_f = \bigotimes_{s \in S} a_{f^s}.$$

As in the definition of an operad, (ii) insures that it is sufficient to specify the action maps $a_n = a_S = a_{p^S} : O_{p^S} \otimes A^{\otimes S} \to A$ for all $S \in \Gamma$, but our formulation is slightly more natural. We also note that algebras over a fixed operad O form a category, which has a forgetfull functor into the category of k-vector spaces. The left-adjoint functor associates to a k-vector space V the free O-algebra $F_O V$ generated by V, which is explicitly given by

$$F_O V = \bigoplus_{S \in \Gamma} \left(O_S \otimes V^{\otimes S}\right)_{\mathsf{Aut}(S)}, \tag{9.2}$$

where the sum is over all the isomorphism classes of finite sets — in other words, over all integers — and $\mathsf{Aut}(S)$ is the symmetric group of all automorphisms of a finite set S.

The reasoning behind these definitions is the following. We want to describe algebras of a certain kind — associative algebras, commutative algebras, Lie algebras, Poisson algebras, etc. To do so, one usually says that an algebra is a vector space A equipped with some multilinear structural maps which satisfy some axioms (associativy, the Jacobi identity, and so forth). However, this is not always convenient — just as describing a concrete algebra by its generators and relations is usually too cumbersome. An operad O encodes *all the polylinear operations* we want our algebra to have. More precisely, given some $f : S_1 \to S_2$, we collect in the vector space O_f all the operations from $A^{\otimes S_1}$ to A^{S_2} which can be obtained from the structural maps by composing them and substituting one into the other; and we take the quotient by all the relations our concrete type of algebraic structure imposes on these compositions. Moreover, we only want to consider those algebraic structures which are defined by operations with values in A itself, not its tensor powers. This is the reason for the condition (ii) in Definition 9.1 and Definition 9.2.

9.2 Examples.

Probably the simplest example of an operad is obtained by taking $O_f = k$, the 1-dimensional vector space, for any $f \in \Gamma^{[2]}$. This operad is denoted by Com. A moment's reflection shows that algebras over Com are nothing but *commutative associative unital algebras*. Indeed, by definition, we must have a unique action map

$$a_S : A^{\otimes S} \to A$$

for any $S \in \Gamma$, and moreover, this map should be functorial with respect to isomorphisms in Γ — in other words, a_{p^S} must the equivariant with respect to the natural action of the symmetric group $\mathsf{Aut}(S)$. Thus first, we must have a commutative multiplication $\mu : A^{\otimes 2} \to A$ corresponding to the generator of $\mathsf{Com}_2 = k$, and second, any way to compose this operation to obtain an operation $A^{\otimes n} \to A$ for any n must give the same result — which for $n = 3$ implies associativity,

$$\mu \circ (\mu \otimes \mathsf{id}) = \mu \circ (\mathsf{id} \otimes \mu).$$

One checks easily that conversely, associativity implies the uniqueness for any $n \geq 3$. The free Com-algebra $F_{\mathsf{Com}} V$ generated by a vector space V is given by (9.2) and coincides with the symmetric algebra $S^{\bullet} V$.

Exercise 9.1. *Check that for a* Com*-algebra A, the action map $a_0 : k = A^{\otimes 0} \to A$ provides a unity in the commutative associative algebra A.*

A slightly more difficult example is the operad Ass which encodes the structure of an associative unital algebra: it is usually described by setting

$$\mathsf{Ass}_S = k[\mathsf{Aut}(S)],$$

the regular representation of the symmetric group $\mathsf{Aut}(S)$. To define the operadic composition, one can, for example, consider the so-called *category Σ of non-commutative sets*: objects are finite sets, morphisms from S' to S are pairs of a map $f : S' \to S$ of finite sets and a total ordering on every preimage $f^{-1}(s)$, $s \in S$. The composition is obvious, and we obviously have the forgetfull functor $\gamma : \Sigma \to \Gamma$ which forgets the total orders. Then we set

$$\mathsf{Ass}_f = k[\{f' \in \Sigma(S', S) \mid \gamma(f') = f\}] \tag{9.3}$$

for any $f \in \Gamma$, $f : S' \to S$, and the composition in Σ induces the composition maps $\mathsf{Ass}_f \otimes \mathsf{Ass}_g \to \mathsf{Ass}_{f \circ g}$. The free algebra $F_{\mathsf{Ass}} V$ generated by a vector space V is the tensor algebra $T^{\bullet} V$.

Let us assume from now on that the base field k has characteristic 0, $\mathsf{char}\, k = 0$. For any vector space V, the diagonal map $V \to V \oplus V$ induces a coproduct $T^{\bullet} V \to T^{\bullet} V \otimes T^{\bullet} V$ which turns the tensor algebra $T^{\bullet} V$ into a cocommutative Hopf algebra. Since $\mathsf{char}\, k = 0$, this means that $T^{\bullet} V$ is the universal envelopping algebra of some Lie algebra $L^{\bullet} V$. In fact, by the universality property of a universal envelopping algebra, $L^{\bullet} V$ is the free Lie algebra generated by V. The universal envelopping algebra $T^{\bullet} V$ acquires a Poincaré-Birkhoff-Witt increasing filtration $K_{\bullet} T^{\bullet} V$, and the associated graded quotient with respect to this filtration is the symmetric algebra generated by $L^{\bullet} V$ — we have a canonical identification

$$\mathsf{gr}^F_{\bullet} T^{\bullet} V \cong S^{\bullet} L^{\bullet} V.$$

This graded quotient is a Poisson algebra, and it is easy to see by spelling out the universal properties that $P_{\bullet} V = \mathsf{gr}^F_{\bullet} T^{\bullet} V$ is actually the free Poisson algebra generated by V.

Now, both the PBW filtration and the isomorphism $\mathsf{gr}^F_{\bullet} T^{\bullet} V \cong P_{\bullet} V$ are functorial in V; this implies that what we actually have is a decreasing filtration $F^{\bullet}\, \mathsf{Ass}$ on the associative operad Ass, and an identification $\mathsf{gr}^{\bullet}_F \mathsf{Ass} \cong \mathsf{Poi}$ between the associated graded quotient of Ass and an operad Poi which encodes the structure of a Poisson algebra (in particular, the PBW filtration on Ass is compatible with the operadic structure). We see that Poi is in fact an operad of graded vector spaces. This is also obvious from the definition: if we assign degree 0 to multiplication and degree 1 to the Poisson bracket, then all the axioms of a Poisson algebra are compatible with these degrees.

The highest degree term of the PBW filtration on Ass — or equivalently, the highest term in the associated graded quotient $\mathsf{gr}^{\bullet}_F \mathsf{Ass} \cong \mathsf{Poi}$ — is the Lie operad Lie; the natural maps $\mathsf{Lie} \to \mathsf{Ass}$,

$\mathsf{Lie} \to \mathsf{Poi}$ encode the fact that both a Poisson algebra and an associative algebra are Lie algebras in a canonical way (in the associative case, the bracket is given by the commutator, $[a, b] = ab - ba$). We note that it is not trivial to describe Lie explicitly. For example, the dimension of Lie_n is $(n-1)!$. If the base field k is algebraically closed, then Lie_n can be described as the representation of the symmetric group Σ_n induced from the non-trivial character of the cyclic subgroup $\mathbb{Z}/n\mathbb{Z} \subset \Sigma_n$ spanned by the long cycle. It is a pleasant exersize to check that this representation is actually defined over k even when k is not algebraically closed.

Finally, the example that interest us most is that of Gerstenhaber algebras. Since the definition of a Gerstenhaber algebra differs from that of a Poisson algebra only in the degree assigned to the bracket, one might expect that Gerstenhaber algebras are controlled by an operad $\mathsf{Gerst}^{\bullet}$ essentially isomorphic to $\mathsf{Poi}^{\bullet}$. This is true, but there is the following subtlety. Both $\mathsf{Poi}^{\bullet}$ and $\mathsf{Gerst}^{\bullet}$ are operads of graded k-vector spaces, but this can means one of two distinct things: either we define the product of graded vector spaces simply as their product with induced grading, or we treat the degree as a homological degree. The difference is in the symmetry isomorphism $\sigma : V_{\bullet} \otimes W_{\bullet} \to W_{\bullet} \otimes V_{\bullet}$ of the tensor product of graded vector spacee $V_{\bullet}$, $W_{\bullet}$: if the degree is homological, then by convention we introduce the sign and define σ by

$$\sigma(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a.$$

Now, $\mathsf{Gerst}^{\bullet}$ and $\mathsf{Poi}^{\bullet}$ are both operads of graded vector spaces, and the difference between them is the following: the action of the symmetric group $\mathsf{Aut}(S)$ on Gerst_S is twisted by the sign representation — for any n, S, we have

$$\mathsf{Gerst}^n_S \cong \mathsf{Poi}^n_S \otimes \varepsilon^{\otimes n}, \tag{9.4}$$

where ε is the one-dimensional sign representation of $\mathsf{Aut}(S)$. But while $\mathsf{Poi}^{\bullet}$ is a graded operad in the usual naive sense, the degree in $\mathsf{Gerst}^{\bullet}$ is homological, and because of this, the isomorphisms (9.4) are still compatible with the operadic structure.

9.3 Little cubes operad.

It turns out, however, that there is a different, more conceptual construction of the Gerstenhaber operad Gerst.

One immediately notes that in the definition of an operad, one can use any symmetric monoidal category instead of the category of k-vector spaces. Thus we can speak not only about operads of vector spaces, or graded vector spaces, but also abouts operads of sets and operads of topological spaces. And historically, it was the operads of topological spaces which appeared first — specifically, the so-called *operad of little n-cubes.*

Let I be the unit interval $[0, 1]$. Fix a positive integer n, and consider the cube I^n of size 1 of dimension n. For any finite set S, say that an *S-cube configuration in I^n* is an open subset in I^n whose complement is the union connected components numbered by elements of S, each being a subcube in I of smaller size, whose faces are parallel to faces of I^n. Let O^n_S be the set of all such configurations. A configuration is completely determined by the centers and the sizes of all the cubes, so that O^n_S is naturally an open subset in $(I^{(n+1)})^S$. This turns it into a topological space.

We now note that the collection O^n_S with a fixed n naturally defines an operad of topological spaces. The composition is given by the following procedure: take an S_1-cube configuration in I^n, rescale it to a smaller size, and plug it into an S_2-cube configuration by filling in one of the connected components of its complement. When the sizes fit, the result is obviously an $(S_1 \cup S_2 \setminus \{s\})$-cube configuration, where $s \in S_2$ is the point which we used for the operation. We leave it to the reader to check that this procedure indeed gives a well-defined operad, and that all the structure maps of this operad are continuous maps.

Definition 9.3. The operad $O^n_\bullet$ is called the *operad of little n-cubes.*

What one is interested in is not the topological spaces O^n_S but their homotopy types, and these have a simpler description. Forgetting the size of a cube defines a projection $O^n_S \to (I^n)^S \setminus \mathsf{Diag}$, the complement to all the diagonals in the power $(I^n)^S$, and this projection is a homotopy equivalence — in other words, O^n_S is homotopy-equivalent to the *configuration space* of injective maps from S to I^n. Equivalently, one can take $\mathbb{R}^n$ instead of the cube I^n. Unfortunately, the structure of the operad is not visible in this model.

If $n = 1$, we can go even further: the configuration space of injective maps from S to the interval I has $|\operatorname{\mathsf{Aut}}(S)|$ connected components, numbered by the induced total order on the set S, and each connected component is a simplex, thus contractible. We conclude that O^1_S is homotopy-equivalent to the (discrete finite) set of total orders on S.

Now, taking the homology with coefficients in k turns any operad of topological spaces into an operad of graded k-vector spacers. In particular, for any $n \geq 1$ we have an operad formed by $H_\bullet(O^n_S, k)$.

Exercise 9.2. *Check that for $n = 1$, $H_\bullet(O^n_\bullet, k)$ is the operad $\mathsf{Ass}_\bullet$. Hint: use the description* (9.3).

Proposition 9.4. *Algebras over the homology operad $H_\bullet(O^2_S, k)$ of the operad $O^2_\bullet$ of little squares are the same as Gerstenhaber algebras, and $H_\bullet(O^2_S, k)$ is isomorphic to the Gerstenhaber operad $\mathsf{Gerst}^\bullet$.*

Proof. This is an essentially well-known but rather non-trivial fact; for example, it implies that $H_n(O^2_n, k)$ is the n-th space Lie_n of the Lie operad — as far as I know, this was first proved by V. Arnold back in the late 60-es.

Let us first construct a map of operads $a_\bullet : \mathsf{Gerst}^\bullet_\bullet \cong H_\bullet(O^2_\bullet, k)$. The component $\mathsf{Gerst}^\bullet_2$ is spanned by the product and the bracket, and O^2_2 is the complement to the diagonal in the product $I^2 \times I^2$, which is homotopy-equivalent to the circle S^1. We define a_2 by sending the product in Gerst^0_2 to the class of a point in $H_0(S^1, k) \cong k$, and the bracket in Gerst^1_2 to the fundamental class in $H_1(S^1, k) \cong k$.

Exercise 9.3. *Check that this extends to a map of operads. Hint: since all the relations in $\mathsf{Gerst}^\bullet$ invlove only three indeterminates, it is sufficient to consider O^2_3.*

Now assume by induction that a_i is an isomorphism for all $i \leq n$. By definition, $\mathsf{Gerst}^\bullet_{n+1}$ is spanned by all expressions involving the product and the bracket in $n + 1$ indeterminates $x_1, \dots, x_{n+1}$. Substituting the unity instead of x_{n+1} gives a map $\mathsf{Gerst}^\bullet_{n+1} \to \mathsf{Gerst}_n$; this map is obviously surjective. Substituting $\{x_{n+1}, x_i\}$ instead of x_i gives a map $\mathsf{Gerst}^\bullet_n \otimes k[S] \to \mathsf{Gerst}^{\bullet+1}_{n+1}$, where S is the set of indeterminates $x_1, \dots, x_n$. Since $\{1, x_i\}$ is by definition equal to 0, we have a sequence

$$\mathsf{Gerst}^{\bullet-1}_n \otimes k[S] \longrightarrow \mathsf{Gerst}^\bullet_{n+1} \longrightarrow \mathsf{Gerst}^\bullet_n \longrightarrow 0. \qquad (9.5)$$

which is exact on the right.

On the geometric side, filling in the $(n + 1)$-st cube in a cube configuration — or equivalently, forgetting the $(n+1)$-st point in a configuration of points in $\mathbb{R}^2$ — defines a projection $O^2_{n+1} \to O^2_n$, and this is a fibration with fiber $E^n = \mathbb{R}^2 \setminus S$, where $S \subset \mathbb{R}^2$ is the configuration of the remaining n distinct points. We have the Leray spectral sequence

$$H_\bullet(O^2_n, H_\bullet(E^2_n, k)) \Rightarrow H_\bullet(O^2_{n+1}, k).$$

The homology $H_\bullet(E^2_n, k)$ is only non-trivial in degrees 0 and 1; the group $H_1(E^2_n, k)$ can be naturally identified with $k[S]$ by sending $s \in S$ to a small circle around its image in $\mathbb{R}^2$. The fundamental group of the base O^2_n is the pure braid group, and it is easy to check that it acts trivially on $H_\bullet(E^2_n, k)$, so that the spectral sequence reads

$$H_\bullet(O^2_n, k) \otimes H_\bullet(E^2_n, k) \Rightarrow H_\bullet(O^2_{n+1}, k).$$

Moreover, replacing $\mathbb{R}^2$ with $\mathbb{C}$, we can treat $O^2_S = \mathbb{C}^S \setminus \mathsf{Diag}$ as a complex algebraic variety whose homology groups have Hodge structures, and in particular, weights. One checks easily that $H_n(O^2_S, k)$ is pure Hodge-Tate of weight $2n$. Therefore the Leray spectral sequence degenerates, so that, taking in account the isomorphism $H_1(E^2_N, k) \cong k[S]$, we have a short exact sequence

$$(9.6) \qquad 0 \longrightarrow H_{\bullet-1}(O^2_n, k) \otimes k[S] \longrightarrow H_\bullet(O^2_{n+1}, k) \longrightarrow H_\bullet(O^2_n, k) \longrightarrow 0.$$

Now, it is obvious from the construction of the map $a_\bullet$ that it is a map between (9.5) and (9.6), so that we have a commutative diagram

$$(9.7) \qquad \begin{array}{ccccccccc} & & \mathsf{Gerst}^{\bullet-1}_n \otimes k[S] & \xrightarrow{f} & \mathsf{Gerst}^\bullet_{n+1} & \longrightarrow & \mathsf{Gerst}^\bullet_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle a_{n+1}} & & \downarrow{\scriptstyle a_n} & & \\ 0 & \longrightarrow & H_{\bullet-1}(O^2_n, k) \otimes k[S] & \longrightarrow & H_\bullet(O^2_{n+1}, k) & \longrightarrow & H_\bullet(O^2_n, k) & \longrightarrow & 0. \end{array}$$

Moreover, we now that a_n is an isomorphism, which implies in particular that the map f in (9.7) is injective. To prove that a_{n+1} is also an isomorphism, it suffices to prove that the top row forms a short exact sequence. But we also have the projection $O^1_{n+1} \to O^1_n$, and it induces a short exact sequence

$$0 \longrightarrow \mathsf{Ass}_n \otimes k[S] \longrightarrow \mathsf{Ass}_{n+1} \longrightarrow \mathsf{Ass}_n \longrightarrow 0$$

which gives (9.5) under taking the associated graded with respect to the Poincaré-Birkhoff-Witt filtration and using the isomorphism $\mathsf{Gerst}^\bullet \cong \mathsf{Poi}^\bullet$. Since this sequence is exact, and its associated graded is exact on the left and on the right, it must also be exact in the middle term for dimension reasons. □

9.4 Braided algebras and Tamarkin's proof.

What we did in Proposition 9.4 was to take two different operads, that of 1-cubes and that of 2-cubes, and identify, up to a sign twist, $H_\bullet(O^2_\bullet, k)$ with a certain associated graded quotient of $H_\bullet(O^1_\bullet, k)$ (which reduces to $H_0(O^1_\bullet, k)$). We now note that $H_\bullet(O^2_\bullet, k)$ can also be treated as an associated graded quotient. Namely, given a topological space X, one can consider its singular chain complex $C_\bullet(X, k)$. Every complex $E_\bullet$ has a "canonical filtration" $F^\bullet E_\bullet$ given by

$$F^i E_j = \begin{cases} 0, & j < i, \\ \mathsf{Ker}\, d, & j = i, \\ E_j, & j > i, \end{cases}$$

where d is the differential. The associated graded quotient $\mathsf{gr}^\bullet_F E_\bullet$ is canonically quasiisomorphic to the sum of homology of the complex $E_\bullet$. In particular, we have

$$\mathsf{gr}^\bullet_F C_\bullet(X, k) \cong H_\bullet(X, k).$$

Thus passing to homology is, up to quasiisomorphism, equivalent to taking the associated graded quotient with respect to the canonical filtration.

Given an operad $X_\bullet$ of topological spaces, we can consider the DG operad formed by $C_\bullet(X_\bullet, k)$. The canonical filtration, being canonical, is automatically compatible with the operadic structure, and the associated graded quotient $\mathsf{gr}_F^\bullet C_\bullet(X_\bullet, k)$ is quasiisomorphic to $H_\bullet(X_\bullet, k)$.

In particular, we can consider the operad $C_\bullet(O^2_\bullet, k)$. Its canonical filtration in fact behaves similarly to the PBW filtration on $\mathsf{Ass} = H_0(O^1_\bullet, k)$, although to define it, we do not need to use the structure of an operad. The associated graded quotient $\mathsf{gr}_F^\bullet C_\bullet(O^2_\bullet, k)$ is quasiisomorphic to the Gerstenhaber operad Gerst.

Definition 9.5. A *braided algebra* is a DG algebra over the DG operad $C_\bullet(O^2_\bullet, k)$.

The term "braided algebra" comes from the relation between O^2_n and the pure braid group B_n of n braids: we have $\pi_1(O^2_n) = B_n$, and one can show that O^2_n has no higher homotopy groups, so that it is homotopy-equivalent to the classifying space of B_n.

We note that as stated, Definition 9.5 is almost useless, since the singular chain complex $C_\bullet(X)$ of a topological space is huge — one cannot expect the DG operad $C_\bullet(O^2_\bullet, k)$ to act on anything reasonable. However, what one can do is to invert quasiisomorphisms and consider DG algebras over some DG operad $O_\bullet$ "up to quasiisomorphism", in the same way as we did for DG Lie algebras. A convenient formalism for this is provided by the so-called *closed model categories* originally introduced by Quillen (a modern reference is the book "Model categories" by M. Hovey). This gives a certain well-defined category $\mathrm{Ho}(O_\bullet)$, and, what is important, it only depends on the defining operad "up to a quasiisomorphism" — a quasiisomorphism $O'_\bullet \to O_\bullet$ between DG operads induces an equivalence $\mathrm{Ho}(O'_\bullet) \cong \mathrm{Ho}(O_\bullet)$. In practice, one is only interested in braided algebras up to a quasiisomorphism, that is, in objects of the category $\mathrm{Ho}(C_\bullet(O^2_\bullet, k))$; and to construct such an algebra, it is sufficient to have a DG algebra over some DG operad quasiisomorphic to $C_\bullet(O^2_\bullet, k)$. It is this structure which one has on the Hochschild cohomology complex of an associative unital algebra A.

Theorem 9.6 (Deligne Conjecture). *For any unital associative k-algebra A, its Hochschild cohomology complex is a DG algebra over a DG operad which is quasiisomorphic to $C_\bullet(O^2_\bullet, k)$.*

This statement has an interesting history. Originally it was a question, not even a conjecture, asked in 1993 by P. Deligne. Almost immediately it was wrongly proved by Getzler and Jones, and independently, also wrongly, by A. Voronov. But in 1998, Tamarkin has discovered his amazingly short proof of the Kontsevich Formality Theorem, which used Deligne conjecture; under close scrutiny, the mistakes were found, and new complete proofs by several groups of people were available by 2000 (among those people I should mention at least Tamarkin, Voronov, J. McClure-J. Smith, and M. Kontsevich-Y. Soibelman). In almost all the proofs, the authors actually construct a single DG operad which works for all associative algebras, but all of them are rather complicated and unnatural. The real reason for this is that what acts naturally on Hochschild cohomology is not an operad but a more complicated object, and this is currently under investigation. However, for practical purposes such as Formality Theorem, any solution is good, since it can used as a black box.

Assuming Deligne Conjecture, Tamarkin's proof of the Formality Theorem is a combination of the following two results.

Theorem 9.7 (Tamarkin,Kontsevich). *The DG operad $C_\bullet(O^2_\bullet, k)$ itself is formal, that is, there exists a chain of quasiisomorphisms connecting it to the Gerstenhaber operad $\mathsf{Gerst}^\bullet = H_\bullet(O^2_\bullet, k)$.*

Theorem 9.8 (Tamarkin). *Let A be the polynomial algebra $k[x_1, \dots, x_n]$ in n variables, equipped with the natural action of the group $GL(n, k)$ which interchanges the variables. Any DG algebra over Gerst which is equipped with a $GL(n, k)$-action and whose cohomology is isomorphic to $HH^\bullet(A)$ as a $GL(n, k)$-equivariant Gerstenhaber algebra is formal.*

It is the second result that was the original discovery of Tamarkin, and its proof was very simple. But then the problems with Deligne Conjecture appeared... in the course of fixing them, Kontsevich suggested that the operad $C_{\bullet}(O^2_{\bullet}, k)$ itself should be formal, and Tamarkin promptly proved it (but this proof was combinatorial and not simple at all). Later on, Kontsevich gave a different proof, also combinatorial. There is also a very simple argument in folklore which deduces Theorem 9.7 from Hodge Theorey, similarly to the classic formality result of Deligne-Griffits-Morgan-Sullivan, but this, to the best of my knowledge, has never been written down. In any case, one thing is very important: the quasiisomorphisms in Theorem 9.7, no matter how one produces them, are very non-trivial, and they usually depend on transcedental things like periods of differential forms or the so-called "Drinfeld associator". In addition, there is no canonical choice of these quasiisomorphisms — one expects that the conjectural "motivic Galois group", or even the usual Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, acts on the set of these quasiisomorphisms in a very non-trivial way. On the other hand, the DG operads which appear in the solutions to Deligne Conjecture are quite canonical, and their action on Hochschild cohomology is elementary and defined over $\mathbb{Q}$.