Lecture 10.

Combinatorics of planar trees. Comparison theorem. Brace operad and its action on the Hochschild cohomology complex.

10.1 Planar trees.

The topic of today's lecture is Deligne Conjecture — we want to construct an operad $O_{\bullet}$ quasiisomorphic to the chain complex operad $C_{\bullet}(O^2_{\bullet}, k)$ of the operad of little squares so that $O_{\bullet}$ acts in a natural way on the Hochschild cohomology complex of an associative algebra A.

We start by introducing a certain combinatorial model of the operad of little squares (or rather, it will be more convenient for us to work with little discs).

By a *planar tree* we will understand an unoriented connected graph with no cycles and one distiguished vertex of valency 1 called *the root*, equipped with a cyclic order on the set of edges attached to each vertex. Given such a tree T, we will denote by $V(T)$ the set of all non-root vertices of T, and we will denote by $E(T)$ the set of all edges of T not adjacent to the root.

More generally, by an *n-planar tree* we will understand an unoriented connected graph with no cycles and n distiguished vertices of valency 1, called *external vertices*, one of which is additionally distinguished and called the root; again, the graph should be equipped with a cyclic order on the set of edges attached to each vertex. We note that this automatically induces a cyclic order on the set of external vertices, so that n-planar trees are naturally numbered by an object $[n]$ of the cyclic category Λ. For an n-planar tree T, $V(T)$ denotes the set of all non-external vertices, and $E(T)$ denotes the set of of edges not adjacent to external vertices.

Given a tree T, we denote by $|T|$ its geometric realization, that is, a CW complex with vertices of T as 0-cells and edges of T as 1-cells. We note that for every planar tree T, $|T|$ can be continuously embedded into the unit disc D so that the root of T goes to $1 \in D$, the external vertices, if any, go to points on the boundary $S^1 \subset D$ and split it into a wheel graph, the rest of $|T|$ is mapped into the interior of the disc, and for every vertex $v \in V(T)$, the given cyclic order on the edges adjacent to v is the clockwise order. Moreover, the set of all such embeddings with a natural topology is contractible, so that the embedding is unique up to a homotopy, and the homotopy is also unique up to a homotopy of higher order, and so on.

Given a tree T and an edge $e \in E(T)$, we may contract e to a vertex and obtain a new tree T^e. The contrations of different edges obviously commute, so that for any n edges $e_1, \dots, e_n \in E(T)$, we have a unique tree $T^{e_1,\dots,e_n}$ obtained by contracting $e_1, \dots, e_n$. By construction, we have a natural map $V(T) \to V(T^{e_1,\dots,e_n})$ and a natural map of realizations $|T| \to |T^{e_1,\dots,e_n}|$.

Assume given a finite set S. By a *tree marked by S* we will understand a planar tree or an n-planar tree T together with an injective map $S \to V(T)$. The vertices in the image of this map are called marked, the other ones are unmarked. A marked tree T is *stable* if every unmarked vertex $v \in V(T) \setminus S$ has valency at least 3. Given a marked tree T which is unstable, we can canonically produce a stable tree T' by first recursively removing all unmarked vertices of valency 1 and edges leading to them, and them removing unmarked vertices of valency 2 and gluing together the corresponding edges. We will call this T' the *stabilization* of T.

Given a stable marked tree T and some edges $e_1, \dots, e_n \in E(T)$, we mark the contraction by composing the map $S \to V(T)$ with the natural map $V(T) \to V(T^{e_1,\dots,e_n})$. If the resulting map is injective, then this is again a stable marked tree.

Exercise 10.1. *Check that for any two trees T, T' stably marked by the same set S, there exists at most one subset $\{e_1, \dots, e_n\} \subset E(T)$ such that $T^{e_1,\dots,e_n} \cong T$. Hint: removing an edge splits a tree T into two connected components; first prove that an edge is uniquely defined by the corresponding partition of the set $V(T)$.*

By virtue of this exercize, for every $[n] \in \Lambda$ the collection of all n-planar trees stably marked by the same finite set S acquires a partial order: we say that $T \geq T'$ if and only if T' can be obtained from T by contraction. We will denote this partially ordered set by $\mathsf{T}_S^{[n]}$, or simply by T_S if $[n] = [1]$. This is our combinatorial model for the configuration space.

Theorem 10.1. *For any $[n] \in \Lambda$, the classifying space $|\mathsf{T}_S^{[n]}|$ of the partially ordered set T_S is homotopy equivalent to the configuration space $D^S \setminus \mathsf{Diag}$ of injective maps $S \to D$ to the unit disc D.*

As a first step of proving this, let us construct a map $|\mathsf{T}_S| \to D$.

Let us denote by B_S the fundamental groupoid of the configuration space $D^S \setminus \mathsf{Diag}$: objects are points, that is, injective maps $f : S \to D$, morphisms from f to f' are homotopy classes of paths, that is, homotopy classes of continuous maps $S \times I \to D$, whose restriction to $S \times \{0\}$, resp. $S \times \{1\}$ is equal to f, resp. f' (here $I = [0,1]$ is the unit interval, and $S \times I$ is equipped with the product topology — it is the disjoint union of S copies of I). Since $D^S \setminus \mathsf{Diag}$ is an Eilenberg-MacLane space of type $K(\pi, 1)$, we have the homotopy equivalence $D^S \setminus \mathsf{Diag} \cong |\mathsf{B}_S|$.

Now consider the following category $\widetilde{\mathsf{T}}_S$. Objects are stable marked trees T together with an embedding $f : |T| \to D$. Maps from $f : |T| \to D$ to $f' : |T'| \to D$ exist only if $T \geq T'$, and they are homotopy classes of continous maps $\gamma : |T| \times I \to D$ such that the restriction $\gamma : |T| \times \{x\} \to D$ is injective for any $x \in [0,1[$, the restriction $\gamma : |T| \times \{0\} \to D$ is equal to the map f, and the restriction $\gamma : |T| \times \{1\} \to D$ is the composition of the natural map $|T| \to |T'|$ and the map $f' : |T'| \to D$.

Then on one hand, we have a forgetfull functor $\widetilde{\mathsf{T}}_S \to \mathsf{T}_S$ which forgets the embedding, and since the space of embeddings is contractible, this is an equivalence of categories.

On the other hand, we have a comparison functor $\widetilde{\mathsf{T}}_S \to \mathsf{B}_S$ which sends an embedded stable marked tree $|T| \subset D$ to the subset of its marked points $S \subset |T| \subset D$, and forgets the rest. Then Theorem 10.1 for $n = 1$ follows immediately from the following.

Proposition 10.2. *The comparison functor $\widetilde{\mathsf{T}}_S \to \mathsf{B}_S$ induces a homotopy equivalence $|\mathsf{T}_S| \cong |\widetilde{\mathsf{T}}_S| \cong |\mathsf{B}_S|$.*

10.2 Stratified spaces and homology equivalences.

Our strategy of proving Proposition 10.2 is the same as in the study of the Gerstenhaber operad in the last lecture: we want to apply induction on the cardinality of S by forgetting points one-by-one and considering the corresponding projections of the configuration spaces.

Thus we assume given a finite set S' and an element $v \in S'$, and we denote $S = S' \setminus \{v\}$. Then forgetting v defines a projection $\mathsf{B}_{S'} \to \mathsf{B}_S$. On the other hand, unmarking v and applying stabilization defines a projection $\mathsf{T}_{S'} \to \mathsf{T}_S$. This is obviously compatible with the comparison functors, so that we have a commutative diagram

$$\begin{array}{ccc} \mathsf{T}_{S'} & \longrightarrow & \mathsf{B}_{S'} \\ \downarrow & & \downarrow \\ \mathsf{T}_S & \longrightarrow & \mathsf{B}_S. \end{array} \tag{10.1}$$

Definition 10.3. An *abelian fibration* $\mathcal{C}$ over a small category Γ is a fibration $\mathcal{C}/\Gamma$ such that all fibers $\mathcal{C}_{[a]}$, $[a] \in \Gamma$ are abelian categories, and all the transition functors $f^* : \mathcal{C}_{[b]} \to \mathcal{C}_{[a]}$, $f : [a] \to [b]$ are left-exact.

Just as in Proposition 5.3, one shows easily that the category of sections $\operatorname{Sec}(\mathcal{C})$ of an abelian fibration $\mathcal{C}/\Gamma$ is an abelian category.

Definition 10.4. A functor $\gamma : \Gamma \to \Gamma'$ between small categories is said to be a *homological equivalence* if

(i) for any abelian fibration $\mathcal{C}/\Gamma'$, the pullback functor $\gamma^* :: \mathcal{D}(\mathrm{Sec}(\mathcal{C})) \to \mathcal{D}(\mathrm{Sec}(\gamma^*\mathcal{C}))$ is a fully faithful embedding, and

(ii) the essential image $\gamma^*(\mathrm{Sec}(\mathcal{C})) \subset \mathrm{Sec}(\gamma^*\mathcal{C})$ consists of such $E \in \mathrm{Sec}(\gamma^*\mathcal{C})$ that for any map $f : [a] \to [b]$ in Γ with invertible $\gamma(f)$, the transition map $E_{[a]} \to f^*E_{[b]}$ is invertible.

Here $\gamma^*\mathcal{C} = \mathcal{C} \times_{\Gamma'} \Gamma$ is the pullback of the abelian fibration $\mathcal{C}/\Gamma'$, $\mathcal{D}(-)$ stand for the derived category, $E|_{[a]}$ is the restriction of E to the fiber $(\gamma^*\mathcal{C})_{[a]} \cong \mathcal{C}_{\gamma([a])}$, and similarly for $E_{[b]}$. For example, if $\Gamma' = \mathsf{pt}$, $\gamma : \Gamma \to \mathsf{pt}$ is the projection to the point, and $\mathcal{C} = k$-Vect, the conditions of the definition say that $\mathcal{D}_{lc}(\Gamma, k)$ is equivalent to the derived category $\mathcal{D}(k\text{-Vect})$. As we saw in Corollary 4.4, this implies that the geometric realization $|\Gamma|$ is contractible.

Exercise 10.2. *Prove that if* $\gamma : \Gamma \to \Gamma'$ *is a homological equivalence, then the induced map* $|\gamma| : |\Gamma| \to |\Gamma'|$ *is a homotopy equivalence.*

The reason we have put the additional abelian fibration $\mathcal{C}$ in Definition 10.4 is that this way, it becomes recursive: we have the following.

Lemma 10.5. *Assume given cofibrations* Γ'_1/Γ_1, Γ'_2/Γ_2, *a functor* $\gamma : \Gamma_1 \to \Gamma_2$, *and a Cartesian functor* $\gamma' : \Gamma'_1 \to \Gamma_1 \times_{\Gamma_2} \Gamma'_2 \to \Gamma'_2$. *Then if* γ *is a homological equivalence, and* γ' *restricts to a homological equivalence on all the fibers, then* γ' *itself is a homological equivalence.*

Exercise 10.3. *Prove this. Hint: first show that for any cofibration* $\pi : \Gamma' \to \Gamma$ *and any abelian fibration* $\mathcal{C}/\Gamma'$, *there exists an abelian fibration* $\pi_*\mathcal{C}$ *whose fibers are given by*

$$(\pi_*\mathcal{C})_{[a]} = \mathrm{Sec}(\mathcal{C}|_{\Gamma'_{[a]}}), \qquad [a] \in \Gamma,$$

where $\mathcal{C}|_{\Gamma'_{[a]}}$ *means the restriction to the fiber* $\Gamma'_{[a]} \subset \Gamma'$, *and that* $\mathrm{Sec}(\mathcal{C}) \cong \mathrm{Sec}(\pi_*\mathcal{C})$.

Exercise 10.4. *Assume given diagrams of categories and functors*

$$\begin{array}{ccc} \Gamma_1 & \longrightarrow & \Gamma_{12} \\ \uparrow & & \uparrow \\ \Gamma_0 & \longrightarrow & \Gamma_2 \end{array} \qquad\qquad \begin{array}{ccc} \Gamma'_1 & \longrightarrow & \Gamma'_{12} \\ \uparrow & & \uparrow \\ \Gamma'_0 & \longrightarrow & \Gamma'_2 \end{array}$$

which are cocartesian in the sense that for any category $\mathcal{C}$, *we have*

$$\mathrm{Fun}(\Gamma_{12}, \mathcal{C}) \cong \mathrm{Fun}(\Gamma_1, \mathcal{C}) \times_{\mathrm{Fun}(\Gamma_0, \mathcal{C})} \mathrm{Fun}(\Gamma_2, \mathcal{C}),$$

and similarly for Γ'. *Assume given a functor* $\gamma = \langle \gamma_0, \gamma_1, \gamma_2, \gamma_{12} \rangle$ *between them. Prove that if* γ_0, γ_1 *and* γ_2 *are homological equivalences, then so is* γ_{12}.

Unfortunately, Lemma 10.5 cannot be used to analyze (10.1) directly, since the projection functor $\mathsf{T}_{S'} \to \mathsf{T}_S$ is not a cofibration. To correct this, we have to "compactify" the categories T_S by allowing non-injective markings $S \to V(T)$ — geometrically, this corresponds to adding the diagonals $\mathsf{Diag} \subset D^S$ to the configuration space $D^S \setminus \mathsf{Diag}$.

So, first, for every finite set S we define the category $\overline{\mathsf{T}}_S$ whose objects are trees T equipped with a map $f : S \to V(T)$ such that the induced embedding $\mathsf{Im}(f) \subset V(T)$ is a stable marking, with maps given by contractions of edges.

Second, we consider the topological space D^S as a space stratified by the diagonals, and we define the category $\overline{\mathsf{B}}_S$ as the its "stratified fundamental groupoid" in the following sense.

Definition 10.6. The *stratified fundamental groupoid* of a topological space X stratified by strata $X_i \subset X$ is the category whose objects are points $x \in X$, and whose maps from $x_1 \in X_1$ to $x_2 \in X_2$ exist only when $X_2 \subset X_1$, and are given by homotopy classes of paths $f : I \to X_1$ from x_1 to x_2 such that $f(I) \cap X_2 = f(1) = p_2$, and $f(I) \cap X_3 = \emptyset$ for any proper substratum $X_3 \subset X_2$.

Explicitly, an object in $\overline{\mathsf{B}}_S$ is given by a not necessarily injective map $f : S \to D$, and maps from f_0 to f_1 are given by homotopy classes of maps $\gamma : f_0(S) \times I \to D$ such that $\gamma : f_0(S) \times \{x\} \to D$ is injective for any $x \in [0, 1[$, $\gamma : f_0 \times \{1\} \to D$ is a map onto $f_1(S) \subset D$, and the composition $\gamma \circ f_0 : S \to f_0(S) \to f_1(S)$ is equal to f_1.

Exercise 10.5. *Let $\langle X, X_i \subset X\rangle$ be a stratified topological space, and let $\pi_1(X)$ be its stratified fundamental groupoid. Prove that the category $\operatorname{Fun}(\pi_1(X)^{opp}, k)$ is equivalent to the category of constructible sheaves of k-vector spaces on X which are locally constant along the open strata. Hint: consider first the case $X = I$, with a single proper stratum $X_1 = \{1\} \subset I$.*

We leave it to the reader to check that the comparison functor (10.1) extends to a functor $\overline{\mathsf{T}}_S \to \overline{\mathsf{B}}_S$, and we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathsf{T}}_{S'} & \longrightarrow & \overline{\mathsf{B}}_{S'} \\ \downarrow & & \downarrow \\ \overline{\mathsf{T}}_S & \longrightarrow & \overline{\mathsf{B}}_S. \end{array} \tag{10.2}$$

10.3 The comparison theorem.

We can now prove the comparison theorem between $\overline{\mathsf{T}}_S$ and $\overline{\mathsf{B}}_S$.

Proposition 10.7. *The comparison functor $\overline{\mathsf{T}}_S \to \overline{\mathsf{B}}_S$ is a homological equivalence for any finite set S.*

Proof. One checks easily that the vertical projections in (10.2) are cofibrations; thus by induction, it suffices to check that the comparison functor induces a homological equivalence on all the fibers.

Fix a tree $T \in \overline{\mathsf{T}}_S$, and consider a tree $T' \in \left(\overline{\mathsf{T}}_{S'}\right)_T$. When we remove the mark $v \in S'$ from T', one of the following four things might happen:

(i) The tree remains stable, with the vertex $v \in V(T') = V(T)$ possibly becoming unmarked.

(ii) An unmarked vertex of valency 2 appears; under stabilization, it is removed, and adjacent edges are glued together to give an edge $e \in E(T)$.

(iii) An unmarked vertex of valency 1 appears; under stabilization, we remove this vertex and the adjacent edge.

(iv) An unmarked vertex of valency 1 appears; under stabilization, we remove it with its edge, and then an unmarked vertex of valency 2 appears, which also has to be removed.

In the case (i), T' is completely determined by specifying $v \in V(T)$, and in the case (ii), by specifying $e \in E(T)$. To describe the combinatorial invariants in (iii), it is convenient to embed the tree T into the disc D and draw a small disc around each vertex $v \in V(T)$. The boundary of this disc is a wheel graph $[n] \in \Lambda$ whose vertices correspond to edges adjacent to v. Edges of these graphs are called *angles* of T, and the set of all angles of T is denoted by $A(T)$. Then in the case (iii), to determine T' we need to specify the other vertex $v \in V(T)$ of the removed edge, and the (unique) angle $a \in A(T)$ which this removed edge intersects. Finally, in the case (iv), T' is determined by the new edge $e \in E(T)$ containing the removed vertex of valency 2, and the "side" of this edge at which the removed edge was attached. The set of these sides is denoted by $S(T)$ (it is of course a 2-fold cover of the set $E(T)$). We note that every side defines an angle attached to each of the two vertices of the corresponding edge.

To sum up: the fiber of the projection $\overline{\mathsf{T}}_{S'} \to \overline{\mathsf{T}}_S$ over a tree $T \in \overline{\mathsf{T}}_S$ is the set

$$F_T = V(T) \cup E(T) \cup A(T) \cup S(T),$$

with some partial order.

Exercise 10.6. *Check that F_T has the following order: an edge $e \in E(T)$ is less than either of its vertices, an angle $a \in A(T)$ is less than the vertex where it lives, and a side $s \in S(T)$ is less than the corresponding edge, and less than the two angles it defines.*

On the other hand, the fiber F_p of the projection $\overline{\mathsf{B}}_{S'} \to \overline{\mathsf{B}}_S$ over an object represented by $p : S \to D$ is the stratified fundamental groupoid of the pair $f(S) \subset D$. To finish the proof, it suffices to prove the following.

Lemma 10.8. *Assume given a possibly unstable marked tree T embedded into the disc D, $|T| \subset D$, and let $p : S \to D$ be the corresponding embedding of the set of markings $S \subset V(T)$. Then the comparison functor $F_T \to F_p$ is a homological equivalence.*

Proof. Choose a vertex $v \in T$ of valency 1, let T' be the tree obtained by removing v and the adjacent edge $e \in E(T)$, and let $p' : S' \to D$ be the embedding of its set of markings $S' \subset V(T')$. Then we have a cocartesian diagram

$$\begin{array}{ccc} F_{T'} & \longrightarrow & F_T \\ \uparrow & & \uparrow \\ F_e & \longrightarrow & F_v, \end{array}$$

where $F_e \subset F_T$ is the subset consisting of e and its two sides, and $F_v \subset F_T$ is the subset consisting of v, all its adjacent edges, all its angles, and all their sides. On the other hand, we can shrink D to a small neighborhood of $|T| \subset D$ and then decompose it into the union of a small disc D_v centered at v and a neighborhood $D_{T'}$ of $|T'| \subset D$ so that the intersection $D_v \cap D_{T'}$ is contractible with no stratification. This gives a cocartesian diagram

$$\begin{array}{ccc} F_{p'} & \longrightarrow & F_p \\ \uparrow & & \uparrow \\ \mathsf{pt} & \longrightarrow & F_v, \end{array}$$

where F_v is the fundamental groupoid of D_v if v is unmarked, and the stratified fundamental groupoid of $\{v\} \subset D_v$ if v is marked. By virtue of Exercize 10.4, we can apply induction. Thus it suffices to prove that F_e is homologically equivalent to a point, and the comparison functor $F_v \to F_v$ is a homological equivalence (both if v is marked and if it is not). We leave it as an exercize. Hint: in the marked case, show first that the partially ordered set $F_v \setminus \{v\}$ is homologically equivalent to the fundamental groupoid of a circle S^1. □

Proof of Proposition 10.2. An immediate corollary of Proposition 10.7: every abelian fibration $\mathcal{C}/\mathsf{B}_S$, resp. $\mathcal{C}/\mathsf{T}_S$ can obviously be extended to $\overline{\mathsf{B}}_S$, resp. $\overline{\mathsf{T}}_S$ by setting $\mathcal{C}_{[a]} = 0$ for any $[a] \in \overline{\mathsf{B}}_S \setminus \mathsf{B}_S$, resp. $[a] \in \overline{\mathsf{T}}_S \setminus \mathsf{T}_S$, and this does not change the category of sections; therefore the comparison functor $\mathsf{T}_S \to \mathsf{B}_S$ is also a homological equivalence, and this implies the claim by Exercize 10.2. □

To finish the proof of Theorem 10.1, it remains to consider n-planar trees for $n \geq 2$. Note that for any $[n] \in \Lambda$ and a fixed embedding $f : [n-1] \to [n]$, we have a natural projection $\pi^f : \mathsf{T}_S^{[n]} \to \mathsf{T}_S^{[n-1]}$ obtained by the removing the external vertex not contained in the image of f and applying stabilization.

Exercise 10.7. *Check that π^f is a cofibration whose fiber E_T over a tree $T \in \mathsf{T}_S^{[n-1]}$ is the partially ordered set of cells of a certain cell decomposition of the open interval $]0,1[$, with the order by given adjacency (the decomposition may depend on T). Deduce that π^f is a homological equivalence.*

This Exercize together with Exercize 10.2 finish the proof of Theorem 10.1.

10.4 Regular partially ordered sets.

By virtue of Theorem 10.1, instead of studying the chain complex $C_{\bullet}(D^S \setminus \mathsf{Diag})$ directly, we may study complexes which compute the homology of the partially ordered set $\mathcal{T}_S$ (considered as a small category). This turns out to be easy, since the partially ordered set $\mathcal{T}_S$ is well-behaved.

Assume given a partially ordered set P. For any $p \in P$, denote by $\delta_p \in \operatorname{Fun}(P^{opp}, k)$ the functor given by $\delta_p(p) = k$, $\delta_p(p') = 0$ if $p \neq p'$.

Definition 10.9. A finite partially ordered set P is called *regular* if for any $p \in P$, we have

$$H_i(P^{opp}, \delta_p) \cong \begin{cases} k, & i = n, \\ 0, & \text{otherwise} \end{cases} \tag{10.3}$$

where n is some integer $n \geq 0$ called the *index* of p and denoted $\mathsf{ind}(p)$.

Exercise 10.8. *Prove that the product $P_1 \times P_2$ of two regular partially ordered sets is regular.*

Exercise 10.9. *Prove that P is regular if and only if for any $p \in P$, so the set $U_p = \{p' \in P | p' \leq q\}$.*

Proposition 10.10. *The partially ordered set $\mathsf{T}_S^{[n]}$ is regular for any finite set S and any $[n] \in \Lambda$, and the index of a tree $T \in \mathsf{T}_S^{[n]}$ is equal to $\mathsf{ind}(T) = n - 2 - v(T)$, where $v(T)$ is the cardinality of $V(T)$.*

Proof. For any tree $T \in \mathsf{T}_S^{[n]}$, the partially ordered set U_T of Exercize 10.9 is isomorphic to

$$U_T \cong \prod_{v \in V(T)} \mathsf{T}_{s(v)}^{[n_v]}, \tag{10.4}$$

where $[n_v]$ is the set of edges adjacent to the vertex v with its given cyclic order, and $s(v)$ is pt if v is marked and $\emptyset$ if v is unmarked. Thus by Exercize 10.8, it suffices to consider the cases $S = \mathsf{pt}$ and $S = \emptyset$. In either of these cases, we use induction on n. The sets $\mathsf{T}_\emptyset^{[1]}$ and $\mathsf{T}_\emptyset^{[2]}$ are empty; the sets $\mathsf{T}_{\mathsf{pt}}^{[1]}$ and $\mathsf{T}_\emptyset^{[3]}$ both consist of one point, thus giving the induction base. For the induction step, choose an embedding $[n-1] \to [n]$, and consider the corresponding projection $\mathsf{T}_S^{[n]} \to \mathsf{T}_S^{[n-1]}$. This is a cofibration. Its fibers E_T have been described in Exercize 10.7, and it is easy to check that they are regular. Moreover, for any $T \leq T' \in \mathsf{T}_S^{[n-1]}$, the corresponding transition map $E_T \to E_{T'}$ is obviously a homological equivalence. To finish the proof of the inductive step and the Proposition, it suffices to apply the following to every $\delta_T \in \operatorname{Fun}(\mathsf{T}_S^{[n]opp}, k)$.

Lemma 10.11. *Assume given a fibration $\gamma : \Gamma' \to \Gamma$ of small categories, and assume that the transistion functor f^* is a homological equivalence for any map $f : [a] \to [b]$ in Γ. Then for any $E \in \operatorname{Fun}(\Gamma', k)$ and any $[a] \in \Gamma$, there exists an isomorphism*

$$(L^{\bullet}\gamma_! E)([a]) \cong H_{\bullet}(\Gamma'_{[a]}, E_{[a]}), \tag{10.5}$$

where $E_{[a]} \in \operatorname{Fun}(\Gamma'_{[a]}, k)$ is the restriction of E to the fiber $\Gamma'_{[a]} \subset \Gamma'$.

Proof. Let $i : \mathsf{pt} \to \Gamma$ be the embedding of the object $[a]$, and let $i' : \Gamma'_{[a]} \to \Gamma'$ be the embedding of the fiber. Then we have the adjunction map

$$i_! \circ \gamma_! \circ i'^* \cong \gamma_! \circ i'_! \circ i'^* \to \gamma_!,$$

which by adjunction induces a base change map $\gamma_! \circ i'^* \to i^* \circ \gamma_!$. Taking derived functors, we obtain a map (10.5) functorially for any E. To prove that it is an isomorphism, it suffices to consider the case of a representable E, $E = k_{[b']}$ for some $[b'] \in \Gamma'$. Then the left-hand side of (10.5) is canonically isomorphic to $k[\Gamma([b],[a])]$, where $[b] = \gamma([b']) \in \Gamma$. On the other hand, since γ is a fibration, we have a canonical identification

$$k_{[b']}|_{\Gamma'_{[a]}} \cong \bigoplus_{f \in \Gamma([b],[a])} (f^*)_! k_{[b']}|_{\Gamma'_{[b]}}.$$

But since f^* is a homological equivalence for any $f \in \Gamma([b],[a])$, we have

$$H_{\bullet}(\Gamma'_{[a]}, (f^*)_! k_{[b']}|_{\Gamma'_{[b]}}) \cong H_{\bullet}(\Gamma'_{[b]}, k_{[b']}|_{\Gamma'_{[b]}}) \cong k,$$

which finishes the proof. □

Exercise 10.10. *Prove that the homology $H_{\bullet}(P,k) = H_{\bullet}(P^{opp},k)$ of a finite partially ordered set P can be computed by a complex $C_{\bullet}(P,k)$ with terms $C_i(P,k) = \bigoplus_{\mathsf{ind}(p)=i} k$. Hint: take a maximal element $p \in P$, let $P' = P \setminus \{p\}$, and consider the short exact sequence*

$$0 \longrightarrow j_!^{opp} k^{P'} \longrightarrow k^P \longrightarrow \delta_p \longrightarrow 0,$$

where $k^P \in \operatorname{Fun}(P^{opp},k)$, $k^{P'} \in \operatorname{Fun}(P'^{opp},k)$ are the constant functors, and $j : P' \to P$ is the embedding.

10.5 The brace operad.

Now consider the partially ordered set T_S. It is regular, so its cohomology can be computed by a complex $C_{\bullet}(\mathsf{T}_S, k)$, which we denote by $C_{\bullet}(S)$ to simplify notation. Unfortunately, there is some abguity in the differentials of the complexes $C_{\bullet}(S)$ (for a discussion, see the Kontsevich-Soibelman paper arxiv:math/0001151). As it turns out, with the appropriate choice of the differentials, the complexes $C_{\bullet}(S)$ form a DG operad, called the *brace operad*, and this operad acts naturally on the Hochschild cohomology complex of any associative algebra A.

Namely, assume given an associative unital algebra A, and assume given an m-cochain $f \in \operatorname{Hom}(A^{\otimes m}, A)$ and l other cochains $g_j \in \operatorname{Hom}(A^{\otimes n_j}, A)$, $1 \leq j \leq l$ of degrees $n_1, \dots, n_l$. Then the *brace* $f\{g_1, \dots, g_l\}$ is the cochain of degree $M = m + n_1 + \dots + n_l - l$ given by

$$f\{g_1,\dots,g_l\}(a_1,\dots,a_M) = \sum_I (-1)^{\varepsilon}_I f(a_1,\dots,a_{i_1-1}, g_1(a_{i_1},\dots,a_{i_1+n_1-1}), a_{i_1+n_1},\dots,$$
$$a_{M-m-n+1}, g_l(a_{M-m-n_l+1},\dots,a_{M-m+i_l}), a_{M-m+i_l+1},\dots,a_M),$$

where the sum is taken over all the multiindices $1 \leq i_1 < \cdots < i_l \leq m$, and ε_I is given by

$$\varepsilon_I = \sum_{1 \leq j \leq l} n_j(i_j - 1).$$

If $m < l$, then the set of multiindices is empty, and the brace is set to be 0.

In other words, the brace is obtained by substituting $g_1, \dots, g_l$ into f in all possible ways, and taking the alternating sum.

Then every tree $T \in \mathsf{T}_S$ defines an S-linear operation α_T on the Hochschild cohomology complex of A by the following inductive rule.

(i) If T is the tree with exactly one vertex of valency ≥ 1, and this vertex is marked, then

$$\alpha_T(f_1, \dots, f_n) = f_1(f_2, \dots, f_n),$$

where $f_1, \dots, f_n$ are cochains numbered by elements in $S = V(T)$, and f_1 corresponds to the marked vertex.

(ii) If in the situation above the vertex is unmarked,

$$\alpha_T(f_1, \dots, f_n) = f_1 \cdot f_2 \cdot \dots \cdot f_n.$$

(iii) In the general case, split T into two trees T_1, T_2 by cutting an edge $e \in E(T)$, marking one of the resulting new vertices, and declaring the other one the new root vertex, and let

$$\alpha_T(f_1, \dots, f_n) = \alpha_{T_1}(\alpha_{T_2}(f_1, \dots, f_l), f_{l+1}, \dots, f_n),$$

where T_1 is the subtree which contains the original root, and α_{T_2} corresponds to the new marked vertex of T_1.

It is not too difficult to check that the brace operation is associative in the appropriate sense, so that the operation in (iii) does not depend on the choice of the edge $e \in E(T)$. To make (ii) similar to (i), we note that since A is associative, we have a preferred cochain $\mu \in \operatorname{Hom}(A^{\otimes n}, A)$ for any $n \geq 0$ given by the product, and

$$\mu\{f_1, \dots, f_n\} = f_1 \cdot f_2 \cdot \dots \cdot f_n.$$

Moreover, it is clear that the collection of the operations α_T is closed under substitution — more precisely, α_T span a suboperad in the endomorphism operad of the Hochschild cohomology complex of the algebra A. This defines an operad structure on the graded vector spaces $C_\bullet(S) = k[\mathsf{T}_S]$.

Theorem 10.12. *With the appropriate choice of the differentials in the complexes $C_\bullet(S)$, the operad structure on $C_\bullet(S)$ defined by the brace operation and the action of this operad on the Hochschild cohomology complexes is compatible with the differential, so that we have a DG operad, and for any associative unital algebra A, its Hochschild cohomology complex is a DG algebra over the DG operad $C_\bullet(S)$.*

I do not give the exact differentials, since I will not prove this result anyway (see the quoted paper of Kontsevich-Soibelman, and also arxiv:math/9910126 of McClure and Smith, where a closely related result is proved). Rather, to finish the lecture, I want to discuss what the result means, and what would a conceptually clear proof look like.

10.6 Discussion.

First of all, we note that Theorem 10.12 *does not prove the Deligne Conjecture.*

Indeed, while we have constructed quasiisomorphisms between the chain complexes of configuration spaces $D^S \setminus \mathsf{Diag}$ and the complexes $C_{\bullet}(S)$ which act on Hochschild cohomology, we did not prove that they are compatible with the operadic structure. A natural way to do this would be to extend Theorem 10.1 to a comparison theorem between operads; this would also take care of all the signs. But it is completely impossible to do this: while the groupoids $\overline{\mathsf{B}}_S$ do form an operad in an approppriate 2-categorical sense, the partially ordered sets T_S *do not.*

Namely, the operadic structure would allow one to replace a marked vertex v in a tree T with another tree T'. But this is only possible if v has valency 1 — otherwise it is not clear what to do with the extra edges coming into v. We can only replace v with an n-planar tree, where n is the valency of v.

This is why there is a sum in our definition of the brace operation — essentially this is an averaging over all possible ways to take care of the extra edges; and this becomes possible only after we pass to the chain complex. What happens is that we consider the canonical quasiisomorphism $C_{\bullet}(\mathsf{T}_S^{[n]}, k) \to C_{\bullet}(\mathsf{T}_S, k) = C_{\bullet}(S)$ obtained by projection, and forcibly invert it.

Considering all the n-planar trees together does not help much: they do not an operad either, because they can be sustituted one into the other only if the valencies match.

To me, the best way to prove Deligne Conjecture would be not to force the pieces into submission, but rather, to formalize the structure that the partially ordered sets T_S and $\mathsf{T}_S^{[n]}$ do possess; this amounts to generalizing the notion of an operad by replacing the category Γ of finite sets with something else — for example, an appropriately defined category of trees, with (10.4) playing the role of the product decomposition (9.1). However, as far as I know, this has not been done. M. Batanin has realized a similar plan, but a different replacement for Γ — he introduces a notion of a "non-Σ 2-operad" which is encoded by the "category of 2-ordinals"; this category is not directly related to trees, but rather, gives another model of the configuration spaces of points on a disc. Recently D. Tamarkin has shown in `arXiv:math/0606553` how to prove the Deligne Conjecture in this language. The other existing approaches to Deligne Conjecture (for example in the papers by Voronov, McClure-Smith, Kontsevich-Soibelman, in fact also in the original paper by Getzler-Jones) are more indirect. What these authors do is the following: they construct a different and much larger DG operad which maps both onto the brace operad and onto the chain complex operad of small discs, and show that both maps are quasiisomorphisms. The construction usually involve doing some very intricate cellular subdivisions of the configuration spaces and a lot of combinatorics. My feeling is that the "final solution" of the Deligne Conjecture is not yet known.

Finally, some bibliographical notes. I have borrowed the formula for the brace operation from the paper `arXiv:math/9910126` of McClure and Smith, together with the signs. The brace operad also appeared there, or rather, a version of it slightly different from the one presented here (I note that the authors use "formulas" instead of planar trees, but these objects are in fact identical). Exactly the same complex as above appears in `arXiv:math/0001151` by Konstevich and Soibelman, and also in other places in the literature. So does the partially ordered set of planar trees. But our proof of the comparison theorem seems to be new. The usual approach is to take a certain cellular subdivision of the configuration space and quote the general theorem which says that if the subdivision is nice enough, then the geometric realization of the partially ordered set of cells in a space is homotopy equivalent to the space itself. An exact subdivision which corresponds to trees also appears in Kontsevich-Soibelman, but without proof. The other references that I know use different subdivisions which give different partially ordered sets, and then use combinatorics of varying degrees of difficulty to identify the result with trees.