Lecture 11.

Deformations of DG algebras and A_∞ algebras. Deformations in the Poisson and the Gerstenhaber case. Formality and deformations. Tamarkin's Theorem.

11.1 The language of A_∞-maps.

In this last lecture, I will try to sketch the proof of D. Tamarkin's theorem which I have already formulated as Theorem 9.8. I start with a discussion of associative DG algebras.

Assume given an associative unital DG algebra $A^\bullet$ over a field k. To define Hochschild cohomology $HH^\bullet(A^\bullet)$, one can naively write down the Hochschild cohomology complex, just as in the case of usual associative algebras, and obtain a bicomplex; Hochschild cohomology $HH^\bullet(A^\bullet)$ is the cohomology of the total complex of this bicomplex. We note that since the complex $\operatorname{Hom}(A^{\bullet\otimes n}, A^\bullet)$ for every $n \geq 1$ has terms both of positive and of negative degrees, there is an ambiguity in taking the total complex of a bicomplex: one can take either the sum, or the product of the diagonal terms. For the definition of Hochschild cohomology, one needs to take the product: the degree-n term of the resulting total complex is given by

$$\prod_{i \geq 0} \operatorname{Hom}^{n-i}(A^{\bullet\otimes i}, A^\bullet),$$

where $\operatorname{Hom}^n(-)$ stands for the term of degree n in the complex of Hom's. More invariantly, one can consider the category of DG modules over $A^\bullet$, and formally invert quasiisomorphisms. The result is a triangulated category $\mathcal{D}(A^\bullet\textsf{-mod})$ known as *derived category of DG-modules over* $A^\bullet$. Analogously, one defines the triangulated category $\mathcal{D}(A^\bullet\textsf{-bimod})$ of DG $A^\bullet$-bimodules. Then we have

$$HH^\bullet(A^\bullet) = \operatorname{RHom}^\bullet_{\mathcal{D}(A^\bullet\textsf{-bimod})}(A^\bullet, A^\bullet),$$

where $A^\bullet$ in the right-hands side is the diagonal bimodule.

Recall now that for ordinary associative algebras, Hochschild cohomology could be also used to describe deformations. What is the situation with DG algebras? It turns out that a similar theory exists, but it describes deformations of DG algebras "up to a quasiisomorphism", as in Lecture 8.

To explain how this works, we first describe briefly a convenient technical tool — the notion of an *A_∞-map*. For a very good overview of this subject with detailed references, I refer the reader to a paper `arXiv:math/0510508` by B. Keller.

Assume given an associative DG algebra $A^\bullet$, with or without unit, and consider the free coalgebra $T_\bullet(A^\bullet)$ generated by $A^\bullet$. Then by Lemma 8.2, $T_\bullet(A^\bullet)$ has a natural structure of a bicomplex, with one differential induced by the differential in $A^\bullet$, and the other induced by multiplication. Its total complex is then a DG coalgebra with counit. For technical reasons, we need to remove the counit, and we denote the corresponding coalgebra by $\overline{T}_\bullet(A^\bullet)$. Explicitly,

$$\overline{T}_\bullet(A^\bullet) = \bigoplus_{i \geq 1} A^{\bullet\otimes i}[i] \tag{11.1}$$

as a graded vector space.

Exercise 11.1. *Prove that if $A^\bullet$ is a DG algebra with unit, then the complex $\overline{T}_\bullet(A^\bullet)$ is acyclic. Hint: show that $\overline{T}_\bullet(A^\bullet)$ is exactly the acyclic bar complex $C'_\bullet(A^\bullet)$ of Lecture 1 (Lemma* 1.3*).*

Exercise 11.2. *Assume that $A^\bullet$ itself is a free associative DG algebra without unit generated by a complex $V^\bullet$, $A^\bullet = \overline{T}^\bullet(V^\bullet) = \bigoplus_{i \geq 1} V^{\bullet\otimes i}[i]$. Prove that the natural map*

$$V^\bullet[1] \to \overline{T}^\bullet(V^\bullet)[1] = A^\bullet[1] \to \overline{T}_\bullet(A^\bullet)$$

is a quasiisomorphism. Hint: using the previous exercize, first show that the complex $T_\bullet(A^\bullet)$ computes $\operatorname{Tor}^{A^\bullet}_\bullet(k,k)$*, where k is the trivial left, resp. right $A^\bullet$-module.*

Definition 11.1. An A_∞*-map* between associative DG algebras $A_1^\bullet$, $A_2^\bullet$ is a DG coalgebra map $\varphi : \overline{T}_\bullet(A_1^\bullet) \to \overline{T}_\bullet(A_2^\bullet)$.

Since the coalgebra $\overline{T}_\bullet(A_2^\bullet)$ is free, an A_∞-map φ is completely defined by the induced map $\varphi : \overline{T}_\bullet(A_1^\bullet) \to A_2^\bullet$, and this can be decomposed as

$$\varphi = \varphi_0 + \varphi_1 + \dots + \varphi_i + \dots$$

according to (11.1). Here φ_0 is simply a map of complexes $\varphi_0 : A_1^\bullet \to A_2^\bullet$. If all the components φ_i, $i \geq 1$ are equal to zero, then $\varphi_0 : A_1^\bullet \to A_2^\bullet$ is just a map which commutes with multiplication — that is, a DG algebra map in the usual sense. In general, however, φ_0 commutes with multiplication only up to a homotopy, and this homotopy is $\varphi_1 : A_1^{\bullet\otimes 2} \to A_2^\bullet[-1]$. This in turn commutes with multiplication in an appropriate sense, but only up to a homotopy given by φ_2, and so on.

Definition 11.2. An A_∞-map φ is *a quasiisomorphism* if so is its component φ_0.

Of course, a quasiisomorphism $\varphi : A_1^\bullet \to A_2^\bullet$ between two DG algebras is also an A_∞-quasiisomorphism. However, while it is often not invertible in any sense as a DG algebra map, the resulting A_∞-map admits an inverse, in the following sense.

Lemma 11.3. *Assume given an A_∞-quasiisomorphism φ from a DG algebra $A_1^\bullet$ to a DG algebra $A_2^\bullet$. Then there exists an A_∞-quasiisomorphism φ^{-1} from $A_2^\bullet$ to $A_1^\bullet$ such that both $\varphi \circ \varphi^{-1}$ and $\varphi^{-1} \circ \varphi$ induce identity maps on cohomology.*

Proof. Since φ is a quasiisomorphism, there exists a map $\varphi_0^{-1} : A_2^\bullet \to A_1^\bullet$ of the underlying complexes which induces an iverse map on cohomology. We extend it to an A_∞-map by induction. Namely, for any DG algebra $A^\bullet$, denote by $\overline{T}_{<i}(A^\bullet) \subset \overline{T}_\bullet(A^\bullet)$ the subcoalgebra consisting of components $A^{\bullet\otimes j}[j]$ with $j \leq i$, and assume given a DG coalgebra map $\varphi_{<i}^{-1} : \overline{T}_{<i}(A_2^\bullet) \to \overline{T}_{<i}(A_1^\bullet)$ which induces a map on cohomology inverse to that induced by φ. Extend $\varphi_{<i}^{-1}$ to a DG coalgebra map $\varphi_{<i}^{-1} : \overline{T}_{<i+1}(A_2^\bullet) \to \overline{T}_{<i+1}(A_1^\bullet)$. Then this extended map $\varphi_{<i}^{-1}$ no longer necessarily commutes with the differential. However, the commutator is a certain map

$$e : A_2^{\bullet\otimes(i+1)} \to A_1^\bullet[-i+1],$$

and using the inductive assumption, one easily checks that e induces a zero map on cohomology. Therefore it is chain-homotopic to 0 by a certain chain homotopy $\varphi_i : A_2^{\bullet\otimes(i+1)} \to A_1^\bullet[-i]$. We now take $\varphi_{<i+1} = \varphi_{<i} + \varphi_i$. □

Proposition 11.4. *Two DG algebras $A_1^\bullet$, $A_2^\bullet$ are quasiisomorphic if and only if there exists an A_∞-quasiisomorphism $\varphi : \overline{T}_\bullet(A_1^\bullet) \to \overline{T}_\bullet(A^2_\bullet)$.*

Proof. Assume that such a φ exists. Then Lemma 8.2 has an obvious dual statement for coalgebras, so that for any DG coalgebra $B^\bullet$, we have a DG algebra $\overline{T}^\bullet(B^\bullet)$ which is free as an algebra. Applying this to DG coalgebras $\overline{T}_{<i}(A^\bullet)$, $i \geq 1$ corresponding to a DG algebra $A^\bullet$, we obtain a DG algebra

$$\widetilde{T}(A^\bullet) = \lim_{\to i} \overline{T}^\bullet(\overline{T}_{<i}(A^\bullet)).$$

Then φ obviously induces a quasiisomorphism $\widetilde{T}(A_1^\bullet) \to \widetilde{T}(A_2^\bullet)$, so that it suffices to prove that $\widetilde{T}(A^\bullet)$ is quasiisomorphic to $A^\bullet$ for any $A^\bullet$. To construct a DG algebra map $\tau : \widetilde{T}(A^\bullet) \to A^\bullet$, we use induction on i and construct a compatible system of DG algebra maps

$$\tau_i : \overline{T}^\bullet(\overline{T}_{<i}(A^\bullet)) \to A^\bullet.$$

Since the left-hand side is a free algebra, at each stage a map is competely defined by its restriction to the generator $\overline{T}_{<i}(A^\bullet)$. When $i = 1$, we have $\overline{T}_{<i}(A^\bullet) = A^\bullet$; as τ_1, we take the map which is identical on generator. Once the map τ_i is constructed to some i, we first extend it to $\overline{T}_{<i+1}(A^\bullet)$ as a linear map commuting with the differential, in any way we like, and then extend further to a DG algebra map $\tau_{i+1} : \overline{T}^\bullet(\overline{T}_{<i+1}(A^\bullet)) \to A^\bullet$ by multiplicativity. Passing to the limit, we obtain a DG algebra map $\tau : \widetilde{T}(A^\bullet) \to A^\bullet$.

To show that τ is a quasiisomorphism, we consider the increasing filtration $F_\bullet \widetilde{T}(A^\bullet)$ induced by the filtration $F_i \overline{T}_\bullet(A^\bullet) = \overline{T}_{<i}(A^\bullet)$ on the generating graded vector space $\overline{T}_\bullet(A^\bullet)$. It is easy to check that this filtration is compatible with the differential, and it suffices to prove that the induced map

$$\mathsf{gr}^\bullet_F \widetilde{T}(A^\bullet) \cong \overline{T}^\bullet(\mathsf{gr}^\bullet_F \tau_\bullet(A^\bullet)) \to A^\bullet$$

is a quasiisomorphism. But the left-hand side is the DG algebra $\widetilde{T}(\overline{A}^\bullet)$, where $\overline{A}^\bullet$ is $A^\bullet$ with the trivial multiplication. Thus we may assume from the very beginning that the multiplication in $A^\bullet$ is trivial. In this case, the claim is an obvious dualization of Exercize 11.2.

Conversely, assume that $A^\bullet_1$ and $A^\bullet_2$ are quasiisomorphic, that is, there exists a chain $A^\bullet_1 \leftarrow A^\bullet_{1,1} \to A^\bullet_{1,2} \leftarrow \cdots \to A^\bullet_{1,n} = A^\bullet_2$ of DG algebras and quasiisomorphisms between them. Then by induction, we may assume that the chain is of length 2, so that we either have a DG quasiisomorphism $\eta : A^\bullet_1 \to A^\bullet_2$, or $\eta : A^\bullet_1 \to A^\bullet_2$. In the first case, φ is induced by η, and in the second case, we take $\varphi = \eta^{-1}$ provided by Lemma 11.3. □

This Proposition considerable simplifies controlling quasiisomorphism classes of DG algebras. In particular, it allows to describe deformations.

Definition 11.5. Assume given a commutative Artin local k-algebra S with maximal ideal $\mathfrak{m}$, $S/\mathfrak{m} \cong k$. An *S-deformation* $\widetilde{A}^\bullet$ of an associative DG algebra $A^\bullet$ is a DG algebra $\widetilde{A}^\bullet$ which is flat over S and equipped with an isomorphism $\widetilde{A}^\bullet \otimes_S k \cong A^\bullet$. Two such deformations $\widetilde{A}^\bullet_1$, $\widetilde{A}^\bullet_2$ are *equivalent* if there exists an S-linear A_∞-quasiisomorphism $\varphi : \overline{T}_\bullet(\widetilde{A}^\bullet_1) \to \overline{T}_\bullet(\widetilde{A}^\bullet_2)$.

Definition 11.6. The *reduced Hochschild cohomology complex* $\overline{DT}^\bullet(A^\bullet)$ of a DG algebra $A^\bullet$ is the DG Lie algebra of derivations of the DG coalgebra without unit $\overline{T}_\bullet(A^\bullet)$. *Reduced Hochschild cohomology groups* $\overline{HH}^\bullet(A^\bullet)$ are the cohomology groups of the complex $\overline{DT}^\bullet(A^\bullet)$.

Exercise 11.3. *Assume given an associative DG algebra $A^\bullet$. Prove that for any S as in Definition* 11.5*, the set of equivalence classes of S-deformations of $A^\bullet$ is in natural one-to-one correspodence with the set of equivalence classes of $\mathfrak{m}$-valued solutions of the Maurer-Cartan equation* (8.8) *in the reduced Hochschild cohomology complex $\overline{DT}^\bullet(A^\bullet)$. Hint: repeat literally the corresponding statement for associative algebras presented in Lecture 8.*

The reason we have to use DG coalgebras without unit in the definition of an A_∞-map is clear from Lemma 11.3 — otherwise, an A_∞-map φ would also have a component φ_{-1}, and the recursive procedure would fail. Because of this, the relevant deformation theory is controlled by the reduced Hochschild cohomology $\overline{DT}^\bullet(A^\bullet)$, not by the full Hochschild cohomology complex $DT^\bullet(A^\bullet)$. The difference between them is the constant term: we have a natural exact triangle

$$\overline{DT}^\bullet(A) \longrightarrow DT^\bullet(A) \longrightarrow A^\bullet \longrightarrow .$$

We note that strictly speaking, we had to consider the reduced Hochschild cohomology even in the deformation theory of the usual associative algebras. However, there it made no difference: if $A^\bullet = A$ is concentrated in degree 0, we have $HH^i(A) = \overline{HH}^i(A)$ for any $i \geq 2$, and the spaces of the Maurer-Cartan solutions are also isomorphic. For a DG algebra $A^\bullet$ which has non-trivial terms in positive degrees, they might be different.

In the interests of full disclosure, let me mention that in some situations, one can also consider A_∞-maps which have non-trivial (-1)-component; these correspond, roughly speaking, to functors between categories of DG modules which are not induced by a map of DG algebra. One can also develop a deformation theory which is controlled by the full Hochschild cohomology complex $DT^\bullet(A^\bullet)$; this is "the deformation theory of the category of DG modules", in some appropriate sense. Deformations of the category of DG modules which do not come from deformation of a DG algebra do exist, and they are sometimes known as "deformations in the gerby direction". However, this lies outside of the scope of the present course.

11.2 Poisson cohomology.

What we really need to study for Theorem 9.8 is Gerstenhaber algebras, not associative ones; thus we need to extend the above formalism to the Gerstenhaber case. For simplicity, we start with the Poisson case. The reference here is, for instance, the Appendix to my joint paper with V. Ginzburg `arXiv:math/0212279`.

Assume given a vector space V. The free Poisson coalgebra $P_\bullet(V)$ generated by V is the associated graded quotient of the free associative coalgebra $T_\bullet(V)$ with respect to the Poincaré-Birkhoff-Witt filtration. It turns out that an analog of Lemma 8.2 holds in the Poisson situation, too.

Lemma 11.7. *Poisson algebra structures on V are in one-to-one correspondence with coderivations $\delta : P_\bullet(V) \to P_{\bullet-1}(V)$ of degree 1 such that $\{\delta, \delta\} = 0$.*

Sketch of a proof. By definition, we have $P_2(V) = \mathsf{gr}_{PBW} V^{\otimes 2} = S^2(V) \oplus \Lambda^2(V)$, the sum of the symmetric and the exterior square of the vector space V. Thus a coderivation δ consists of two components, $\delta_0 : S^2(V) \to V$ and $\delta_1 : \Lambda^2(V) \to V$. The component δ_0 defines the multiplication, and δ_1 defines the Poisson bracket. The commutator $\{\delta, \delta\}$ has three components, $\{\delta_0, \delta_0\}$, $\{\delta_1, \delta_1\}$ and $2\{\delta_1, \delta_0\}$; their vanishing means, respectively, that the multiplication is associative, the bracket satisfies the Jacobi identity, and that the bracket satisfies the Leibnitz rule with respect to the multiplication. The proof is a direct computation which I leave as an exercize (or see the quoted paper `arXiv:math/0212279`). □

Thus given a Poisson algebra A, we have a canonical differential on the free Poisson coalgebra $P_\bullet(A)$, and we can consider the DG Lie algebra $DP^\bullet(A)$ of all coderivations of $P_\bullet(A)$.

Definition 11.8. *Poisson cohomology $HP^\bullet(A)$* of the Poisson algebra A is the cohomology of the complex $DP^\bullet(A)$.

As in Lecture 8, we can also consider the DG Lie algebra $DT^\bullet(A)$ of coderivations of the tensor coalgebra $T_\bullet(A)$, and this is nothing but the Hochschild cohomology complex of the algebra A. The PBW filtration on $T_\bullet(A)$ induces a filtration on $DT^\bullet(A)$, and we have $\mathsf{gr}^\bullet_{PBW} DT^\bullet(A) \cong DP^\bullet(A)$. The component $DL^\bullet(A) = \mathsf{gr}^0_{PBW} DT^\bullet(A)$ is particularly important; it depends only on the multiplcation in A, and it coincides with the DG Lie algebra of coderivation of the free Lie coalgebra $L^\bullet(A)$ generated by A. This is known as the *tangent complex* of the commutative algebra A, and it computes the so-called *Harrison cohomology* of A. We note that the differential in $DL^\bullet(A)$ is A-linear, so that it is a DG Lie algebra of A-modules (in fact, free A-modules). As such, it is quasiisomorphic to the complex

$$\mathrm{RHom}^\bullet_A(\Omega_\bullet(A), A),$$

where $\Omega_\bullet(A)$ is the *cotangent complex* of A first constructed by L. Illusie. The whole Lie algebra $DP^\bullet(A)$ also has the structure of an A-module, and coincides with the total complex of the bicomplex

$$\mathsf{gr}^{\bullet}_{PBW} DT^{\bullet}(A) \cong \Lambda^{\bullet}_A DL^{\bullet}(A).$$

One differential in this bicomplex is induced by the differential in $DL^{\bullet}(A)$, thus by multiplication in A — explicitly, the multiplication gives a class $\mu \in DL^1(A)$, and the differential is given by $\alpha \mapsto \{\mu, \alpha\}$. The other differential in the bicomplex comes from the Poisson bracket in A — the bracket gives a class

$$\Theta \in \operatorname{Hom}(\Lambda^2 A, A) \subset \Lambda^2_A(DL^0(A)), \tag{11.2}$$

and the differential is given by $\alpha \mapsto \{\Theta, \alpha\}$.

In general, it is very diffucult to compute $DP^{\bullet}(A)$ and the Harrison complex $DL^{\bullet}(A)$. However, the situation becomes much simpler when the algebra A is smooth — that is, in the situaton of the Hochschild-Kostant-Rosenberg Theorem. In this case, the cotangent complex $\Omega_{\bullet}(A)$ reduces to the module $\Omega(A)$ of Kähler differentials of A/k, and this module is flat. Therefore $DL^{\bullet}(A)$ has non-trivial cohomology only in degree 1, and this cohomology is canonically identified with the module $\mathcal{T}(A)$ of derivations of the algebra A (that is, vector fields on $X = \operatorname{Spec} A$). The higher quotients $\mathsf{gr}^{\bullet}_{PBW} DT^{\bullet}(A)$ are then isomorphic to modules of polyvector fields on X, so that the PBW filtration is in fact split — $\mathsf{gr}^{\bullet}_{PBW} DT^{\bullet}(A)$ is quasiisomorphic to the same space of polyvector fields $H^0(X, \Lambda^{\bullet}\mathcal{T}_X)$ as the full Hochschild cohomology complex $DT^{\bullet}(A)$. Under this identification, the class Θ from (11.2) corresponds to the Poisson bivector $\Theta \in H^0(X, \Lambda^2\mathcal{T}_X)$. To sum up:

Proposition 11.9. *Assume given a smooth Poisson algebra A of finite type over a characteristic-0 field k. Then the Poisson cohomology complex $DP^{\bullet}(A)$ is quasiisomorphic to the complex with terms*

$$H^0(X, \Lambda^{\bullet}\mathcal{T}_X)$$

and with differential given by $a \mapsto [\Theta, a]$, where $\Theta \in H^0(X, \Lambda^2\mathcal{T}_X)$ is the Poisson bivector. □

I will not prove this Proposition. Let me just mention that it is rather easy to reduce the statement to the case when $A = S^{\bullet}(V)$ is the symmetric algebra generated by a k-vector space V — in other words, a polynomial algebra — and then the crucial fact is the quasiisomorphism $L^{\bullet}(\overline{S}_{\bullet}(V)) \cong V$, analogous to the quasiisomorphism of Exercize 11.2 (here $\overline{S}_{\bullet}(-)$ means the free commutative coalgebra without unit).

One way to establish this quasiisomorphism uses a more careful analysis of the Hochschild-Kostant-Rosenberg map of Lecture 2, which shows how it interacts with the symmetric group actions on the terms $A^{\otimes n}$ of the Hochschild complex; the reader can find such a proof, for instance, in Loday's book.

Another and slightly more conceptual proof uses the notion of "Koszul duality of operads" introduced in Ginzburg-Kapranov `arXiv:0709.1228`. One of the statements there is that the Lie and the commutative operad are "Koszul dual", and this includes, as a part of the package, canonical quasiisomorphisms $L^{\bullet}(\overline{S}^{\bullet}(V)) \cong V$ and $\overline{S}^{\bullet}(L_{\bullet}(V)) \cong V$. The second quasiisomorphism is semi-obvious, since the left-hand side $\overline{S}^{\bullet}(L_{\bullet}(V))$, with the degree-0 term $S^0(L_{\bullet}(V))$ added, is nothing but the standard Chevalley complex which computes Lie algebra homology $H_{\bullet}(L^{\bullet}(V), k)$. Then the first quasiisomorphism, which we actually need, can be deduced by the general formalism of Koszul duality. I refer the reader to `arXiv:0709.1228` for further details.

Assuming Proposition 11.9, we see that for a smooth algebra A — in particular, for a polynomial algebra $S^{\bullet}(V)$ — the Poisson cohomology can be computed by the very explicit complex whose terms are polyvector fields. This complex was first discovered by J.-L. Brylinski in the early 80es, so that it is sometimes called the *Brylinski complex*. But when the Poisson bivector Θ is non-degenerate, so that the smooth Poisson variety $X = \operatorname{Spec} A$ is actually symplectic, the Poisson cohomology becomes even simpler.

Exercise 11.4. *Prove that for any smooth Poisson variety X, the map $\Omega^1_X \to \mathcal{T}_X$ given by contraction with the Poisson bivector Θ extends to a multiplicative map*

$$\Omega^\bullet_X \to \Lambda^\bullet \mathcal{T}_X$$

from the de Rham complex of X to the Brylinski complex $\langle \Lambda^\bullet \mathcal{T}_X, [-, \Theta] \rangle$.

Applying this in the affine symplectic case $X = \operatorname{Spec} A$, we see that $\Omega^1_X \to \mathcal{T}_X$ is actually an isomorphism, so that the Brylinski complex is quasiisomorphic to the de Rham complex, and the Poisson cohomology $HP^\bullet(A)$ is isomorphic to the de Rham cohomology $H^\bullet_{DR}(X)$ (in particular, it does not depend on the symplectic/Poisson structure at all). When $A = S^\bullet(V)$ is the polynomial algebra generated by a symplectic vector space V, with the Poisson structure induced by the symplectic form on V, we have

$$HP^i(A) = H^i_{DR}(X) = \begin{cases} k, & i = 0, \\ 0, & i \geq 1, \end{cases}$$

where $X = \operatorname{Spec} A$ is the affine space.

The Gerstenhaber case works in exactly the same way, except that we now have to care of the gradings, and use reduced cohomology.

Definition 11.10. The *Gerstenhaber cohomology complex $DG^\bullet(A^\bullet)$*, resp. the *reduced Gerstenhabe cohomology complex $\overline{DG}^\bullet(A^\bullet)$* of a Gerstenhaber DG algebra $A^\bullet$ is the DG Lie algebra of coderivations of the free Gerstenhaber coalgebra with, resp. without unit generated by $A^\bullet[1]$.

There is also a version of the A_∞-formalism for DG Gerstenhaber algebra, and the classification theorem for deformations of DG Gerstenhaber algebras up to a quasiisomorphism; this is completely parallel to the associative case and left to the reader. The end result is that deformations "up to a quasiisomorphims" of a DG Gerstenhaber algebra $A^\bullet$ are controlled by the DG Lie algebra $\overline{DG}^\bullet(A^\bullet)$.

Exercise 11.5. *Let $A^\bullet = S^\bullet(V^\bullet)$ be graded polynomial algebra generated by a graded vector space V, with the Gerstenhaber structure induced by a non-degenerate graded sympletic form $\Lambda^2(V^\bullet) \to k[-1]$. Show that the reduced Gerstenhaber cohomology complex $\overline{DG}^\bullet(A^\bullet)$ is quasiisomorphic to the quotient $A^\bullet/k$, where $k \to A^\bullet$ is the unit map $\lambda \mapsto \lambda \cdot 1$.*

11.3 Tamarkin's Theorem.

We can now explain how to prove Tamarkin's Theorem, ar rather, the following version of it.

Proposition 11.11. *Let $A^\bullet = S^\bullet(V)$ be the polynomial algebra generated by a vector space V, and assume given a DG Gerstenhaber algebra $B^\bullet$ whose cohomology is isomorphic to the Hochschild cohomology Gerstenhaber algebra $HH^\bullet(A)$. Aswsume in addition that $B^\bullet$ admits an action of the group $GL(V)$ such that the isomorphism $H^\bullet(B^\bullet) \cong HH^\bullet(A)$ is $GL(V)$-equivariant. Then the DG Gerstenhaber algebra $B^\bullet$ is formal, that is, quasiisomorphic to $HH^\bullet(A)$.*

Proof. Consider the canonical filtration $F_\bullet B^\bullet$ on the Gerstenhaber algebra $B^\bullet$. Then we have a canonical quasiisomorphism $\operatorname{gr}^F_\bullet B^\bullet \cong HH^\bullet(A)$, and this quasiisomorphism, being canonical, is compatible with the Gerstenhaber algebra structure and with the $GL(V)$-action. There is a standard way to interpret the associated graded quotient $\operatorname{gr}^F_\bullet B^\bullet$ as a special fiber of a certain deformation of the algebra $B^\bullet$ (known as "the deformation to the normal cone"). Namely, consider the Rees algebra

$$\widetilde{B}^\bullet = \bigoplus_i F_i B^\bullet$$

defined by the canonical filtration. This is also a Gerstenhaber algebra which has an additional grading by i. Moreover, the embeddings $F_i B^\bullet \subset F_{i+1} B^\bullet$ give a certain endomorphism of $\widetilde{B}^\bullet$ of degree 1 which we denote by h. Then $\widetilde{B}^\bullet$ is a graded Gerstenhaber algebra over $S = k[h]$. Its generic fiber $\widetilde{B}^\bullet \otimes_S k[h, h^{-1}]$ is isomorphic to $B^\bullet \otimes k[h, h^{-1}]$, while its special fiber $\widetilde{B}^\bullet / h$ is isomorphic to $\mathsf{gr}^\bullet_F B^\bullet$. Thus we have a $GL(V)$-equivariant S-deformation of the Gerstenhaber algebra $\mathsf{gr}^\bullet_F B^\bullet \cong HH^\bullet(A)$, and we have to show that this deformation is trivial up to a quasiisomorphism. But by the Hochschild-Kostant-Rosenberg Theorem, we have $HH^\bullet(A) = S^\bullet(V \oplus V^*[-1])$, and it is easy to check that the Gerstenhaber structure is induced by the natural pairing $V \otimes (V^*[-1]) \to k[-1])$ (it suffices to check this on the generators $V \oplus V^*[-1]$, and this is a trivial exercize). Applying Exercize 11.5, we conclude that

$$\overline{DG}^\bullet(HH^\bullet(A)) \cong HH^\bullet(A)/k.$$

In the right-hand side, the $GL(V)$-invariant part is trivial in degrees ≥ 2, so that *every* $GL(V)$-equivariant deformation of the DG Gerstenhaber algebra $HH^\bullet(A)$ is trivial up to a quasiisomorphism. □

As we have noted already in Lecture 9, this reduces Kontsevich Formality Theorem to Theorem 9.7, the formality of the chain operad of little discs (and the Deligne Conjecture). Indeed, once these both are established, we know that the Hochschild cohomology complex $DT^\bullet(A^\bullet)$ is a DG Gerstenhaber algebra. It is obviously $GL(V)$-equivariant, thus formal by Proposition 11.11.