Lectures 1 and 2: General introduction, DG algebras.

1.1 Motivations.

Let me start by recalling the standard correspondence between algebra and geometry which goes back essentially to the work of Gelfand and Naimark in the 1940ies. It can be conveniently summarized in the following table.

Geometry	Algebra
Space X	Algebra A of functions on X
A point $x \in X$	An ideal $I \subset A$

Here "space" can mean different things in different contexts. In the original work of Gelfand and Naimark, X could be any compact topological space; the algebra A was then a commutative Banach algebra of functions on X, and points in X correspond to closed maximal ideals in A. The famous Gelfand-Naimark theorem asserts that this is a one-to-one correspondence. It also makes sense to consider C^∞-manifolds and the algebras of C^∞-functions on them (and if the manifold is compact, every maximal ideal is automatically closed). Grothendieck transported the correspondence into the context of algebraic geometry; here A can be any commutative algebra at all, with no topology on it, and X is then, more or less by definition, an affine scheme. Grothendieck also showed that to obtain a well-behaved theory, it makes sense to add non-maximal prime ideals and the corresponding non-closed points to the scheme X.

In a nutshell, non-commutative geometry attempts to build up some sort of analogous dictionary valid for *non-commutative* algebras.

The desire to do so came originally from physics – one of the ways to interpret the formalism of quantum mechanic is to say that instead of the algebra of functions on a symplectic manifold M ("the phase space"), we should consider a certain non-commutative deformation of it. Another strong motivation is representation theory. In fact, Gelfand and Naimark treated their theorem as a classification of irreducible representations of a Banach algebra – it just happenned that since the algebra is commutative, these representations are one-dimensional and correspond to maximal ideals; it is natural to try to extend everything to non-commutative algebras such as, for example, the universal envelopping algebra of a Lie algebra $\mathfrak{g}$.

A priori, there is no reason at all why anything should work, and in fact, some of it very obviously does not: one can consider various types of ideals in general non-commutative algebras, but none of them form a "space" in a natural way – there were many attempts over the year to invent something, but none were too convincing. So, the second line in our table breaks down completely, and the story might have ended here: there is no such thing as a non-commutative space.

But if we cannot have a "space" in a conventional sense, maybe we can have something usually associated to it – for example, certain properties such as compactness or smoothness, or certain invariants such as homology and cohomology?

As we know now, and it took a long time to realize this, we can have quite a lot – and this is the only reason why speaking about "non-commutative geometry" makes any sense. Here is a partial list of things that exist in the non-commutative setting.

- Smoothness and compactness.
- Differential forms and polyvector fields.
- Algebraic K-theory.

- De Rham differential on differential forms, Lie bracket on vector fields.
- Hodge theory (in its algebraic form given by Deligne).
- Cartier isomorphism and Frobenius action on cristalline cohomology in positive characteristic.

This list is roughly in chronological order, and one can see how strangely the theory developed. For example, there is a 20-year gap between the discovery of the non-commutative versions of differential forms and the de Rham differential, and such an advanced thing as algebraic K-theory appeared in the middle of this gap. In fact, K-theory is the only item on the list which obviously should be there: Quillen's definition of the K-theory of an algebraic variety $X = \operatorname{Spec} A$ involves only the abelian category $A\text{-}\mathsf{mod}$ of A-modules, and it works for a non-commutative ring A without any changes whatsoever. Before we can discuss the other items on the list, however, we need to explain more precisely what we mean by "non-commutative setting".

1.2 The notions of a non-commutative variety.

Actually, there are several levels of abstraction at which non-commutative geometry can be built. Namely, we can take as our definition of a "non-commutative variety" one of the following four.

(1) An associative algebra A.

(2) A differential graded (DG) algebra $A^{\bullet}$.

(3) An abelian category $\mathcal{C}$.

(4) A triangulated category $\mathcal{D}$ "with some enhancement".

The relation between these levels is not linear, but rather as follows:

$$\begin{array}{ccc} (1) & \longrightarrow & (2) \\ \downarrow & & \downarrow \\ (3) & \longrightarrow & (4). \end{array} \tag{1.1}$$

Given an associative ring A, we can treat it as a DG algebra placed in degree 0 – this is the correspondence (1) $\Rightarrow$ (2). Or else, we can consider the category $A\text{-}\mathsf{mod}$ of left A-modules – this is the correspondence (1) $\Rightarrow$ (3). Given a DG algebra $A^{\bullet}$, we can construct the derived category $\mathcal{D}(A^{\bullet})$ of left DG $A^{\bullet}$-modules, and given an abelian category $\mathcal{C}$, we can consider its derived category $\mathcal{D}(\mathcal{C})$ – this is (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4).

Of course, in any meaningful formalism, the usual notion of a (commutative) algebraic variety has to be included as a particular case. In the list above, (1) is the level of an affine algebraic variety $X = \operatorname{Spec} A$. Passing from (1) to (3) gives the category of A-modules, or, equivalently, the category of quasicoherent sheaves on X. This makes sense for an arbitrary, not necessarily affine scheme X – thus on level (3), we can work with any scheme X by replacing it with its category of quasicoherent sheaves. We can then pass to level (4), and take the derived category $\mathcal{D}(X)$.

What about (2)? As it turns out, an arbitrary scheme X also appears already on this level: by results of Bondal-Van den Bergh and Keller, the derived category $\mathcal{D}(X)$ of quasicoherent sheaves on a scheme X is equivalent to the derived category $\mathcal{D}(A^{\bullet})$ of a certain (non-canonical) DG algebra $A^{\bullet}$. The rough slogan for this is that "every scheme is derived-affine".

Here are some other examples of non-commutative varieties that one would like to consider.

(i) Given a scheme X, one can consider a coherent sheaf $\mathcal{A}$ of algebras on X and the category of sheaves of $\mathcal{A}$. This is only "slightly" non-commutative, in the sense that we have an honest commutative scheme, and the non-commutative algebra sheaf is of finite rank over the commutative sheaf $\mathcal{O}_X$ (e.g. if $X = \operatorname{Spec} B$ is affine, then $\mathcal{A}$ comes from a non-commutative algebra which has B lying its center, and is of finite rank over this center). However, there are examples where this is useful. For instance, in the so-called *non-commutative resolutions* introduced by M. Van den Bergh, X is usually singular; generically over X, $\mathcal{A}$ is a sheaf of matrix algebras, so that its category of modules is equivalent to the category of coherent sheaves on X, but near the singular locus of X, $\mathcal{A}$ is no longer a matrix algebra, and it is "better behaved" than $\mathcal{O}_X$ – e.g. it has finite homological dimension.

(ii) Many interesting categories come from representation theory – representation of a Lie algebra, or of a quantum group, or versions of these in finite characteristic, and so on.

(iii) In sympletic geometry, there is the so-called *Fukaya category* and its versions (e.g. the "Fukaya-Seidel category"). These only exist at level (4) above, and they are very hard to handle; still, the fully developed theory should apply to these categories, too.

Let us also mention that even if one is only interested in the usual schemes X, looking at them non-commutatively is still non-trivial, because *there are more maps between schemes X, X' when they are considered as non-commutative varieties.* E.g. on level (4), a map between triangulated categories is essentially a triangulated functor, or maybe a pair of adjoint triangulated functors, depending on the specific formalism used – but in any approach, a Fourier-Mukai transform, for instance, gives a well-defined non-commutative map. Flips and flops in the Minimal Model Program are also expected to give non-commutative maps.

Passing to a higher level of abstraction in (1.1), we lose some information. A single abelian category can be equivalent to the category of modules for different rings A (this is known as Morita equivalence – e.g. a commutative algebra A is Morita-equivalent to its matrix algebra $M_n(A)$, for any $n \geq 2$). And a single triangulated category can appear as the derived category of quasicoherent sheaves on different schemes (e.g. related by the Fourier-Mukai transform) and the derived category of DG modules over different DG algebras (e.g. related by Koszul duality, the DG version of Morita equivalence). However, it seems that the information lost is inessential; especially if we think of various homological invariants of a non-commutative variety, they all are independent of the specifics lost when passing to (4). While this is not a self-evident first principle but rather an empirical observation, it seems to hold – again as a rough slogan, "non-commutative geometry is derived Morita-invariant". Thus it would be highly desirable to develop the theory directly on level (4) and not bother with irrelevant data.

However, at present it is not possible to do this. The reason is the well-known fact that the notion of triangulated category is "too weak". Here are some instances of this.

(i) "Cones are not functorial". Thus for a triangulated category $\mathcal{D}$, the category of functors $\operatorname{Fun}(I, \mathcal{D})$ for even the simplest diagrams I – e.g. the category of arrows in $\mathcal{D}$ – is not triangulated.

(ii) Triangulated categories do not patch together well. For instance, if we are given two triangulated categories $\mathcal{D}_1$, $\mathcal{D}_2$ equipped with triangulated functors to a triangulated category $\mathcal{D}$, the fibered product $\mathcal{D}_1 \times_{\mathcal{D}} \mathcal{D}_2$ is not triangulated.

(iii) Given two triangulated categories $\mathcal{D}_1$, $\mathcal{D}_2$, the category of triangulated functors $\operatorname{Fun}_{tr}(\mathcal{D}_1, \mathcal{D}_2)$ is not triangulated.

It is the consensus of all people working in the field that the correct notion is that of a triangulated category with some additional structure, called "enhancement"; however, there is no consensus as to what a convenient enhancement might be, exactly. Popular candidates are "DG-enhancement", "A_∞-enhancement" and "derivator enhancement". At present, the only sufficiently developed notion of enhancement seems to be the DG approach, but using it is not much different from simply working in the context of DG algebras, that is, on our level (2).

Thus in these lectures, I will not attempt to work in the full generality of (4), and stick to (1) and (2).

However, it is important to keep in mind that (4) is the correct level. In particular, everything should and will be "derived-Morita-invariant" – DG algebras or abelian categories that have equivalent derived categories are indistinguishable from the non-commutative point of view.

1.3 The derived category of a DG algebra.

Let me now describe in some detail the basics of the theory of DG algebras. I will mostly follow the approach of B. Toën, and also a recent paper by M. Kontsevich and Ya. Soibelman; a very good overview of the state-of-the-art in the year 2005 can be found in B. Keller's talk at ICM Madrid (it is also very good for references, except that Keller's own huge contribution to the subject is grossly understated).

We fix once and for all a commutative base ring k. In applications, k is usually a field, but it is sometimes convenient, for example, to let $k = \mathbb{Z}$.

Let $A^\bullet$ be an associative DG algebra over k. Assume that A^i is a flat k-module for every integer i. DG modules over $A^\bullet$ form an abelian category $A^\bullet\text{-}\mathsf{mod}$. A map $f : M \to M'$ between DG modules M, M' is a *quasiisomorphism* if it is a quasiisomorphism of the underlying complexes of k-modules. The *derived category* $\mathcal{D}(A^\bullet)$ is obtained by formally inverting all quasiisomorphisms in $A^\bullet\text{-}\mathsf{mod}$: we have

$$\mathcal{D}(A^\bullet) = A^\bullet\text{-}\mathsf{mod}[W^{-1}],$$

where W is the class of all quasiisomorphisms. This inversion procedure is purely formal: by definition, $A^\bullet\text{-}\mathsf{mod}[W^{-1}]$ is the category such that any functor $A^\bullet\text{-}\mathsf{mod} \to \mathcal{C}$ to some category $\mathcal{C}$ such that every mao $f \in W$ is sent to an invetible map factors unqiuely through the natural functor $A^\bullet\text{-}\mathsf{mod} \to A^\bullet\text{-}\mathsf{mod}[W^{-1}]$ (there may be some set-theoretic problems with this definition, but we will ignore them). For any two objects $M, M' \in A^\bullet\text{-}\mathsf{mod}$, maps from M to M' in $\mathcal{D}(A^\bullet)$ are represented by diagrams of the form $M \leftarrow M_1 \to M_2 \leftarrow M_3 \to \dots \leftarrow M_n \to M'$, where all arrows going in the wrong direction are quasiisomorphisms. In practice, working with such diagrams is incovenient; one way to control the localization is by introducing the following notion.

Definition 1.1. A DG module $M \in A^\bullet\text{-}\mathsf{mod}$ is *h-projective* if for any surjective quasiisomorphism $N \to N'$ of DG $A^\bullet$ modules the induced map

$$\mathsf{Hom}(M, N) \to \mathsf{Hom}(M, N')$$

is surjective.

For example, if $A^\bullet = A$ is simply an algebra, placed in degree 0, then a DG A-module is a complex of A-modules; when such a complex is bounded from above, it is h-projective if and only if all its terms are projective A-modules. For a general DG algebra, h-projective DG modules play the role of projective resolutions. In particular,

(i) for any DG $A^\bullet$-modules M, there exists an h-projective DG module P and a quasiisomorphism $P \to M$ (this is called an *h-projective replacement* for M),

(ii) for any two h-projective replacements P, P', there exists a third one P'' and quasiisomorphisms $P'' \to P$, $P' \to P'$, and

(iii) for any h-projective replacement $P \to M$, maps from M to some M' in the derived category $\mathcal{D}(A^\bullet)$ can be represented by diagrams $M \leftarrow P \to M'$.

Here (ii) and (iii) immediately follow from the definition. To see (i) in the case when M is h-projective over k, one can use for example the bar construction to obtain a free resolution $P_\bullet$ of M in the abelian category $A^\bullet\text{-}\mathsf{mod}$ such that

(i) P_i is a free $A^\bullet$-module for any $i \geq 0$, and

(ii) the cone of the augmentation map $P_\bullet \to M$ admits a k-linear contracting homotopy.

One then lets P be the total complex of the bicomplex $P_\bullet$; the first property insures that P is h-projective, and the second insures that the map $P \to M$ induced by the augmentation map is a quasiisomorphism.

Using h-projective replacements, it is easy to show that the derived category $\mathcal{D}(A^\bullet)$ is a triangulated category. Here is another application. Assume given a map $A_1^\bullet \to A_2^\bullet$ between two DG algebras. Then by restriction, we obtain a natural functor

$$\mathcal{D}(A_2^\bullet) \to \mathcal{D}(A_1^\bullet).$$

Using h-projective replacement, one immediately shows that this functor admits a left-adjoint $D(A_1^\bullet) \to \mathcal{D}(A_2^\bullet)$. When both DG algebras are just algebras placed in degree 0, this is the derived functor of the usual tensor product. By abuse of notation, we will denote this adjoint functor by the usual tensor product sign,

$$M \mapsto M \otimes_{A_1^\bullet} A_2^\bullet.$$

If a map $A_1^\bullet \to A_2^\bullet$ is a quasiisomorphism, then the restriction functor is an equivalence of categories, with the inverse equivalence given by the tensor product. Again, this is easily proved using h-projective replacements. We note, however, that the statement is a sort of an accident, and it is *not* true if one replaces DG algebras with something more complicated. In particular, an analogous statement for DG coalgebras is false.

A third useful application of h-projective replacements is the existence of "homotopy colimits" of DG modules.

What this means is the following. For any category $\mathcal{C}$ with a chosen class of morphisms W, one can consider the formal localization $\mathcal{C}[W^{-1}]$. Assume given a small category I, and consider the category $\mathcal{C}^I$ of functors from I to $\mathcal{C}$. Let W_I be the class of such morphisms f in $\mathcal{C}^I$ that $f(i) \in W$ for any $i \in I$. Then sending $c \in \mathcal{C}$ to the constant functor with value c gives a functor $\mathcal{C} \to \mathcal{C}^I$, and this induces a tautological functor $\tau : \mathcal{C}[W^{-1}] \to \mathcal{C}^I[W_I^{-1}]$. The *homotopy colimit functor*

$$\mathsf{hocolim}_I : \mathcal{C}^I[W_I^{-1}] \to \mathcal{C}[W^{-1}] \qquad (1.2)$$

is by definition the left-adjoint functor to τ. This defines $\mathsf{hocolim}$ uniqiely; whether it exists can be a complicated question. Fortunately, in the specific case of DG modules and the class of quasiisomorphism, it is easy to show that $\mathsf{hocolim}$ always exists (one uses a version of h-projective replacements for the category $A^\bullet\text{-}\mathsf{mod}^I$). This becomes especially simple if the category I is filtered in the sense of Grothendieck – for example, if I is a partially ordered set which is *directed* in the following sense:

- for any $i_1, i_2 \in I$, there exists $i \in I$ such that $i_1, i_2 \leq i$.

In this case, we simply have

$$\mathsf{hocolim}_I M^I \cong \varinjlim_I M^I$$

for any functor $M^I : I \to \mathcal{A}^\bullet\text{-}\mathsf{mod}$.

1.4 Perfect DG modules.

We can now give the following fundamental definition. We use shorthand "directed inductive system" of objects in a category $\mathcal{C}$ for a functor $I \to \mathcal{C}$ from a directed partially ordered set I.

Definition 1.2. A DG module M over a DG algebra $A^\bullet$ is *perfect* if for any directed inductive system N_i, $i \in I$ of DG $A^\bullet$-modules, the natural map

$$\lim_{\underset{I}{\to}} \mathsf{Hom}(M, N_i) \to \mathsf{Hom}(M, \mathsf{hocolim}_I(N_i)) \tag{1.3}$$

is an isomorphism (where Hom is taken in the derived category $\mathcal{D}(\mathcal{A}^\bullet)$.

In other words, any map

$$M \to \mathsf{hocolim}_I N_i \cong \lim_{\underset{I}{\to}} N_i$$

in the derived category $\mathcal{D}(A^\bullet)$ factors through a map $f_i \to N_i$ for some i, and any two such factorizations f_i $f_{i'}$ become equal after passing to some $i'' \geq i, i'$.

We note right away that a cone of a map between perfect DG modules is also obviously perfect; thus the full subcategory

$$\mathcal{D}_{perf}(A^\bullet) \subset \mathcal{D}(A^\bullet)$$

spanned by perfect modules is a triangulated subcategory.

Definition 1.2 is the standard definition of an object "of finite type" in a category (with colimits replaced by homotopy colimits). For example, if k is a field, then a complex of k-vector spaces is perfect if and only if its homology is non-trivial only in a finite number of degrees, and all these non-trivial homology vector spaces are finite-dimensional. More generally, for a arbitrary commutative k, a perfect DG module is the same as a perfect complex in the sense of Grothendieck; in particular, a complex is perfect if and only if it is quasiisomorphic to a finite-length complex of finitely generated projective k-modules. Equivalently, it is a direct summand of a finite-length complex of free finitely generated k-modules.

To obtain an analogous characterization for general perfect DG modules, one introduces the following.

Definition 1.3. A DG $A^\bullet$-module M is a *finite cell module* if there exists a finite sequence of DG modules

$$0 \cong M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_n \longrightarrow M$$

such that for any i, $0 \leq i \leq n$, the cone of the map $M_i \to M_{i+1}$ is quasiisomorphic to a shift of the free DG $A^\bullet$-module $A^\bullet$.

In other words, a finite cell module is obtained by attaching, one-by-one, a finite number of copies of the (shifted) free module (called "cells"). The name comes by analogy with a finite CW complex in topology.

Lemma 1.4. *A DG $A^\bullet$-module is perfect if and only if it is quasiisomorphic to a retract of a finite cell module M' – in other words, there exists a finite cell module M' and maps $a : M \to M'$, $a' : M' \to M$ in $\mathcal{D}(A^\bullet)$ such that $a' \circ a = \mathsf{id}$.*

Remark 1.5. In the notation of the lemma, the map $p = a \circ a' : M' \to M'$ is a projector, a.k.a. idempotent – that is, $p^2 = p$, – and M is completely defined by M' and p. Indeed, it is a general categorical fact that for any idempotent endomorphism $p : c' \to c'$ of an object in a category $\mathcal{C}$, $p^2 = p$, there exists at most one object $c \in \mathcal{C}$ equipped with a decomposition $p = a \circ a'$, $a : c \to c'$, $a' : c' \to c$, $a' \circ a = \mathsf{id}$; when such a $c \in \mathcal{C}$ exists, it is called the *image* of the projector p. One may wonder whether *any* projector $p : M' \to M'$ in $\mathcal{D}(A^\bullet)$ has such an image. The answer is "yes": it is easy to show that the category $\mathcal{D}(A^\bullet)$ is *idempotent-complete.*

Proof. One direction is easy: any shift of the free module $A^\bullet$ is obviously perfect, and since $\mathcal{D}_{perf}(A^\bullet) \subset \mathcal{D}(A^\bullet)$ is a triangulated subcategory, any finite cell module is perfect by induction. Then one notes that a retract of a perfect module is also perfect: indeed, if M is a retract of M', then for any DG module N, the set $\mathsf{Hom}(M, N)$ is a retract of $\mathsf{Hom}(M', N)$.

To go in the other direction, one first notes that any DG module M is the homotopy colimit of a directed inductive system of finite cell modules,

$$M \cong \mathsf{hocolim}_I M_i$$

for some directed partially ordered set I (if $A^\bullet = k$, we can replace M by its h-projective replacement, which is then a direct limit of all its perfect subcomplexes; in the general case, use the bar resolution replacement). Thus in particular, we have a quasiisomorphism

$$M \to \mathsf{hocolim}_I M_i.$$

Then by the definition of a perfect module, it must factor through a map $f : M \to M_i$ for some i. Take $a = f$ and $M' = M_i$. □

This lemma shows that perfect modules are essentially "defined by a finite amount of data". Here is one useful corollary of this principle. Note that by the lemma, tensor product sends perfect DG modules to perfect DG modules: for any map $A^\bullet \to B^\bullet$ and any perfect $M \in \mathcal{D}(A^\bullet)$, the tensor product

$$M \otimes_{A^\bullet} B^\bullet \in \mathcal{D}(B^\bullet)$$

is perfect.

Lemma 1.6. *Assume given a direct inductive system $A_i^\bullet$, $i \in I$ of DG algebras over k, and let $A^\bullet = \lim_{\overrightarrow{I}} A_i^\bullet$. Then the natural functor*

$$\lim_{\overrightarrow{I}} \mathcal{D}_{perf}(A_i^\bullet) \to \mathcal{D}_{perf}(A^\bullet)$$

is an equivalence of categories.

In other words, any perfect DG $A^\bullet$-modules M is of the form $M_i \otimes_{A_i^\bullet} A^\bullet$ for some i and some perfect $A_i^\bullet$-module M_i, and for any two $A_i^\bullet$-modules M, M', the natural map

$$\lim_{\overrightarrow{j \in I, j \geq i}} \mathsf{Hom}(M \otimes_{A_i^\bullet} A_j^\bullet, M' \otimes_{A_i^\bullet} A_j^\bullet) \to \mathsf{Hom}(M \otimes_{A_i^\bullet} A^\bullet, M' \otimes_{A_i^\bullet} A^\bullet) \tag{1.4}$$

is an isomorphism.

Proof. When M and M' are shifts of $A_i^\bullet$, (1.4) is an isomorphism by definition. By the five-lemma and induction on the number of cells, it is also an isomorphism for finite cell modules; then by retraction, it is an isomorphism for all perfect modules, as required.

To prove that a finite cell module M comex from M_i for some $A_i^\bullet$, represent is a a cone of a map $\mu : M' \to \mathcal{A}^\bullet[l]$ for some l, where M' has less cells; by induction on the number of cells, M' is defined over some $A_i^\bullet$, and by (1.4), the map μ is also defined over some $A_j^\bullet$, possibly with $j > i$.

Finally, a general perfect module is the image of a projector $p : M \to M$ with a finite cell module M; then M is defined over some $A_i^\bullet$, p is defined over some $A_j^\bullet$, $j \geq i$, and increasing j, we can achieve that $p^2 = p$. □

1.5 Smooth and compact DG algebras.

Consider now the category of all DG algebras. Here again, it makes sense to formally invert quasiisomorphisms; this gives the "homotopy category of DG algebras", or "the category of DG algebras up to a quasiisomorphism", denoted $\mathsf{Ho}\text{-}\mathsf{DG}$. Again, it is hard to work with localization directly. A technique analogous to the "h-projective replacement" is provided by the formalism of closed model categories introduced by Quillen. In this lectures, I don't want to go into the details of the technique. Let me just say that repeating Definition 1.1 literally with DG modules replaced by DG algebras gives a notion of a "cofibrant DG algebra", and every DG algebra has a cofibrant replacement with the same properties as h-projective replacement for DG modules. Explicitly, a DG algebra $A^\bullet$ is cofibrant if, for example,

(i) $A^\bullet$ is free as a graded algebra, $A^\bullet - k[x_1, x_2, \dots]$, with generators x_i numbered by elements in a set I (and placed in various degrees), and

(ii) the set I can be ordered so that the differential sends x_i into the subalgebra $k[\{x_j\}] \subset A^\bullet$ generated by elements x_j with $j < i$.

There may be other cofibrant DG algebras; however, to get a cofibrant replacement, it suffices to consider DG algebras of this form. In particular, one notes that such a cofibrant DG algebra is a complex of *free* k-modules, just as we required in Section 1.3. Thus when working with DG algebras up to a quasiisomorphism, we may always assume that this requirement is satisfied. Moreover, since a quasiisomorphism of DG algebras induces an equivalence of their derived categories, the derived category $\mathcal{D}(A^\bullet)$ is well-defined for objects of the category $\mathsf{Ho}\text{-}\mathsf{DG}$, and for any map $f : A^\bullet \to B^\bullet$ in $\mathsf{Ho}\text{-}\mathsf{DG}$, we have the restriction functor $f^* : \mathcal{D}(B^\bullet) \to \mathcal{D}(A^\bullet)$ and its left-adjoint tensor product functor $\mathcal{D}(A^\bullet) \to \mathcal{D}(B^\bullet)$.

As far as I know, the following definition has been first introduced in this form by M. Kontsevich and Ya. Soibelman; in any case, it has now become a de-facto standard.

Definition 1.7. (i) A DG algebra $A^\bullet$ is (homologically) *compact* if $A^\bullet$ is a perfect DG module over k.

(ii) A DG algebra $A^\bullet$ is (homologically) *smooth* if $A^\bullet$ is a perfect DG module over $A^{\bullet opp} \otimes A^\bullet$ (in other words, a perfect DG $A^\bullet$-bimodule).

For example, if $A^\bullet$ is such that $\mathcal{D}(A^\bullet) \cong \mathcal{D}(X)$, the derived category of quasicoherent sheaves on a scheme X, then $A^\bullet$ is homologically compact if and only if X is proper (this easily follows from standard finiteness results in algebraic geometry). Smoothness is more delicate. It claims, roughly speaking, that the diagonal bimodule $A^\bullet$ "has finite homological dimension", so that the algebra $A^\bullet$ itself has "finite homological dimension". In the case when $A^\bullet$ is a commutative algebra of finite type placed in degree 0, this is equivalent to the smoothness of $X = \operatorname{Spec} A$ by a famous theorem of Serre. The same holds in general: if $\mathcal{D}(A^\bullet) \cong \mathcal{D}(X)$ for some scheme X of finite type over k, then $A^\bullet$ is homologically smooth if and only if X is smooth.

To analyse smoothness and compactness, it is convenient to introduce the following notion, due to B. Toën.

Definition 1.8. A DG $A^\bullet$-module is *pseudoperfect* if it is perfect as a DG module over k. More generally, given another DG algebra $B^\bullet$, a DG module over $A^\bullet \otimes B^\bullet$ is *pseudoperfect* if it is perfect as a DG module over $B^\bullet$.

Lemma 1.9. (i) *$A^\bullet$ is compact if and only if for any DG algebra $B^\bullet$, any perfect $M \in \mathcal{D}(A^\bullet \otimes B^\bullet)$ is pseudoperfect.*

(ii) $A^\bullet$ *is smooth if and only if for any DG algebra* $B^\bullet$*, any pseudoperfect* $M \in \mathcal{D}(A^\bullet \otimes B^\bullet)$ *is perfect.*

Proof. (i) is clear: being pseudoperfect is obviously stable under taking shifts, cones and retracts, so that it suffices to check the condition for $M = A^\bullet \otimes B^\bullet$, when it is obvious.

For (ii), we note that for any pseudoperfect $M \in \mathcal{D}(A^\bullet \otimes B^\bullet)$ and perfect $N \in \mathcal{D}(A^\bullet \otimes A^{\bullet opp})$, the tensor product

$$N \otimes_{A^\bullet} M \in \mathcal{D}(A^\bullet \otimes B^\bullet)$$

is perfect (again it suffices to check this for $N = A^\bullet \otimes A^{\bullet opp}$, when the claim is obvious). Thus it suffices to check the condition for $B^\bullet = A^{\bullet opp}$ and $M = A^\bullet$, the diagonal $A^\bullet$-bimodule. Then it becomes exactly the definition of "homologically smooth". ☐

A DG algebra is called *saturated* if it is homologically smooth and homologically compact. By Lemma 1.9, $A^\bullet$ is saturated if an only if for any $B^\bullet$, perfect and pseudoperfect objects in $\mathcal{D}(A^\bullet \otimes B^\bullet)$ coincide. In the homological version of non-commutative geometry, it is saturated algebras that are the most important ones; they play the role of smooth and proper algebraic varieties.

1.6 Finite-type conditions for DG algebras.

The definitions of smooth and compact DG algebras come from geometry: they are arranged so as to give the standard notions for those DG algebras which correspond to schemes. One can also consider the category of DG algebras as a closed model category with nice properties, and introduce a finiteness condition which is natural from the general categorical point of view. This has been done by B. Toën who defined "homotopically finitely presented" algebras. Namely, one can prove that the category of DG algebras with the class of quasiisomorphisms between them admits homotopy colimits in the general sense of (1.2). Then one can literally repeat Definition 1.2 with DG modules replaced by DG algebras.

Definition 1.10. A DG algebra $A^\bullet$ is *homotopically finitely presented* (hfp for short) if for any any inductive system $B_i^\bullet$, $i \in I$ of DG algebras, with I being a directed partially ordered set, the natural map

$$\lim_{\substack{\to\\ I}} \mathsf{Hom}(A^\bullet, B_i^\bullet) \to \mathsf{Hom}(A^\bullet, \mathsf{hocolim}_I(B_i^\bullet))$$

is an isomorphism (where Hom is taken in the category of DG algebras up to a quasiisomorphism).

A more natural name would have been "algebras of finite type"; unfortunately, this contradicts the intuition – if a DG algebra is simply a commutative algebra A placed in degree 0, then it is hfp if and only if it if of finite type *and smooth*. Hfp is ugly but unambiguous.

Lemma 1.4 also has a counterpart for DG algebras (in fact, it holds more generally for a rather large class of closed model categories). Namely, let us say that a *finite cell algebra* is a DG algebra $A^\bullet$ which is finitely generated as an algebra, $A^\bullet = k[x_1, \dots, x_n]$, and the differential sends each x_i into the subalgebra $k[x_1, \dots, x_{i-1}] \subset A^\bullet$. This coincides with the explicit description of cofibrant DG algebras given earlier, except that the number of generators is required to be finite. Then we have the following result of B. Toën.

Lemma 1.11. *A DG algebra* $A^\bullet \in \mathsf{Ho}\text{-}\mathsf{DG}$ *is hfp if and only if it is a retract of a finite cell algebra.*

Proof. As in Lemma 1.4, one immediately checks by induction on the number of generators that a finite cell algebra is hfp, and this property is obviously stable under retraction.

Conversely, it is easy to show that every DG algebra $A^\bullet$ is quasiisomorphic to the homotopy colimit of a directed inductive system of finite cell algebras $A_i^\bullet$, that is, we have a quasiisomorphism

$$A^\bullet \to \mathsf{hocolim}_I A_i^\bullet,$$

and as in Lemma 1.4, if $A^\bullet$ is hfp, this quasiisomorphism factors through some $A_i^\bullet$. This gives the desired retraction. $\square$

As in the case of perfect modules, this Lemma essentially says that a hfp DG algebra $A^\bullet$ is defined by a finite amount of data: a finite cell algebra $A_i^\bullet$ and an idempotent $p : A_1^\bullet \to A_1^\bullet$, $p^2 = p$. However, there is a non-trivial point here: is the homotopy category of DG algebras idempotent-complete? Fortunately, this is true; the corresponding statement has been proved recently by B. Toën (Bull. London Math. Soc. **40** (2008), 642–650).

The relation between smoothness, compactness and hfp is as follows:

$$\text{Saturated} \;\Rightarrow\; \text{Hfp} \;\Rightarrow\; \text{smooth}. \tag{1.5}$$

To prove the first implication, let us make the following observation. Given a map $f : A^\bullet \to B^\bullet$ between DG algebras in the homotopy category $\mathsf{Ho\text{-}DG}$, one can consider $B^\bullet$ as a module over $A^{\bullet opp} \otimes B^\bullet$: here $B^\bullet$ acts on itself on the left, and $A^\bullet$ acts on the right by means of the map f (formally, one takes the diagonal $B^\bullet$-bimodule and restricts it to $A^{\bullet opp} \otimes B^\bullet$ be the map $f^{opp} \otimes \mathsf{id}$). This bimodule is in a sense the *graph* of the map f, so we will denote it $\mathsf{graph}(f)$. By definition, $\mathsf{graph}(f) \cong B^\bullet$ as a $B^\bullet$-module. Conversely, given an object $M \in \mathcal{D}(A^\bullet \otimes B^\bullet)$ equipped with an isomorphism $M \cong B^\bullet$ in $\mathcal{D}(B^\bullet)$, there exists a unique map $f : A^\bullet \to B^\bullet$ such that $M \cong \mathsf{graph}(f)$. Thus one can describe morphisms in $\mathsf{Ho\text{-}DG}$ by means of their graphs.

Lemma 1.12. *A saturated DG algebra $A^\bullet$ is hfp.*

Proof. Assume given a directed inductive system of DG algebras $B_i^\bullet$, $i \in I$, and a map

$$f : A^\bullet \to B^\bullet = \mathsf{hocolim}_I B_i^\bullet.$$

Since $B^\bullet$ is perfect as a module over itself, $\mathsf{graph}(f) \in \mathcal{D}(A^{\bullet opp} \otimes B^\bullet)$ is by definition pseudoperfect. Since $A^\bullet$ is smooth, it is also perfect by Lemma 1.9 (ii). Then by Lemma 1.6, we have

$$\mathsf{graph}(f) \cong M_i \otimes_{A^{\bullet opp} \otimes B_i^\bullet} A^{\bullet opp} \otimes B^\bullet$$

for some perfect $M_i \in \mathcal{D}(A^{\bullet opp} \otimes B_i^\bullet)$. However, the DG algebra $A^\bullet$ is also compact; therefore by Lemma 1.9 (i) M_i is pseudoperfect. Then again by Lemma 1.6, we can increase $i \in I$ and achieve that the $B^\bullet$-module isomorphism $\mathsf{graph}(f) \cong B^\bullet$ comes from a $B_i^\bullet$-module isomorphism $M_i \cong B_i^\bullet$. Then $M_i \cong \mathsf{graph}(f_i)$ for some map $f_i : A^\bullet \to B_i^\bullet$, and we are done. $\square$

To prove the second implication in (1.5), one argues as follows. For any DG bimodule M over a DG algebra $A^\bullet$, the direct sum $A_M^\bullet = A^\bullet \oplus M$ is naturally a DG algebra, the split square-zero extension of $A^\bullet$ by M. It comes equipped with a natural map $\rho : A_M^\bullet \to A^\bullet$. By a *derivation* δ from $A^\bullet$ to M we will understand a map $\delta : A^\bullet \to A_M^\bullet$ in $\mathsf{Ho\text{-}DG}$ such that $\rho \circ \delta = \mathsf{id}$. We will denote by $\mathsf{Der}(A^\bullet, M)$ the space of such derivations. By the *bimodule of differentials* $I(A^\bullet)$ of a DG algebra $A^\bullet$ we will understand the cone of the natural multiplication map $A^{\bullet opp} \otimes A^\bullet \to A^\bullet$, so that we have an exact triangle

$$I(A^\bullet) \longrightarrow A^{\bullet opp} \otimes A^\bullet \longrightarrow A^\bullet \longrightarrow .$$

We note that both $I(A^\bullet)$ and $\mathsf{Der}(A^\bullet, M)$ are well-defined on the homotopy category $\mathsf{Ho\text{-}DG}$. Therefore we may replace $A^\bullet$ with a cofibrant replacement, so that all maps $A^\bullet \to B^\bullet$ in $\mathsf{Ho\text{-}DG}$ are represented by an actual DG algebra map $A^\bullet \to B^\bullet$. Then it is easy to construct an isomorphism

$$\mathsf{Hom}(I(A^\bullet), M) \cong \mathsf{Der}(A^\bullet, M) \tag{1.6}$$

for any $M \in \mathcal{D}(A^\bullet)$ – this is done by the same argument as in the case of usual associative algebras.

Lemma 1.13. *A hfp DG algebra $A^\bullet$ is smooth.*

Proof. We have to prove that the diagonal bimodule $A^\bullet$ is perfect. Since the free bimodule $A^{\bullet opp} \otimes A^\bullet$ is by definition perfect, it suffices to prove that $I(A^\bullet)$ is perfect. Substituting (1.6) into (1.3), we see that it suffice to prove that for any directed inductive system M_i of DG bimodules over $A^\bullet$, we have

$$\mathsf{Der}(A^\bullet, \mathsf{hocolim}_I M_i) \cong \lim_{\stackrel{i}{\to}} \mathsf{Der}(A^\bullet, M_i).$$

But the inductive system M_i gives an inductive system $A^\bullet_{M_i}$ of DG algebras, and the claim follows immediately from the hfp property of the DG algebra $A^\bullet$. $\square$

1.7 Further results.

To conclude, let me mention some further results and conjectures. First of all, there is the following beautiful theorem of B. Toën (the same paper Bull. London Math. Soc. **40** (2008), 642–650).

Theorem 1.14. *Assume given a directed direct system k_i of commutative rings k_i, with the limit $k = lim_{\stackrel{i}{\to}} k_i$, and a DG algebra $A^\bullet$ over k. Assume that $A^\bullet$ is saturated. Then for some i, there exists a satirated DG algebra $A^\bullet_i$ over k_i and a quasiisomorphism $A^\bullet_i \otimes_{k_i} k \cong A^\bullet$.*

The proof is not difficult but quite elegant. Roughly speaking, Toën first proves an analogous statement for hfp algebras instead of saturated ones, by essentially the same argument as Lemma 1.6; this gives $A^\bullet_i$ as a hfp, hence smooth DG algebra, and then an additional argument shows that it can also be made compact. We note that an analogous statement for algebraic varieties is usually proved by invoking the Noetherian property of the rings of functions. In the non-commutative world, this leads nowhere since non-commutative rings are typically non-Noetherian – Hilbert's Basis Theorem only works in the commutative case. Toën's theorem then shows that Noetherianness survives, in some form, on the derived level. This is a very useful result in practice, since it allows to prove statemnts for saturated DG algebras by "reduction to positive characteristic".

One expects other results "of Noetherian type"; the most useful of them is probably the following. Say that a full triangulated subcategory $\mathcal{D}_0 \subset \mathcal{D}$ is *left-admissible* if the embedding functor admits a left-adjoint.

Conjecture 1.15. *Assume given a saturated DG algebra $A^\bullet$ over a Noetherian commutative ring k. Then every increasing sequence*

$$\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}(A^\bullet)$$

of left-admissible triangulated subcategories in the derived category $\mathcal{D}(A^\bullet)$ stabilizes.

This has a lot of applications in, for example, algebraic geometry of Fano varities (especially in the work of A. Kuznetsov), and is expected to have a lot more. Unfortunately, it seems that we do not have any approaches to this conjecture.

What one can also do is return to the "naive" approach to non-commutative geometry: start with a "non-commutative variety", by which we understand here a saturated DG algebra $A^\bullet$, and obtain a usual commutative algebraic variety. As we know, it is hopeless to work with ideals, but we can work with modules and, more generally, DG modules. Even when $A^\bullet$ corresponds to an algebraic variety X, there is no way to distiguish which DG $A^\bullet$-modules came from skyscraper sheaves on X, so that we cannot recover X from $\mathcal{D}(A^\bullet)$. However, we can construct a much larger variety that parametrizes *all* reasonable DG $A^\bullet$-modules; "reasonable" here means perfect, for lack of anything better. The problem is, this variety will not be a variety anymore. First of all, it will be a stack, since points have automorphisms, and moreover, there are higher homotopies between these automorphisms, so that we end up with a sort of "∞-stack", whatever that means. Secondly, even if we ignore automorphisms, our moduli space will be a DG scheme: its structure sheaf will be a commutative DG algebra rather than simply a sheaf of rings. So, before we can even start defining the hypothetical moduli space of perfect objects, we need to define the category it will live in.

This formidable task, fortunately, has been accomplished already by B. Toën and G. Vezzosi; the resulting objects are called "$\mathcal{D}^-$-stacks". Moreover, the $\mathcal{D}^-$-stack $\mathsf{Mod}(A^\bullet)$ parametrizing of perfect objects in $\mathcal{D}(A^\bullet)$ has been constructed by B. Toën and M. Vaquié.

Conjecturally, $\mathsf{Mod}(A^\bullet)$ should contain all the homological information one can extract from $A^\bullet$. At present, the construction seems to be too inexplicit to allow any computations, or even exact statements; but this is an area of active research, and we expect significant progress.