Lectures 3 and 4: Cyclic homology.

Today's topic is cyclic homology and everything related to it. A very good general reference for the subject is J.-L. Loday's book "Cyclic homology"; another old but useful reference is a long paper "Additive K-theory" by B. Feigin and B. Tsygan in Lecture Notes in Mathematics, vol. 1289.

1.1 Lie algebra of matrices.

Last time, I finished by saying that "all homological invariants of a non-commutative variety should be contained in the Toën-Vaquié moduli space of perfect objects". This is a relatively new point of view. Today, I want to start with a certain explicit computation which is rather old – in fact it was the origin of cyclic homology in the beginning of 1980es. How one is related to the other is not completely clear, but we do have a conjectural picture (with many gaps); this I will explain in the end of the lecture.

So, assume given a unital associative algebra A over a field k of characteristic 0, and consider the Lie algebra

$$\mathfrak{g} = \mathfrak{gl}_\infty(A) = \lim_{\substack{\to \\ n}} \mathfrak{gl}_n(A),$$

the direct limit of the matrix Lie algebras $\mathfrak{gl}_n(A)$. What we want to do is to compute Lie algebra homology $H_\bullet(\mathfrak{g}, k)$ with trivial coefficients.

This computation was done independently by J.-L. Loday and D. Quillen, and B. Tsygan some time in the early 1980es. The reason the question was interesting then was its relation to K-theory. By definition, K-theory is related to the infinite discrete group $BGL_\infty(A)$, but its homology is very hard to compute, and sometimes – say for unipotent groups – Lie algebra gives a good approximation to the group. The group $BGL_\infty(A)$ is very far from unipotent, but let's do the computation anyway and see what happens. With hindsight, it seems a rather naive idea, but it did work – the computation resulted in a real breakthrough.

The computation is done by applying classical invariant theory. We start by considering the standard Chevalley complex $C_\bullet(\mathfrak{gl}_\infty(A), k)$; its terms are the exterior powers

$$C_i(\mathfrak{gl}_\infty(A), k) = \Lambda^i(\mathfrak{gl}_\infty(A)).$$

To simplify it, we note that

- the adjoint action of a Lie algebra $\mathfrak{g}$ on its homology is trivial.

This can be proved in many different ways; for example, for any element $\xi \in \mathfrak{g}$, the Cartan homotopy formula shows that the action of ξ on $H_\bullet(\mathfrak{g}, k)$ is chain-homotopic to 0. Since A is unital, we have a natural embedding $\mathfrak{gl}_\infty(k) \subset \mathfrak{gl}_\infty(A)$, and the induced $\mathfrak{gl}_\infty(k)$-action on $H_\bullet(\mathfrak{gl}_\infty(A), k)$ is also trivial.

Now, the action of $\mathfrak{gl}_\infty(k)$ on $\mathfrak{gl}_\infty(A)$ is completely reducible, and so is its action on its tensor powers and on the terms of the Chevalley complex. Therefore when computing $H_\bullet(\mathfrak{gl}_\infty(A), k)$, we may just replace the terms with the invariants of the $\mathfrak{gl}_\infty(k)$-action. Explicitly, the i-th term becomes

$$(\Lambda^i \mathfrak{gl}_\infty(A))^{\mathfrak{gl}_\infty(k)} = (\mathfrak{gl}_\infty(A)^{\otimes i} \otimes \mathsf{sgn})_{\Sigma_i}^{\mathfrak{gl}_\infty(k)}, \tag{1.1}$$

where Σ_i is the permutation group on i letters, and sgn is its one-dimensional sign representation. Let us first compute the $\mathfrak{gl}_\infty(k)$-invariants. We have

$$\mathfrak{gl}_\infty(A)^{\otimes i} = \lim_{\substack{\to \\ n}} \mathfrak{gl}_n(A)^{\otimes i} = \lim_{\substack{\to \\ n}} A^{\otimes i} \otimes \mathrm{End}(k_n^{\otimes i}),$$

where $k_n = k^{\oplus n}$ is the sum of n copies of k, and End means endomorphisms as a vector space. The algebra $\mathfrak{gl}_\infty(k)$ only acts on this second factor $\operatorname{End}(k_n^{\otimes i})$, and the invariant theory shows that

$$\lim_{\substack{\to \\ n}} \operatorname{End}(k_n^{\otimes i})^{\mathfrak{gl}_n(k)} = \lim_{\substack{\to \\ n}} \operatorname{End}_{\mathfrak{gl}_n(k)}(k_n^{\otimes i}) \cong k[\Sigma_i],$$

the group algebra of the symmetric group Σ_i, with the conjugation action of Σ_i (more precisely, all $\mathfrak{gl}_n(k)$-invariant linear maps $k_n^{\otimes i} \to k_n^{\otimes i}$ are given by linear compositions of permutations of factors in the tensor product, and if $n \geq i$, these permutations are linearly independent). We conclude that (1.1) is isomorphic to

$$(A^{\otimes i} \otimes k[\Sigma_i] \otimes \mathsf{sgn})_{\Sigma_i}.$$

Conjugacy classes of permutations $\sigma \in \Sigma_i$ are numbered by Young diagrams – this is the cycle decomposition of a permutation; thus for any representation V of the group Σ_i, we have a decomposition

$$(V \otimes k[\Sigma_i])_{\Sigma_i} \cong \bigoplus_\lambda V_{\mathsf{Stab}(\lambda)},$$

where the sum is over Young diagrams λ, and $\mathsf{Stab}(\lambda) \subset \Sigma_i$ is the stabilizer of the corresponding permutation with respect to the conjugation action. For a diagram $\lambda = \langle a_1 \geq a_2 \geq \dots \geq a_l \rangle$, we have

$$\mathsf{Stab}(\lambda) = ((\mathbb{Z}/a_1\mathbb{Z}) \times \dots \times (\mathbb{Z}/a_l\mathbb{Z})) \rtimes \mathsf{Aut}(\lambda),$$

where $\mathsf{Aut}(\lambda)$ is equal to

$$\mathsf{Aut}(\lambda) = \Sigma_{b_1 - b_2} \times \dots \times \Sigma_{b_{m-1} - b_m},$$

$\langle b_1 \geq \dots \geq b_m \rangle$ being the dual Young diagram. Applying this to $A^{\otimes i} \otimes \mathsf{sgn}$ and summing over i, we obtain an isomorphism

$$C_\bullet(\mathfrak{gl}_\infty(A), k)^{\mathfrak{gl}_\infty(k)} \cong S^\bullet(CC_\bullet(A)[1]), \tag{1.2}$$

where $S^\bullet$ stands for graded-symmetric power, and $CC_\bullet(A)$ is given by

$$CC_i(A) = (A^{\otimes i})_{\mathbb{Z}/i\mathbb{Z}}. \tag{1.3}$$

The cyclic group $\mathbb{Z}/i\mathbb{Z}$ acts on $A^{\otimes i}$ by the longest permutation twisted by $(-1)^{i+1}$.

This computes the invariant part of the Chevalley complex in a nice and short way. As it turns out, this is also compatible with the differential. Namely, equip $CC_\bullet(A)$ with the differential b given by the following formula

$$b(a_0 \otimes a_1 \otimes \dots \otimes a_i) = \sum_{0 \leq j < i} (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_i + a_i a_0 \otimes \dots \otimes a_{i-1}. \tag{1.4}$$

It is easy to see that $b^2 = 0$, so that $CC_\bullet(A)$ becomes a complex; the symmetric power of b is then a differential on the right-hand side of (1.2).

Proposition 1.1. *The isomorphims* (1.2) *can be chosen so that the Chevalley differential on the left-hand side is mapped into the differential induced by b on the right-hand side.*

Proof. The Chevalley complex of any Lie algebra is naturally a commutative coalgebra. Let us treat the right-hand side of (1.2) as a free graded commutative coalgebra generated by $CC_\bullet(A)[1]$. We will construct (1.2) as a coalgebra map. Since the right-hand side is free, it is equivalent to constructing a map of complexes

$$\gamma : C_\bullet(\mathfrak{gl}_\infty(A), k)^{\mathfrak{gl}_\infty(k)} \to CC_\bullet(A)[1].$$

Represent elements in $\mathfrak{gl}_\infty(A)$ as $a \otimes m$, where $a \in A$, and $m \in \mathfrak{gl}_\infty(k)$ is a matrix with only finite number of non-zero entries. Then we can define γ by

$$\gamma((a_1 \otimes m_1) \wedge \cdots \wedge (a_i \otimes m_i)) = \sum_{\sigma \in \Sigma_i} (\mathsf{sgn}\,\sigma)\,\mathsf{tr}(m_1 \otimes \cdots \otimes a_i) \otimes (a_1 \otimes \cdots \otimes a_i),$$

where $\mathsf{sgn}\,\sigma$ is the sign of the permutation σ. It is elementary to check that this commutes with the differentials and gives an isomorphism (1.2). □

Remark 1.2. In this exposition, I follow B. Feigin and B. Tsygan's "Additive K-theory". The complex $CC_\bullet(A)$ was discovered independently by A. Connes and B. Tsygan in about 1981-1982. For Tsygan, this came out of the computation above. I don't how Connes discovered it (it looks like something out of thin air, a stroke of genius). The computation of Loday and Quillen appeared slightly later, and used Connes' results.

1.2 Cyclic homology – definitions.

The computation in the last subsection gave a rather strange complex $CC_\bullet(A)$. How does one interpret it? First of all, if one drops the cyclic group invariants in (1.3), then the resulting complex is very well known.

Definition 1.3. The *Hochschild homology complex* $CH_\bullet(A)$ of an associative algebra A is given by

$$CH_i(A) = A^{\otimes i},$$

with the differential b given by (1.4). The *Hochschild homology* $HH_\bullet(A)$ of the algebra A is the homology of the complex $CH_\bullet(A)$.

We note that this definition makes sense for an algebra A flat over an arbitrary commutative base ring k (and in particular, over a field of positive characteristic). Moreover, there is the following important theorem which allows to compute $HH_\bullet(A)$ in the commutative case.

Theorem 1.4 (Hochschild-Kostant-Rosenberg, 1962). *Assume that A is a finitely generated commutative algebra over k, and that $X = \operatorname{Spec} A$ is smooth over k. Then there exist canonical isomorphisms*

$$HH_i(A) \cong \Omega^i(A/k),$$

where $\Omega^i(A/k) = \Lambda^i\Omega(A/k)$ are the modules of differential forms. If k contains $\mathbb{Q}$, the isomorphism can be lifted to a map of complexes $P : CH_\bullet(A) \to \bigoplus_i \Omega^i(A/k)[i]$ given by

$$P(a_0 \otimes a_1 \otimes \cdots \otimes a_i) = \frac{1}{i!} a_0 da_1 \wedge \cdots \wedge a_i$$

for any $a_0 \otimes \cdots \otimes a_i \in A^{\otimes i+1}$. □

(Actually, Hochschild, Kostant and Rosenberg only proved the characteristic 0 part, but the general case is not much more difficult).

It would be desirable to express the complex $CC_\bullet(A)$ in similar terms. To do so, one argues as follows. In char 0, cyclic groups have no higher homology. So, instead of considering coinvariants with respect to the cyclic group, let us consider full group homology, and let us compute it by the standard periodic complex. The result is the following diagram.

(1.5)
$$\begin{tikzcd}
\dots \arrow[r] & A \arrow[r, "\mathrm{id}"] & A \arrow[r, "0"] & A \\
\dots \arrow[r] & A \otimes A \arrow[u, "b"] \arrow[r, "\mathrm{id}+\tau"] & A \otimes A \arrow[u, "b'"] \arrow[r, "\mathrm{id}-\tau"] & A \otimes A \arrow[u, "b"] \\
\dots & \dots \arrow[u, "b"] & \dots \arrow[u, "b'"] & \dots \arrow[u, "b"] \\
\dots \arrow[r] & A^{\otimes n} \arrow[u, "b"] \arrow[r, "\mathrm{id}+\tau+\dots+\tau^{n-1}"] & A^{\otimes n} \arrow[u, "b'"] \arrow[r, "\mathrm{id}-\tau"] & A^{\otimes n} \arrow[u, "b"] \\
& \arrow[u, "b"] & \arrow[u, "b'"] & \arrow[u, "b"]
\end{tikzcd}$$

Here for any $i \geq 1$, $\tau : A^{\otimes i} \to A^{\otimes i}$ is generator of the group $\mathbb{Z}/i\mathbb{Z}$ (the longest permutation multiplied by $(-1)^{i+1}$). The differential b in odd-numbered columns is the differential (1.4), so that all odd-numbered columns are copies of the Hochschild homology complex $CH_{\bullet}(A)$. The differential b' in even-numbered columns is given by

(1.6)
$$b(a_0 \otimes a_1 \otimes \dots \otimes a_i) = \sum_{0 \leq j < i} (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_i.$$

In other words, this is b without the last term. It is easy to see that b' is actually *acyclic*; in fact, it is homotopic to 0, and the contracting homotopy is given by

(1.7)
$$h(a_1 \otimes \dots \otimes a_i) = 1 \otimes a_1 \otimes \dots \otimes a_i.$$

Why we need b' instead of b is explained by the following strange result.

Lemma 1.5. *The diagram* (1.5) *is a bicomplex (that is, the vertical and the horizontal differential anticommute).*

Proof. Direct computation. □

Definition 1.6. The *cyclic homology* $HC_{\bullet}(A)$ of the algebra A is the homology of the total complex of the bicomplex (1.5).

Just as Definition 1.3, this definition makes sense and is valid for any base ring k. If k is a field of characteristic 0, cyclic homology may be computed by the complex $CC_{\bullet}(A)$.

1.3 Cyclic homology – properties.

We note that the explicit contracting homotopy h of (1.7) allows us to write a slightly smaller bicomplex which has the same total homology as (1.5) (this is en elementary exercize in linear algebra). Namely, we can get rid of all the even-numbered columns. The results is the bicomplex

$$
\begin{tikzcd}
 & & & A \\
 & & A \arrow[r, "B"] & A^{\otimes 2} \arrow[u, "b"'] \\
 & A \arrow[r, "B"] & A^{\otimes 2} \arrow[u, "b"'] \arrow[r, "B"] & A^{\otimes 3} \arrow[u, "b"'] \\
A \arrow[r, "B"] & A^{\otimes 2} \arrow[u, "b"'] \arrow[r, "B"] & A^{\otimes 3} \arrow[u, "b"'] \arrow[r, "B"] & A^{\otimes 4}, \arrow[u, "b"'] \\
{} \arrow[u, "b"'] & {} \arrow[u, "b"'] & {} \arrow[u, "b"'] & {} \arrow[u, "b"']
\end{tikzcd}
\tag{1.8}
$$

with the horizontal differential $B : A^{\otimes n} \to A^{\otimes(n+1)}$ given by

$$B = (\mathsf{id} - \tau) \circ h \circ (\mathsf{id} + \tau + \cdots + \tau^{n-1}).$$

This bicomplex allows for comparison with the Hochschild-Kostant-Rosenberg Theorem, with the following result.

Lemma 1.7. *In the assumptions of HKR Theorem, assume that k contains $\mathbb{Q}$. Then we have*

$$P \circ B = d,$$

where d is the de Rham differential.

Proof. Direct computation. □

By the HKR Theorem, we also have $P \circ b = 0$, so that in the assumptions of the Lemma, the complex (1.8) is quasiisomorphic to the bicomplex

$$
\begin{tikzcd}
 & & & A \\
 & & A \arrow[r, "d"] & \Omega^1_A \arrow[u, "0"'] \\
 & A \arrow[r, "d"] & \Omega^1_A \arrow[u, "0"'] \arrow[r, "d"] & \Omega^2_A \arrow[u, "0"'] \\
A \arrow[r, "d"] & \Omega^1_A \arrow[u, "0"'] \arrow[r, "d"] & \Omega^2_A \arrow[u, "0"'] \arrow[r, "d"] & \Omega^3_A \arrow[u, "0"'] \\
{} \arrow[u, "0"'] & {} \arrow[u, "0"'] & {} \arrow[u, "0"'] & {} \arrow[u, "0"']
\end{tikzcd}
$$

The differential B is called the *Connes-Tsygan* differential, or the *Rinehart* differential; it lifts the de Rham differential to the general non-commutative setting.

Remark 1.8. G. Rinehart was a student of Hochschild who was given the problem of finding the expression for de Rham differential in terms of Hochschild homology (this was early 1960es, right after the HKR Theorem). The indended application was to non-smooth commutative algebras A.

He solved the problem and published it in 1963, in the case of non-smooth commutative algebra A. After that he didn't publish much, and actually died quite young. His formula didn't use commutativity at all; at the time, no-one cared. This history came to light only after B was rediscovered by Connes and Tsygan. Ironically, in the case of non-smooth commutative algebras, cyclic homology has not been really useful (many important results require A to have finite homological dimension).

On obvious feature of the bicomplexes (1.5) and (1.8) is that they are periodic – shifting say (1.5) by two columns to the left is an endomorphism, and its cone is quasiisomorphic to $CH_\bullet(A)$. This gives the so-called *Connes' long exact sequence*

$$HH_\bullet(A) \longrightarrow HC_\bullet(A) \xrightarrow{u} HC_{\bullet-2} \longrightarrow ,$$

where u is the periodicity map. Moreover, we can extend (1.5) also to the right; the result is an infinite bicomplex, and the cohomology of its total complex is called the *periodic cyclic homology* of the algebra A and denote $HP_\bullet(A)$. Note that here there is an ambiguity in taking the total complex – we can either take the sum or the product of the terms. We take the product (if we take the sum, the result will trivial in char 0). The third possibility is to extend (1.5) to the right but not to the left – this gives the so-called *negative cyclic homology* $HC_\bullet^-(A)$. In all these procedures, one can of course use (1.8) instead of (1.5) – then u shifts by one column to the left and by one row downward.

Lemma 1.7 looks especially nice when applied to $HP_\bullet(A)$ – it gives an isomorphism

$$HP_\bullet(A) \cong H^\bullet_{DR}(\operatorname{Spec} A)((u)),$$

where $H^\bullet_{DR}(-)$ stands for de Rham cohomology, and $((u))$ stands for "formal Laurent series in an indeterminate u of degree 2". Thus we can recover the de Rham cohomology of $X = \operatorname{Spec} A$ from the periodic cyclic homology $HP_\bullet(A)$, with one note: because of periodicity, we loose the grading (all that survives is the decomposition into the odd and the even-degree part).

Finally, recall that for a smooth compact algebraic variety X, considering the stupid filtration on the de Rham complex gives the so-called *Hodge-to-de Rham spectral sequence* which starts from Hodge cohomology $H^p(X, \Omega^q)$ and converges to $H^{p+q}_{DR}(X)$. In the non-commutative case, we have exactly the same thing – taking the stupid filtration on (1.8) in the horizontal direction gives spectral sequences

$$HH_\bullet(A)[u^{-1}] \Rightarrow HC_\bullet(A) \qquad HH_\bullet(A)((u)) \Rightarrow HP_\bullet(A). \tag{1.9}$$

It has been recently proved that if A is replaced by a smooth and compact DG algebra $A^\bullet$ over a field k of characteristic 0 which is concentrated in non-negative degrees, then the spectral sequence degenerates.

1.4 Small category interpretation.

Definitions 1.3 and 1.6 are unsatisfactory in many respects. To begin with, while computing homological invariants by means of an explicit complex is often unavoidable, it is unpleasant to have to *define* them in this way.

For Hochschild homology, there is one obvious improvement of the definition. Namely, denote by Δ the category of non-empty finite totally ordered sets; we will denote by $[n] \in \Delta$ the set of integers $\{1, \dots, n\}$. Recall that a *simplicial object* in a category $\mathcal{C}$ is a functor from the opposite category Δ^{opp} to $\mathcal{C}$. If $\mathcal{C}$ is abelian, then by a famous theorem of Dold and Kan, there exists an equivalence

$$\mathsf{DK} : \Delta^{opp}(\mathcal{C}) \cong C_{\geq 0}(\mathcal{C})$$

between the category $\Delta^{opp}(\mathcal{C})$ of simplicial objects in $\mathcal{C}$ and the category $C_{\geq 0}(\mathcal{C})$ of complexes in $\mathcal{C}$ concentrated in non-negative homological degrees. Moreover, for any $M \in \Delta^{opp}(\mathcal{C})$, the *standard complex* $M_{\bullet}$ is given by $M_i = M([i+1])$, with the differential $d : M_i \to M_{i-1}$ given by

$$d = \sum_{1 \leq j \leq i} (-1)^j d_j, \tag{1.10}$$

where d_j is the "face map" corresponding to the injective map $[i-1] \to [i]$ whose image does not contain j. The standard complex $M_{\bullet}$ is *not* isomorphic to $\mathsf{DK}(M)$, but there is a natural quasiisomorphism between.

Now, our formula (1.4) for the differential b is obviously of the form (1.10), and it is not difficult to see that the Hochschild homology complex $CH_{\bullet}(A)$ is the standard complex of a canonical simplicial k-module $A_{\sharp} \in \Delta^{opp}(k\text{-}\mathsf{mod})$ associated to the algebra A.

It was Alain Connes who figured out how to interpret the cyclic homology complex (1.5) in a similar way. To do this, recall that for any small category I, the category $\mathrm{Fun}(I, k)$ of functors from I to k-mod is abelian, and the direct limit functor from $\mathrm{Fun}(I, k\text{-}\mathsf{mod})$ to k-mod is right-exact; taking its derived functors, we obtain the *homology of the category* I with coefficients in some $E \in \mathrm{Fun}(I, k\text{-}\mathsf{mod})$:

$$H_{\bullet}(I, E) = L^{\bullet} \varinjlim_{I} E.$$

If $I = \Delta^{opp}$, so that E is a simplicial k-module, then $H_{\bullet}(\Delta^{opp}, E)$ can be computed by the standard complex. In the case $E = A_{\sharp}$, this gives

$$HH_{\bullet}(A) \cong H_{\bullet}(\Delta^{opp}, A_{\sharp}).$$

Connes' idea is to extend this to $HC_{\bullet}(A)$ by extending the category Δ^{opp}. Namely, he introduced a special small category known as *the cyclic category* and denoted by Λ. Objects $[n]$ of Λ are indexed by positive integers n, just as for Δ^{opp}. Maps between $[n]$ and $[m]$ can be defined in various equivalent ways; for example, there is the following topological description.

- The object $[n]$ is thought of as a "wheel" – the circle S^1 with n distinct marked points, called *vertices* (equivalently, a circle with a fixed finite cellular decomposition). A continuous map $f : [n] \to [m]$ is *good* if it sends marked points to marked points, has degree 1, and is *monotonous* in the following sense: for any connected interval $[a, b] \subset S^1$, the preimage $f^{-1}([a, b]) \subset S^1$ is connected. Morphisms from $[n]$ to $[m]$ in the category Λ are homotopy classes of good maps $f : [n] \to [m]$.

There are also combinatorial descriptions of the category Λ, or explicit descriptions in terms of generators and relations. It will be convenient to denote the set of vertices of an object $[n] \in \Lambda$ by $V([n])$. We will also denote by $\sigma : [n] \to [n]$ the rotation of the wheel by $\frac{2\pi}{n}$.

A moment's reflection shows that for any $[n], [m] \in \Lambda$, the set of maps $\Lambda([n], [m])$ is a finite set. Moreover, the category Λ naturally contains Δ^{opp}. Namely, fix an element $v \in V([n])$ for any any object $[n]$, and consider only those maps which send the fixed vertex to the fixed vertex; this defines a subcategory in Λ equivalent to Δ^{opp}.

We now note that for any associative unital algebra A, the simplicial k-module $A_{\sharp}$ extends to a functor from Λ to k-mod (such functors are called *cyclic k-modules*). By abuse of notation, we will still denote the extended functor by $A_{\sharp}$, and we define it as follows. For any $[n] \in \Lambda$, $A_{\#}([n]) = A^{\otimes n}$, where we think of the factors A in the tensor product as being numbered by vertices $v \in V([n])$, and for any map $f : [n] \to [m]$, the corresponding map $A_{\#}(f) : A^{\otimes n} \to A^{\otimes m}$ is given by

$$A_{\#}(f)\left(\bigotimes_{i \in V([n])} a_i\right) = \bigotimes_{j \in V([m])} \prod_{i \in f^{-1}(j)} a_i. \tag{1.11}$$

We note that for any $j \in V([m])$, the finite set $f^{-1}(j)$ has a natural total order given by the clockwise order on the circle S^1. Thus, although A need not be commutative, the product in the right-hand side is well-defined. If $f^{-1}(j)$ is empty for some $j \in V([m])$, then the right-hand side involves a product numbered by the empty set; this is defined to be the unity element $1 \in A$.

Proposition 1.9. *We have a natural functorial isomorphism*

$$HC_\bullet(A) \cong H_\bullet(\Lambda, A_\sharp). \tag{1.12}$$

Proof. We first note that the definition of the bicomplex (1.5) only uses those maps between tensor powers $A^{\otimes \bullet}$ that appear as $A_\sharp(f)$ for various maps f in the category Λ. Thus an analogous complex can be defined for an arbitrary $E \in \operatorname{Fun}(\Lambda, k)$. Denote its homology by $HC_\bullet(E)$; all we need to do is to construct an isomorphism $HC_\bullet(E) \cong H_\bullet(\Lambda, E)$ which is functorial in E. It is easy to construct a comparison map

$$\rho : HC_0(E) \to \lim_{\overset{\Lambda}{\to}} E.$$

Homology is a derived functor, and $HC_\bullet(E)$ is obviously a δ-functor in the sense of Grothendieck (that is, it has the homology long exact sequence for any short exact sequence in $\operatorname{Fun}(\Lambda, k)$). Therefore the map ρ canonically extends to a map (1.12). Then by the long exact sequence, it suffices to prove that this map is an isomorphism for a set of projective generators of the abelian category $\operatorname{Fun}(\Lambda, k)$. For example, for any small category I, a good set of projective generators of the category $\operatorname{Fun}(I, k)$ is given by the representable functors k_i^I, $i \in I$ given by

$$k_i^I(i') = k[I(i, i')].$$

Thus we may restrict our attention to $E \in \operatorname{Fun}(\Lambda, k)$ of the form $E = k_{[n]}^\Lambda$, $[n] \in \Lambda$. By general nonsense, $H_\bullet(I, k_i) = k$ for any I and $i \in I$, so that for such E, in the right-hand side of (1.12) we have k in degree 0 and 0 in higher degrees. On the other hand, the action of the cyclic group $\mathbb{Z}/m\mathbb{Z}$ generated by the rotation $\sigma \in \Lambda([m],[m])$ on $\Lambda([n],[m])$ is obviously stabilizer-free, and we have

$$\Lambda([n],[m])/\tau \cong \Delta^{opp}([n],[m])$$

– every $f : [n] \to [n]$ can be uniquely decomposed into a map sending the fixed vertex to the fixed vertex, and a rotation. The rows of the complex (1.5) compute

$$H_\bullet(\mathbb{Z}/m\mathbb{Z}, k_{[n]}([m])) \cong k\left[\Delta^{opp}([n],[m])\right],$$

and the first term in the corresponding spectral sequence is the standard complex for $k_{[n]}^\Delta \in \operatorname{Fun}(\Delta^{opp}, k)$. Therefore this complex computes $H_\bullet(\Delta^{opp}, k_{[n]}^{\Delta^{opp}})$. This is again k in degree 0 and 0 in higher degrees. □

1.5 Morita-invariance.

The definition of $HH_\bullet(A)$ and $HC_\bullet(A)$ in terms of homology of small categories is still unsatifactory; in particular, it is not clear whether either of them is Morita-invariant. For Hochschild homology, there is a well-known alternative definition which makes Morita-invariance obvious.

Consider the category A-$\mathsf{bimod} = (A^{opp} \otimes A)$-$\mathsf{mod}$ of A-bimodules. This is a unital monoidal category, with the unit object A (the monoidal structure is given by the tensor product $\otimes_A$). Let $\mathsf{tr} : A\text{-}\mathsf{bimod} \to k\text{-}\mathsf{mod}$ be the functor given by

$$\mathsf{tr}(M) = M/\{am - ma \mid a \in A, m \in M\}.$$

Equivalently, we can define

$$\mathsf{tr}(M) = A \otimes_{A^{opp} \otimes A} M,$$

where A in the right-hand side is understood as a *right* $A^{opp} \otimes A$-modules. The functor tr is obviously right-exact, so we can consider it derived functors.

Lemma 1.10. *We have* $HH_{\bullet}(A) \cong L^{\bullet}\,\mathsf{tr}(A)$.

Proof. Consider the complex $\langle A^{\otimes \bullet}, b' \rangle$ which appears in even-numbered columns of (1.5). This complex is acyclic. All its terms are A-bimodules in a natural way, and b' is an A-bimodule map. Therefore $\langle A^{\otimes \geq 2}, b' \rangle$ gives a free resolution of the zero term A (this is the bar resolution). Applying tr to this resolution gives the Hochschild homology complex $CH_{\bullet}(A)$. ☐

Now, if we have two algebras A, B and an equivalence $\pi : A\text{-}\mathsf{mod} \cong B\text{-}\mathsf{mod}$, then let $P = \pi(A) \in B\text{-}\mathsf{mod}$ be the image of the free A-module A. Since $\operatorname{End}_B(P) = \operatorname{End}_A(A) = A^{opp}$, P is actually a module over $A^{opp} \otimes B$, and the equivalence is given by

$$\pi(M) \cong P \otimes_A B.$$

Analogously, the inverse equivalence is induced by a $B^{opp} \otimes A$-module P^{opp}. Then $P \otimes P^{opp}$ identifies the categories of bimodules, and this is compatible with the monoidal structures, the unit objects, and the functor tr. Hence the derived functors $L^{\bullet}\,\mathsf{tr}$ are also identified by π, so that we have $HH_{\bullet}(A) \cong HH_{\bullet}(B)$.

In the case of cyclic homology, we do not have such a simple alternative description, and the one we have is obviously *not* Morita-invariant on the level of cyclic k-modules – in the assumptions above, the objects $A_{\sharp}$ and $B_{\sharp}$ are usually different. What we have to do is to show that they become isomorphic after we apply $H_{\bullet}(\Lambda, -)$.

One way to do it is to re-interpret $A_{\sharp}$ as something with more structure than just a cyclic k-module – this is an approach described in my paper arXiv:math/0702068, and I want to give a brief sketch of it here. The main observation is that for $n = 1$, $A_{\sharp}([1]) = A$ is naturally an A-bimodule. Analogously, for any $n \geq 1$, $A^{\otimes n}$ is a bimodule over the algebra $A^{\otimes n}$.

What about the transition maps $A_{\sharp}(f)$ for various maps $f : [n] \to [m]$ in the category Λ? The idea is to associate to any such map a right-exact functor $f_* : A^{\otimes n}\text{-}\mathsf{bimod} \to A^{\otimes m}$. To do this, note that for any A-bimodules $M_1, \dots, M_n \in A\text{-}\mathsf{bimod}$ which are flat over k, we have a natural $A^{\otimes n}$-bimodule $M_1 \boxtimes \cdots \boxtimes M_n$; by abuse of notation, denote

$$M_1 \boxtimes \cdots \boxtimes M_n = \bigotimes_{1 \leq j \leq n} M_j,$$

let the integers $1, \dots, n$ number the vertices in $V([n])$, and set

$$f_* \left(\bigotimes_{j \in V([n])} M_j \right) = \bigotimes_{j \in V([m])} \prod_{i \in f^{-1}(j)} M_i,$$

where $\prod$ stands for tensor product over A. This is exactly the same formula as in (1.11), but it is now applied to the monoidal category $A\text{-}\mathsf{bimod}$ rather that to the algebra A. Since $A\text{-}\mathsf{bimod}$ is a unital monoidal category, and the sets $f^{-1}(j)$ are totally ordered, we again have no problem in defining the iterated tensor product. It is also not difficult to check that bimodules of the form $M_1 \boxtimes \cdots \boxtimes M_n$ generate the category $A^{\otimes n}\text{-}\mathsf{bimod}$ in a suitable sense, and the functor f_* canonically extends to a right-exact functor on the whole $A^{\otimes n}\text{-}\mathsf{bimod}$. For a composable pair of morphisms f, g, we have a natural isomorphism $f_* \circ g_* \cong (f \circ g)_*$.

We now introduce the following.

Definition 1.11. A *cyclic A-bimodule $M_\bullet$* is a collection of

(i) an $A^{\otimes n}$-bimodule M_n for any $[n] \in \Lambda$, and

(ii) a map $f_*(M_n) \to M_m$ for every map $f : [n] \to [m]$ in Λ,

such that for any composable pair of maps $f : [n] \to [m]$, $g : [m] \to [l]$, the diagram

$$\begin{array}{ccc} (g_* \circ f_*)(M_n) & \longrightarrow & g_*(M_m) \\ \downarrow & & \downarrow \\ (g \circ f)_*(M_n) & \longrightarrow & M_l \end{array}$$

is commutative.

One checks that the category $A\text{-}\mathsf{bimod}^\Lambda$ of cyclic A-bimodules is abelian (this is not difficult, the main point is that the functors f_* are right-exact). The object $A_\sharp$ has a natural structure of a cyclic A-bimodule. Moreover, for any $[n] \in \Lambda$, let $\mathsf{tr}_n : A^{\otimes n}\text{-}\mathsf{bimod} \to k\text{-}\mathsf{mod}$ be the functor defined by

$$\mathsf{tr}_n(M) = M/\{am - m\sigma(a) \mid a \in A^{\otimes n}, m \in M\}, \tag{1.13}$$

where $\sigma : A^{\otimes n} \to A^{\otimes n}$ is the longest cyclic permutation. It is easy to see that for $M = M_1 \boxtimes \cdots \boxtimes M_n$, we have a natural isomorphism

$$\mathsf{tr}_n(M) \cong \mathsf{tr}(M_1 \otimes_A \cdots \otimes_A M_n),$$

and this isomorphism is compatible with a cyclic permutation of $M_1, \dots, M_n$ – it is this trace-like property of the functor tr which motivates our notation. More generally, the functors tr_n are compatible with all the transition functors f_*, and taken together, the functors tr_n give a right-exact functor

$$\mathsf{tr} : A\text{-}\mathsf{bimod}^\Lambda \to \operatorname{Fun}(\Lambda, k),$$

so that we can consider its derived functors $L^\bullet \mathsf{tr}$. Here is then the main result of the paper arXiv:math/0702068.

Proposition 1.12. *There exists a canonical isomorphism of functors*

$$HC_\bullet(A) \cong H_\bullet(\Lambda, L^\bullet \mathsf{tr}(A_\sharp)).$$

The proof is again not difficult but a bit technical, so I refer an interested reader to the original paper (or to my Tokyo lectures).

At a first glance, the proposition does not give much: to obtain cyclic homology, we still have to take the homology of the category Λ. However, unlike $A_\sharp \in \operatorname{Fun}(\Lambda, k)$, the object $L^\bullet \mathsf{tr}(A_\sharp)$ in the derived category $\mathcal{D}(\Lambda, k)$ of the category of cyclic k-modules is Morita-invariant. Indeed, an equivalence $\pi : A\text{-}\mathsf{mod} \cong B\text{-}\mathsf{mod}$ represented by an $A^{opp} \otimes B$-module P induces equivalences $A^{\otimes n}\text{-}\mathsf{bimod} \cong B^{\otimes n}\text{-}\mathsf{bimod}$, $A\text{-}\mathsf{bimod}^\Lambda \cong B\text{-}\mathsf{bimod}^\Lambda$, and one easily checks that this equivalence is compatible with the trace functor tr and sends $A_\sharp$ to $B_\sharp$. Thus we obtain a canonical isomorphism $HC_\bullet(A) \cong HC_\bullet(B)$.

1.6 Cyclic homology as TQFT.

It is interesting to note the following property of the object $L^{\bullet}\,\mathsf{tr}(A_\sharp)$ of Proposition 1.12: for any map $f : [n] \to [m]$ in Λ, the corresponding transition map

$$L^{\bullet}\,\mathsf{tr}(A_\sharp)([n]) \to L^{\bullet}\,\mathsf{tr}(A_\sharp)([m])$$

is a quasiisomorphism. In fact, by general nonsense we have

$$L^{\bullet}\,\mathsf{tr}(A_\sharp)([n]) \cong L^{\bullet}\,\mathsf{tr}_n(A^{\otimes n}),$$

so that naively, one would expect that $L^{\bullet}\,\mathsf{tr}(A_\sharp)([n]) \cong HH_{\bullet}(A^{\otimes n})$. However, $\mathsf{tr}_n : A^{\otimes n}\text{-}\mathsf{bimod} \to k\text{-}\mathsf{mod}$ is not the same as the trace functor for the algebra $A^{\otimes n}$; the reason is the twist by σ in (1.13). In fact, for every $[n] \in \Lambda$ and any $i \geq 0$, we have

$$L^{i}\,\mathsf{tr}_n(A^{\otimes n}) \cong HH_{\bullet}(A).$$

Thus $L^{\bullet}\,\mathsf{tr}(A_\sharp) \subset \mathcal{D}(\Lambda, k)$ actually lies inside the full subcategory

$$\mathcal{D}_{const}(\Lambda, k) \subset \mathcal{D}(\Lambda, k)$$

spanned by objects $E \in \mathcal{D}(\Lambda, k)$ which are "locally constant" in the sense that $E(f)$ is a quasiisomorphism for any $f : [n] \to [m]$.

If a locally constant object $E \in \mathcal{D}(\Lambda, k)$ is concentrated in a single homological degree, then it is simply constant: $E \cong \rho^* M$ for some k-module M, where $\rho : \Lambda \to \mathsf{pt}$ is the tautological projection. Analogously, for any $E_{\bullet} \in \mathcal{D}(\Lambda, k)$, the corresponding functor $\widetilde{E_{\bullet}} \in \operatorname{Fun}(\Lambda, \mathcal{D}(k\text{-}\mathsf{mod}))$ is constant; in the case $E_{\bullet} = L^{\bullet}\,\mathsf{tr}(A_\sharp)$, we have

$$\widetilde{E_{\bullet}} \cong \rho^* CH_{\bullet}(A).$$

However, an object of the derived category of functors contains strictly more information than a functor into the derived category. It is this extra information that remembers the Connes-Tsygan differential B and the Hodge-to-de Rham spectral sequence (1.9).

Here are some alternative equivalent descriptions of the category $\mathcal{D}_{const}(\Lambda, k)$.

(i) Let $\mathcal{D}\mathsf{F}(k\text{-}\mathsf{mod})$ be the filtered derived category of the category of k-modules, – that is, the category obtained from the category of filtered complexes of k-modules by inverting the quasiisomorphisms which are strictly compatible with the filtration, – and let $\mathcal{D}\mathsf{F}^{per}(k\text{-}\mathsf{mod})$ be its periodic version – namely, the category of filtered complexes $K_{\bullet}$ of k-modules equiped with a quasiisomorphism $K_{\bullet} \cong K_{\bullet}[2](1)$, where (1) means the renumbering of the filtration by 1. Then $\mathcal{D}_{const}(\Lambda, k) \cong \mathcal{D}\mathsf{F}(k\text{-}\mathsf{mod})$. The equivalence sends $E \in \mathcal{D}_{const}(\Lambda, k)$ to $H_{\bullet}(\Lambda, E)$ with the filtration $F^{\bullet}$ induced by the spectral sequence (1.9).

(ii) $\mathcal{D}_{const}(\Lambda, k)$ is also equivalent to the derived category of complexes of sheaves of k-modules on $\mathbb{C}\mathbf{P}^{\infty}$ with locally constant homology sheaves.

(iii) Consider the equivariant derived category $\mathcal{D}_{U(1)}(\mathsf{pt})$ of $U(1)$-equivariant sheaves of k-modules on a point. Then $\mathcal{D}_{const}(\Lambda, k)$ is equivalent to the full triangulated subcategory spanned by the trivial object k.

Informally, one can treat an object $E_{\bullet} \in \mathcal{D}_{const}(\Lambda, k)$ as a "topological quantum field theory in dim 1" – one associates a complex $E_{\bullet}([n])$ to any compact 1-manifold equipped with a cellular decomposition, and a quasiisomorphism $E(f)$ for any "good" map f between such manifolds, and this is "lifted to the derived category level" in the appropriate sense. Such a structure can be treated as an additional structure on the complex $E_{\bullet}([1])$. However, all the non-triviality is really contained in the "lifting to the derived category level". In the equivariant derived category interpretation of $\mathcal{D}_{const}(\Lambda, k)$, one can also say that cyclic homology appears because of "$U(1)$-action on $HH_\bullet(A)$" (but again, this has to be understood in the derived category sense).

From this perspective, it seems natural to try to expand the definition of a "good" map and consider maps between circles with marked point which do not correspond to maps in the category Λ. I know of only such result: étale degree-p coverings $U(1) \to U(1)$ turn out to be very helpful in the study of cyclic homology over a field of characteristic p, since they control the so-called *Cartier isomorphism*, a version of the Frobenius action on de Rham cohomology. However, this is only useful in positive characteristic, or over $\mathbb{Z}_p$.

1.7 Moduli space of perfect objects.

Let me now return to what I started with: the Toën-Vaquié moduli space of perfect objects in a derived category $\mathcal{D}(A^\bullet)$.

First of all, let me mention that all the today's results and constructions immediately generalize to DG algebras: all one needs to do is to plug in a associative DG algebra $A^\bullet$ instead of an associative algebra A (and maybe take the total complex of a tricomplex instead of a bicomplex whenever necessary). This also holds for the results about Morita-invariance: although there may exist an equivalence $\mathcal{D}(A^\bullet) \cong \mathcal{D}(B^\bullet)$ between derived categories of two DG algebras which is not induced by a kernel $P \in \mathcal{D}(A^{\bullet opp} \otimes B^\bullet)$, such an equivalence is "bad" anyway – it is only the functors which are given by kernels that should be considered as maps in non-commutative geometry. To sum up: for any DG algebra $A^\bullet$, we have functorial homology groups $HH_\bullet(A^\bullet)$, $HC_\bullet(A^\bullet)$, $HP_\bullet(A^\bullet)$, $HC^-_\bullet(A^\bullet)$; all these groups are derived Morita-invariant, and they are related by the same exact and spectral sequences as in the algebra case.

Now, as I have mentioned earlier, for any saturated DG algebra $A^\bullet$, Toën and Vaquier define the "moduli space of perfect DG $A^\bullet$-modules", denoted by $\mathcal{M}(A^\bullet)$. This is not an algebraic variety but a more complicated gadget called "smooth $\mathcal{D}^-$-stack" (subject of the so-called "derived algebraic geometry"). However, it is close enough to usual algebraic varieties so that it has various types of cohomology that algebraic varieties have. In particular, one can define the de Rham cohomology $H^\bullet_{DR}(\mathcal{M}(A^\bullet))$. Here is the main conjecture.

Conjecture 1.13. *Assume that k is a field of characteristic 0. Then there exists a functor $\overline{HP}_\bullet(-)$ from the category of DG algebras over k up to Morita-equivalence to $\mathcal{D}(k)$ such that*

(i) *$\overline{HP}_\bullet(-)$ is equipped with a functorial periodicity map $u : \overline{HP}_\bullet(-) \to \overline{HP}_{\bullet-2}(-)$ and a functorial map $\overline{HP}_\bullet(-) \to HP_\bullet(-)$ of $k[u]$-modules,*

(ii) *the induced map*

$$\overline{HP}_\bullet(A^\bullet)(u^{-1}) = \lim_{\substack{\to \\ u}} \overline{HP}_\bullet(A^\bullet) \to HP_\bullet(A^\bullet) \tag{1.14}$$

is a quasiisomorphism, and

(iii) *there exists a functorial isomorphism*

$$H^\bullet_{DR}(\mathcal{M}(A^\bullet)) \cong S^\bullet\left(\overline{HP}_\bullet(A^\bullet)\right)^*$$

for any saturated DG algebra $A^\bullet$.

This conjecture is a slight refinement and/or reformulation of a conjecture made by B. Toën and mentioned by M. Kontsevich and Ya. Soibelman.

We note that the appearance of symmetric power in (iii) is very natural: the moduli space $\mathcal{M}(A^\bullet)$ has a structure of a "commutative h-space" induced by the direct sum on the category $\mathcal{D}(A^\bullet)$, and one expects its cohomology to be a commutative cocommutative Hopf algebra, thus a free commutative coalgebra. The interesting invariants are the primitive elements with respect to the coalgebra structure. In the language of rational homotopy theory, these correspond to homotopy rather than homology of $\mathcal{M}(A^\bullet)$, and it is these groups that are expected to coincide with $\overline{HP}_\bullet(A^\bullet)$. We also note that it is obviously necessary to introduce an axiliary theory $\overline{HP}_\bullet(-)$, since de Rham cohomology is by definition non-trivial only in positive degrees, while $HP_\bullet(-)$ is 2-periodic. The interpretation of the periodicity map u in terms of $\mathcal{M}(A^\bullet)$ is known, at least in some cases, but it is very non-trivial; roughly speaking, it corresponds to the "Bott periodicity element" in algebraic K-theory.

As it happens, when $k = \mathbb{C}$, the moduli space $\mathcal{M}(A^\bullet)$ also has what one can call "Betti cohomology" – that is, one can define the underlying analytic space of $\mathcal{M}(A^\bullet)$ and consider its cohomology in the usual topological sense. There is also a comparison theorem which says that Betti cohomology with coefficients in $\mathbb{C}$ coincides with the de Rham cohomology. As a result, $\overline{HP}_\bullet(A^\bullet)$ acquires a real structure. This real structure can be transported to $HP_\bullet(A^\bullet)$ by the isomorphism (1.14). Then there is the following further conjecture.

Conjecture 1.14. *Assume that $k = \mathbb{C}$ and the DG algebra $A^\bullet$ is saturated. Then the natural map $\overline{HP}_i(A^\bullet) \to HP_i(A^\bullet)$ is injective for any i. Moreover, let*

$$W_i HP_\bullet(A^\bullet) = u^i \overline{HP}_\bullet(A^\bullet) \subset HP_\bullet(A^\bullet),$$

and let $F^\bullet HP_\bullet(A^\bullet)$ be the natural descreasing filtration induced by the Hodge-to-de Rham spectral sequence (1.9)*. Then for any integer i, the triple*

$$\langle HP_i(A^\bullet), F^\bullet, W_\bullet \rangle$$

together with the real structure on $W_\bullet(A^\bullet)$ is a pure $\mathbb{R}$-Hodge structure of weight $-i$.

At present, we have no evidence for Conjecture 1.14 (properly speaking, it should be called a "hope" rather than a "conjecture"). The only reason to hope is that in the p-adic setting, a p-adic analog of the notion of a Hodge structure does exist on $HP_\bullet(A^\bullet)$. As for Conjecture 1.13, again, there is no direct evidence, except for some related results of Friedlander in the case $D^\bullet(A^\bullet) = \mathcal{D}(X)$ for some smooth projective algebraic variety X (which basically prove a similar statement with de Rham cohomology replaced by Betti cohomology). However, there is the folllowing intriguing analogy with the computation I have started with – namely, with that of $H_\bullet(\mathfrak{gl}_\infty(A), k)$.

For simplicity, let us take $A^\bullet = A$, an associative algebra placed in degree 0, and let us assume that it is finite-dimensional over k. The moduli space $\mathcal{M}(A)$ then parametrizes all finite-length complexes of finitely generated projective A-modules. Let

$$\mathcal{M}^{free}(A) \subset \mathcal{M}(A)$$

be the subspace parametrizing complexes of *free* A-modules, and let

$$\overline{\mathcal{M}}^{free}(A) \subset \mathcal{M}^{free}(A)$$

be the subspace paramterizing those complexes that are actualy concentrated in degree 0. Then $\overline{\mathcal{M}}^{free}(A)$ is easy to describe: this is an Artin stack given by

$$\overline{\mathcal{M}}^{free}(A) = \coprod_n [\mathsf{pt} / GL_n(A)],$$

where $GL_n(A)$ is the algebraic group of $n \times n$-matrices with entries in A. Direct sum of free A-modules induces a structure of a symmetric h-monoid on this stack. Then the first expectation is that passage to $\mathcal{M}^{free}(A)$ corresponds to the "group completion" of this monoid in the sense of algebraic topology, and on the level of de Rham cohomology, we have

$$H^{\bullet}_{DR}(\mathcal{M}^{free}(A)) = \lim_{\leftarrow} H^{\bullet}_{DR}([\mathsf{pt}/GL_n(A)]).$$

Roughly speaking, the right-hand side is the "de Rham cohomology of the stack $[\mathsf{pt}/GL_\infty(A)]$". Passing from $\mathcal{M}^{free}(A)$ to $\mathcal{M}(A)$ corresponds to adding the images of projectors, and the second expectation is that it does not change the de Rham cohomology too much – if we are very lucky, the change is only in degree 0. Thus finally, what we end up with is complex of primitive elements in the de Rham cohomology of the stack $[\mathsf{pt}/GL_\infty(A)]$. We can now replace the stack with its loop space in the sense of algebraic topology – on the space of primitive elements, this corresponds simply to a shift by 1. This reduces the problem to studying the de Rham cohomology

$$H^{\bullet}_{DR}(GL_\infty(A)) = \lim_{\leftarrow} H^{\bullet}_{DR}(GL_n(A)),$$

where $GL_n(A)$ is considered simply as an algebraic variety.

Now, the de Rham complex of an algebraic group G actually coincides with

$$C^{\bullet}(\mathfrak{g}, \mathcal{O}_G),$$

the Chevalley complex of its Lie algebra $\mathfrak{g}$ with coefficients in the algebra $\mathcal{O}_G$ of functions on G. Thus (iii) of Conjecture 1.13 reduces to

$$H^{\bullet}(\mathfrak{gl}_\infty(A), \mathcal{O}_{GL_\infty(A)}) \cong S^*(\overline{HP}_{\bullet}(A))^*.$$

And this is already a formula of the type (1.2). In fact, the only difference is coefficients: in (1.2), the coefficients are simply the trivial $\mathfrak{gl}_\infty(A)$-module k, and here it is replaced with $\mathcal{O}_{GL_\infty(A)}$. It is hoped that, nevertheless, one can still compute the cohomology by some sort of invariant theory procedure, and the result is an expression for $\overline{HP}_{\bullet}(A)$ similar to the complex $\mathrm{CC}_{\bullet}(A)$ for $HC_{\bullet}(A)$.