Lectures 5 and 6: Hochschild cohomology and Deligne conjecture.

1.1 Two definitions of Hochschild cohomology.

As in the last lecture, we start with associative algebras. Fix a base commutative ring k and an associative unital algebra A flat over k. Recall that A-bimod stands for the abelian category of A-bimodules.

Definition 1.1. The *Hochschild cohomology* $HH^{\bullet}(A)$ of the algebra A is given by

$$HH^{\bullet}(A) = \operatorname{Ext}^{\bullet}_{A\text{-}\mathsf{bimod}}(A, A).$$

We note that this definition is manifestly Morita-invariant, since it only depends on the monoidal category A-bimod with its unit object $A \in A$-bimod. Unlike the case of Hochschild homology, no additional data such as a trace functor are needed. Moreover, A appears as a left $A^{opp} \otimes A$-module on both sides in $\operatorname{Ext}^{\bullet}(-,-)$.

In the situation of the HKR Theorem (k contains $\mathbb{Q}$, A is commutative and smooth over k), the argument of Hochschild-Kostant-Rosenberg also provides an identification

$$HH^{\bullet}(A) = H^0(X, \Lambda^{\bullet}\mathcal{T}_X)$$

between Hochschild cohomology classes and the polyvector fields on $X = \operatorname{Spec} A$.

In the general case, one can compute Hochschild cohomology by using the bar resolution. This gives the *Hochschild cohomology complex* $\mathrm{C}H^{\bullet}(A)$,

$$\mathrm{C}H^i(A) = \mathsf{Hom}(A^{\otimes i}, A),$$

consisting of *Hochschild cochains*, – that is, i-linear A-valued polyvector forms on A, – and with the differential δ given by

$$\delta(f)(a_0, \dots, a_i) = a_0 f(a_1, \dots, a_i) - \sum_{0 \leq j < i} (-1)^j f(a_0, \dots, a_j a_{j+1}, \dots, a_i) + (-1)^{i+1} f(a_0, \dots, a_{i-1}) a_i.$$

For example, if $f = a$ is a 0-cochain, then $\delta(f)(b) = ab - ba$, and if $f : A \to A$ is a 1-cochain, then

$$\delta(f)(a, b) = a f(b) - f(ab) + f(a) b.$$

Thus $HH^0(A)$ is the center of the algebra A, Hochschild 1-cocycles are derivations of the algebra A, and $HH^1(A)$ is the space of all derivations modulo the inner ones.

By definition, $HH^{\bullet}(A)$ is an algebra (with respect to Yoneda product). To write down the multiplication in an explicit form, it is useful to notice that there is a second product on $HH^{\bullet}(A)$ induced by the tensor product on A-bimod. Both products commute; – that is, we have

$$(\alpha_1\beta_1) \otimes (\alpha_2\beta_2) = (-1)^{\deg \alpha_2 \deg \beta_1} (\alpha_1 \otimes \alpha_2)(\beta_1 \otimes \beta_2)$$

for any Hochschild cohomology classes α_1, α_2, β_1, β_2. It is an easy and purely formal exercise to deduce from this that both products are graded-commutative and coincide (this is known as the "Eckman-Hilton argument"). The tensor product of two cochains f, g of degrees i, j is given by

$$f_1 f_2(a_1, \dots, a_{i+1}) = f(a_1, \dots, a_i) g(a_{i+1}, \dots, a_{i+j}),$$

and this also can be used to compute the Yoneda product on $HH^{\bullet}(A)$.

However, it is interesting to note that the complex $\mathrm{C}H^\bullet(A)$ has a completely different definition. Namely, for any free k-module V, let $T_\bullet(V)$ be the free graded associative coalgebra generated by V placed in homological degree 1, and let $\mathrm{D}T^\bullet(V)$ be the Lie algebra of coderivations of the coalgebra $T_\bullet(V)$. Since $T_\bullet(A)$ is free, we have

$$\mathrm{D}T^i(V) = \mathsf{Hom}(V^{\otimes i+1}, V).$$

Lemma 1.2. *Assume that 2 is invertible in k. An element $\mu \in DT^1(V) = \mathsf{Hom}(V^{\otimes 2}, V)$ gives an associative product on V if and only if $\{\mu, \mu\} = 0$ with respect to the Lie bracket on $DT^\bullet(V)$.*

Proof. By the Leibnitz rule, the element $\{\mu, \mu\} \in \mathsf{Hom}(V^{\otimes 3}, V)$ is given by

$$\{\mu, \mu\}(v_1, v_2, v_3) = 2(\mu(\mu(v,1,v_2), v_3) - \mu(v_1, \mu(v_2, v_3)));$$

this vanishes exactly when μ is associative. □

Now take $V = A$, with $\mu : A^{\otimes 2} \to A$ being the product map. Then we have

$$\mathrm{C}H^\bullet(A) \cong \mathrm{D}T^{\bullet-1}(A),$$

and one checks easily that the Hochschild differential δ is given by

$$\delta(f) = \{\mu, f\}.$$

In particular, $\mathrm{D}T^\bullet(A)$ with this differential is a DG Lie algebra, so that $HH^\bullet(A)$ is automatically a graded Lie algebra (with the Lie bracket of degree -1). This Lie bracket was discovered by Gerstenhaber, and it is known as *Gerstenhaber bracket*. In the HKR case, the bracket is the so-called *Schouten bracket* of polyvector fields, a generalization of the usual bracket of vector fields.

1.2 Relation to deformation theory.

In degree 1 – or in fact, on the whole space of Hochschild 1-cocycles – the Gerstenhaber bracket is just the usual commutator bracket of derivations. More interestingly, we can use the Gertenhabe bracket to interpret the second Hochschild cohomology group $HH^2(A)$. Namely, assume given a k-algebra $\widetilde{k}$ which is flat and finitely generated as a k-module; assume also that $\widetilde{k}$ is equipped with an augmentation map $\widetilde{k} \to k$ whose kernel $\mathfrak{m} \subset \widetilde{k}$ is a nilpotent ideal. By a *$\widetilde{k}$-deformation* of the algebra A we will understand a flat $\widetilde{k}$-algebra $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/\mathfrak{m}\widetilde{A} \cong A$. If we further assume that A is not only flat but also projective as a k-module, then $\widetilde{A}$ is projective as a $\widetilde{k}$-module, and we can choose a $\widetilde{k}$-module isomorphism

$$\widetilde{A} \cong A \otimes_k \widetilde{k}. \tag{1.1}$$

Under this isomorphism, the multiplication in $\widetilde{A}$ is given by an element $\mu' \in \mathrm{D}T^1(A) \otimes_k \widetilde{k}$ of the form $\mu' = \mu + \gamma$, where μ is the multiplication in A, and

$$\gamma \in \mathrm{D}T^2(A) \otimes_k \mathfrak{m}$$

is a certain element satisfying the *Maurer-Cartain equation*

$$\delta(\gamma) = \frac{1}{2}\{\gamma, \gamma\}. \tag{1.2}$$

Thus pairs of a deformation $\widetilde{A}$ and an isomorphism (1.1) are in one-to-one correspondence with degree-1 $\mathfrak{m}$-valued solutions of the Maurer-Cartain equation in the DG Lie algebra $\mathrm{D}T^\bullet(A)$.

Moreover, assume that k contains $\mathbb{Q}$. Then the Lie algebra $\mathrm{D}T^0(A) \otimes \mathfrak{m}$ acts naturally on the set $\mathsf{MC}(\mathfrak{m})$ of such solutions, and by our assumptions, this Lie algebra is unipotent, so that the action extends to the action of the corresponding unipotent algebraic group $\mathsf{Aut}(\mathfrak{m})$. Two solution $\gamma, \gamma' \in \mathsf{MC}(\mathfrak{m})$ correspond to the same deformation $\widetilde{A}$ (with different isomorphisms (1.1)) if and only if they lie in the same $\mathsf{Aut}(\mathfrak{m})$-orbit. The set of isomorphism classes of $\widetilde{k}$-deformations is then naturally identified with the quotient

$$\mathsf{MC}(\mathfrak{m}) / \mathsf{Aut}(\mathfrak{m}).$$

This can also be interpreted as a description of the groupoid of deformations.

In the particular case when the ideal $\mathfrak{m}$ is square-zero, the right-hand side of (1.2) vanishes, the group $\mathsf{Aut}(\mathfrak{m})$ is simply the vector space $\mathrm{D}T^0(A) \otimes_k \mathfrak{m}$ considered as an abelian group, and the result becomes especially simple: isomorphisms classes of $\widetilde{k}$-deformations are in one-to-one correspondence with elements in the set

$$\mathrm{C}H^2_{cl}(A) \otimes_k \mathfrak{m} / \delta(\mathrm{C}H^1(A) \otimes_k \mathfrak{m}) = HH^2(A) \otimes_k \mathfrak{m},$$

where $\mathrm{C}H^2_{cl}(A) \subset \mathrm{C}H^2(A)$ denotes the space of Hochschild 2-cocycles. This is the classic interpretation of Hochschild cohomology classes of degree 2 – they parametrize square-zero deformations of the algebra A.

1.3 Gerstenhaber algebras and the Deligne conjecture.

Two definitions of Hochschild cohomology give rise to two natural operations on $HH^{\bullet}(A)$, the commutative Yoneda product and the Gerstenhaber bracket. It is natural to ask whether there are any compatibility conditions between the two. It turns out that there are, and they are very similar to the compatibility conditions between the product and the Poisson bracket in symplectic geometry. In modern language, this is axiomatized as follows.

Definition 1.3. A *Gerstenhaber algebra* is a graded-commutative algebra $B^{\bullet}$ equipped with a graded Lie bracket $\{-,-\}$ of degree -1 such that

$$\{a, bc\} = \{a, b\}c + (-1)^{\deg b \deg c}\{a, c\}b \tag{1.3}$$

for any $a, b, c \in B^{\bullet}$.

Proposition 1.4. *For any associative algebra A, the Hochschild cohomology $HH^{\bullet}(A)$ equipped with the Yoneda product and the Gerstenhaber bracket satisfies* (1.3)*, so that $HH^{\bullet}(A)$ is a Gerstenhaber algebra in the sense of Definition* 1.3. □

This can be proved, for example, by an easy direct computation (and this is how it was originally proved by Gerstenhaber).

The notion of a Gerstenhaber algebra admits the following very important geometric interpretation. Let D be the unit disc. For any n, let $D^{[n]} = D^n \setminus \mathsf{Diag}$ be the n-fold self product D^n with all the diagonals removed – in other words, $D^{[n]}$ is the space of confugurations of n *distinct* points on D. Then $D^{[n]}$ is a topological space, so we can consider its homology $H_{\bullet}(D^{[n]}, k)$. The symmetric group Σ_n acts on $D^{[n]}$, hence also on its homology.

Lemma 1.5. *For any $n \geq 1$ and any Gerstenhaber algebra $B^{\bullet}$ flat over k, there is a natural Σ_n-invariant map*

$$H_{\bullet}(D^{[n]}, k) \otimes B^{\bullet \otimes n} \to B^{\bullet}. \tag{1.4}$$

I will not prove this lemma (one proof is presented in my Tokyo lectures, but in fact, the statement is very well-known). Let me just illustrate what happens for $n = 2$ and $n = 3$. For $n = 2$, the space $D^{[2]} = D^2 \setminus \mathsf{Diag}$ is homotopy-equivalent to a circle, so that $H_i(D^{[2]}, k)$ is k for $i = 0, 1$ and 0 otherwise. The homology class in degree 0 gives the product, and the class in degree 1 gives the bracket. For $n = 3$, one can check that

$$H_i(D^{[3]}, k) = \begin{cases} k, & i = 0, \\ k^{\oplus 3}, & i = 1, \\ k^{\oplus 2}, & i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

The class in degree 0 again gives the product (since the product is commutative and associative, there is exactly one way to compose it with itself to obtain a map $B^{\bullet \otimes 3} \to B^{\bullet}$). The two classes in degree 2 give maps obtained by composing the bracket with itself; *a priori*, an antisymmetric bracket could induces three linearly independent trinary operations, but the Jacobi identity cuts it down to two. The classes in degree 1 correspond to composing the bracket and product in various ways; again, we have less operations than dictated by symmetry due to the relation (1.3).

In fact, if one sets $\mathsf{Gerst}_n = H_{\bullet}(D^{[n]}, k)$, then composition of the operations can be encoded as various natural operations on the spaces Gerst_n, and this completely defines the structure of a Gerstenhaber algebra (in modern language, the spaces Gerst_n form an *operad*, the notion invented by P. May in the 1970es in the theory of n-fold loop spaces). However, we will not need this in these lectures. Rather, I would prefer to concentrate on a question originally asked by Deligne. Namely, taking for granted the Gerstenhaber algebras structure on the Hochschild cohomology $HH^{\bullet}(A)$, what is the natural structure on the Hochschild complex $CH^{\bullet}(A)$?

A moments' reflection shows that $CH^{\bullet}(A)$ is *not* a "DG Gerstenhaber algebra" (or at least, not in an obvious way) – the Yoneda product is only commutative "up to a coboundary", not on the nose. However, in light of Lemma 1.5, the following seems a good guess.

- Let $C_{\bullet}(D^{[n]}, k)$ be the singular chain complex of the configuration space $D^{[n]}$, and assume given an associative k-algebra A flat over k. Is there a natural map

  $$C_{\bullet}(D^{[n]}, k) \otimes CH^{\bullet}(A)^{\otimes n} \to CH^{\bullet}(A) \tag{1.5}$$

  which induces (1.4) after passing to cohomology?

This is more-or-less what Deligne asked in 1993 (he was more precise and also included the operad structure). Afterwards it became known as "Deligne conjecture". Several wrong proofs of the statement appeared right away, and the question was considered settled. Then in 1998, the question suddenly became very important because it was found to be crucial to D. Tamarkin's short proof of Kontsevich's formality theorem. The mistakes were discovered, and correct proofs were found by several groups of people (McClure and Smith were probably the first, and I also want to mention an influential paper by Kontsevich and Soibelman).

I would like to sketch one construction here. However, before I start, we need to discuss what is the exact statement that we are trying to prove.

Indeed, at the very least, one notes immediately that the complex $C_{\bullet}(D^{[n]}, k)$ is huge, so that there is no hope of constructing directly a map (1.5). Instead, one should treat $C_{\bullet}(D^{[n]}, k)$ as a complex given only "up to quasiisomorphism", in a suitable sense. The usual way to formalize things is again by using the language of operads. One notes that the topological spaces $D^{[n]}$ – or rather, the configuration spaces of "n little subdiscs in a disc" which are homotopy equivalent to them – form an well-known operad of topological spaces called "the operad of little discs" (this is one of the original operads considered by May). Then $C_{\bullet}(D^{[n]}, k)$ also becomes an operad. What people do is construct a different operad, which is on one hand, quasiisomorphic to $C_{\bullet}(D^{[n]}, k)$, and on the other hand, does act naturally on $CH^{\bullet}(A)$. This different operad is usually combinatorial in nature; constructing it amounts to finding good combinatorial models for the spaces $D^{[n]}$ (usually a good cellular decomposition).

However, as it happens, the language of operads is not very well suitable for the problem: while there are simple and nice combinatorial models for the space $D^{[n]}$, they are not compatible with the structure of the operad of little discs. As a result of this, all the proofs of Deligne conjecture have to interpolate between several quasiisomorphic "combinatorial" operads of varying complexity, and none of the papers is shorter than 50 pages. They also use a lot of cellular subdivisions and other technicalities which seem to be irrelevant to the problem.

So, what I would like to sketch here is an alternative approach based on the notion of a "factorization 2-algebra". This notion is partially inspired by factorization algebras and chiral algebras of Beilinson and Drinfeld, another group of objects for whom the operadic formalism does not work too well.

1.4 Factorization 2-algebras: definition.

From now on, it will be convenient to mark the points constituting a configuration in D. Thus for any finite set S, denote by D^S the space of all maps $\kappa : S \to D$, with its natural topology. For any map $f : S_1 \to S_2$ between finite sets, we have a corresponding map $\iota_f : D^{S_2} \to D^{S_1}$ which sends $\kappa : S_2 \to D$ to $\kappa \circ f : S_1 \to D$. If f is surjective, then ι_f is a closed embedding which identifies D^{S_2} with a diagonal in D^{S_1}.

Moreover, for any surjective map $f : S \to S'$, denote by $D_f \subset D^S$ the subset of maps $\kappa : S \to D$ such that

- for any $s_1, s_2 \in S$ with $f(s_1) \neq f(s_2)$ we have $\kappa(s_1) \neq \kappa(s_2)$.

In other words, D_f is the complement to all the diagonals $f_1^*(D^{S_1}) \subset D^S$, $f_1 : S \to S_1$ such that f does not factor through f_1. In particular, $D_f \subset D^S$ is open; we will denote by $j_f : D_f \hookrightarrow D^S$ the corresponding open embedding. In the extreme case $f = \mathsf{id}$, $D_f = D_{\mathsf{id}} \subset D^S$ is the complement to all the diagonals, and if $f : S \to \mathsf{pt}$ is the map to the point, then D_f is the whole D^S.

Assume given a surjective map $f : S \to S'$, and consider the natural decomposition

$$S = \coprod_{s \in S'} f^{-1}(s).$$

This decomposition induces a homeomorphism

$$D^S \to \prod_{s \in S'} D^{f^{-1}(s)};$$

restricting it to $D_f \subset D^S$, we obtain an open embedding

$$\varphi_f : D_f \to \prod_{s \in S'} D^{f^{-1}(s)}$$

which we will call the *factorization map* associated to f.

Recall that for any topological space X equipped with a good enough stratification, we have the category $\mathrm{Shv}(X, k)$ of constructible sheaves of k-modules on X which are locally constant along the open strata of the stratification. We also have the triangulated category $\mathcal{D}_c(X, k)$ of complexess of sheaves of k-modules with homology sheaves in $\mathrm{Shv}(X, k)$, and a natural comparison functor $\mathcal{D}(\mathrm{Shv}(X, k)) \to \mathcal{D}_c(X, k)$. If the open strata of the stratification have homotopy types $K(\pi, 1)$ for some groups π, then this comparison functor is an equivalence. In particular, the stratification by diagonals on D^S is good enough, so that we have the categories $\mathrm{Shv}(D^S,k)$, $\mathcal{D}_c(D^S,k)$ for any finite S. Moreover, the open strata of the stratification are products of spaces $D^{[m]}$ for various $m \geq 1$, and these spaces are of type $K(\pi,1)$ – in fact, $D^{[m]}$ is the classifying spaces of the pure braid group B_m. Therefore $\mathcal{D}(\mathrm{Shv}(D^S,k)) \cong \mathcal{D}_c(D^S,k)$.

Definition 1.6. A *factorization* 2*-algebra* $B^\bullet$ over k is a collection of the following data:

(i) for any finite set S, a complex $B^\bullet(S)$ in $\mathrm{Shv}(D^S,k)$,

(ii) for any surjective map $f : S \to S'$, a quasiisomorphism

$$B^\bullet(S) \cong \iota_f^! B^\bullet(S'),$$

subject to a natural associativity condition for any pair of composable maps,

(iii) for any surjective map $f : S \to S'$, a quasiisomorphism

$$j_f^* B^\bullet(S) \to \varphi_f^* \bigotimes_{s \in S'} B^\bullet(f^{-1}(s)), \tag{1.6}$$

again subject to a natural associativity condition, where $\otimes$ in the right-hand side actually stands for $\boxtimes$.

The meaning associativity in (ii) is obvious (the functor $\iota_f^!$ is *a priori* defined on the category $\mathcal{D}_c(D^S,k)$, but since $\mathcal{D}(\mathrm{Shv}(D^S,k)) \cong \mathcal{D}_c(D^S,k)$, we may treat it a functor on complexes of constructible sheaves). In (iii), what we mean is the following: given two surjective maps $f : S \to S'$, $f' : S' \to S''$ with the composition $f'' = f' \circ f : S \to S''$, we can first subdivide S with respect to the map f'', and then further subdivide the pieces $f''^{-1}(s)$, $s \in S''$ with respect to the maps $f^s : f''^{-1}(s) \to f'^{-1}(s)$ induced by f – the end result is the same as simply subdividing with respect to f, so that

$$\varphi_f = \left(\prod_{s \in S''} \varphi_{f^s} \right) \circ \varphi_{f''}.$$

The quasiisomorphisms (1.6) should be compatible with this decomposition.

1.5 Factorization 2-algebras: discussion.

We note right away that Definition 1.6 is nearly identical to the definition of a *chiral algebra* on D given by Beilinson and Drinfeld, and analogous to their notion of a *factorization algebra* on D. The crucial difference is that they treat D and D^S as complex varieties, and work with $\mathcal{D}$-modules rather than constructible sheaves. Since the Riemann-Hilbert correspondence only applies to holonomic $\mathcal{D}$-modules with regular singularities, their notion is much more general; and indeed, interesting chiral algebras usually correspond to $\mathcal{D}$-modules which are very far from holonomic. Nevertheless, factorization 2-algebras in the sense of (1.6) do give examples of “DG chiral algebras” in the sense of Beilinson and Drinfeld. This explains the adjective “factorization” in my terminology.

As for “2-algebra”, this is usually used in the literature to denote an “algebra over the DG operad of singular chain complexes of the operad of small discs”, or rather, such an algebra considered “up to a quasiisomorphism”. In other words, it is exactly the structure that the Deligne Conjecture expects to have on the Hochschild cohomology complex $CH^\bullet(A)$ of an associative algebra A. Here I have the following conjecture.

Conjecture 1.7. *There exists a closed model structure on the category of factorization 2-algebras whose weak equivalences are quasiisomorphisms, and the corresponding homotopy category is equivalent to the homotopy category of 2-algebras.*

In other words, factorization 2-algebras considered up to a quasiisomorphism are the same thing as the usual 2-algebras considered up to a quasiisomorphism, so that we simply have a different description of the same notion.

Unlike the conjectures in the last lecture, this one is pretty straightforward — it is more or less clear how to prove it, and it has not been done so far simply because of laziness. The main part is the passage from factorization 2-algebras to 2-algebras in the usual sense. Since I did not even define an operad, leave alone 2-algebras, I cannot explain it fully in these lectures; however, let us see how the action maps (1.5) appear.

Recall that for any topological good space X with an open subset $U \subset X$ and its closed complement $Z \subset X$, a complex $\mathcal{E}^\bullet$ of constructible sheaves on X is defined by the following "gluing data":

(i) complexes of constructible sheaves $\mathcal{E}^\bullet_U = j^*\mathcal{E}$, $\mathcal{E}^\bullet_Z = i^!\mathcal{E}$ on U and Z, and

(ii) a gluing map

$$i^! j_! \mathcal{E}^\bullet_U \to \mathcal{E}^\bullet_Z, \tag{1.7}$$

where $j : U \hookrightarrow X$, $i : Z \hookrightarrow X$ are the embedding maps.

If Z is not the full complement but its proper closed subset, we still have the gluing map (1.7).

Assume given a factorization 2-algebra $B^\bullet$. Then in particular, we have the complex $B^\bullet(S)$ of sheaves in $\operatorname{Shv}(D^S, k)$, where S is the set with n elements. When $n = 1$, $S = \mathsf{pt}$, there is no stratification, so that $B^\bullet(\mathsf{pt})$ is a complex of constant sheaves on the disc D; our 2-algebra $A^\bullet$ corresponding to $B^\bullet$ will be given by $A^\bullet = B^\bullet(\mathsf{pt})[-2]$. Fix some $n \geq 2$, and let $U = D_{\mathsf{id}} = D^{[n]} \subset D^S$, $Z = D \subset D^S$ be the maximal open and the minimal closed strata in the stratification, with the embedding maps $j = j_{\mathsf{id}} : U \hookrightarrow D^S$, $i : D \hookrightarrow D^S$. Then by Definition 1.6 (ii), we have a fixed quasiisomorphism $B^\bullet(S)_Z \cong B^\bullet(\mathsf{pt})$, and by Definition 1.6 (iii), we have a fixed quasiisomorphism

$$B^\bullet(S)_U \cong \varphi^*_{\mathsf{id}} B^\bullet(\mathsf{pt})^{\boxtimes n}.$$

In particular, $B^\bullet(S)_U$ is a complex of constant sheaves. Thus we have a natural quasiisomorphism

$$i^! j_! B^\bullet(S)_U \cong C_\bullet(D^{[n]}, k) \otimes B^\bullet(\mathsf{pt})^{\boxtimes n}[2(1-n)],$$

and the gluing map (1.7) gives the action map (1.5).

1.6 Planar trees.

In order to construct a factorization 2-algebra structure on the Hochschild cohomology complex $CH^\bullet(A)$, we need a combinatorial approximation of the configuration spaces D^S. The situation is analogous to what we did with Hochschild homology in the last lecture: while all the additional structures on $HH_\bullet(A)$ such as the Connes-Tsygan differential can be encoded in a purely geometric notion of a "$U(1)$-action on $HH_\bullet(A)$", in order to construct it, we need the combinatorial category Λ. In the Hochschild homology case, the relevant combinatorics was that of cellular decompositions of a circle. For Hochschild cohomology, we need planar trees.

By a *planar tree* we will understand an unoriented connected graph with no cycles and one distiguished vertex of valency 1 called *the root*, equipped with a cyclic order on the set of edges attached to each vertex. Given such a tree T, we will denote by $V(T)$ the set of all non-root vertices of T, and we will denote by $E(T)$ the set of all edges of T not adjacent to the root. We note that the choice of the root vertex uniquely determines an orientation of the tree: all edges are oriented towards the root. Thus for any vertex $v \in V(T)$ of valency $n+1$, we have n incoming and one outgoing edge.

Given a tree T, we denote by $|T|$ its geometric realization, that is, a CW complex with vertices of T as 0-cells and edges of T as 1-cells. For every planar tree T, $|T|$ can be continuously embedded into the unit disc D so that the root of T goes to $1 \in D$, the rest of $|T|$ is mapped into the interior of the disc, and for every vertex $v \in V(T)$, the given cyclic order on the edges adjacent to v is the clockwise order. Moreover, the set of all such embeddings with its natural topology is contractible, so that the embedding is unique up to a homotopy, and the homotopy is also unique up to a homotopy of higher order, and so on.

Given a tree T and an edge $e \in E(T)$, we may contract e to a vertex and obtain a new tree T^e. The contractions of different edges obviously commute, so that for any n edges $e_1, \dots, e_n \in E(T)$, we have a unique tree $T^{e_1,\dots,e_n}$ obtained by contracting $e_1, \dots, e_n$. By construction, we have a natural map $V(T) \to V(T^{e_1,\dots,e_n})$ and a natural map of realizations $|T| \to |T^{e_1,\dots,e_n}|$.

Assume given a finite set S. By a *tree marked by* S we will understand a planar tree T together with a map $\tau : S \to V(T)$. The vertices in the image of this map are called marked, the other ones are unmarked. A marked tree T is *stable* if every unmarked vertex $v \in V(T) \setminus \tau(S)$ has valency at least 3. Given a stable marked tree T and some edges $e_1, \dots, e_n \in E(T)$, we mark the contraction by composing the map $S \to V(T)$ with the natural map $V(T) \to V(T^{e_1,\dots,e_n})$. This is again a stable marked tree. Moreover, it is easy to check the following.

Lemma 1.8. *For any two trees T, T' stably marked by the same set S, there exists at most one subset $\{e_1, \dots, e_n\} \subset E(T)$ such that $T^{e_1,\dots,e_n} \cong T$.* □

(The main observation for the proof is that removing an edge splits a tree T into two connected components, and an edge is uniquely defined by the corresponding partition of the set $V(T)$.)

By virtue of this Lemma, the collection of all planar trees stably marked by the same finite set S acquires a partial order: we say that $T \geq T'$ if and only if T' can be obtained from T by contraction. We will denote this partially ordered set by T_S. This is our combinatorial model for the configuration space D^S.

For any surjective map $f : S \to S'$, we have an obvious order-preserving map $\iota_f : \mathsf{T}_{S'} \to \mathsf{T}_S$ (a marked tree $\langle T, \tau \rangle$ goes to $\langle T, \tau \circ f \rangle$). Moreover, for any such map f, let $\mathsf{T}_f \subset \mathsf{T}_S$ be the subset of marked trees $T \in \mathsf{T}_S$ such that

- $T \geq f^* T'$ for some $T' \in \mathsf{T}_{\mathsf{id}} \subset \mathsf{T}_{S'}$,

and let $j_f : \mathsf{T}_f \hookrightarrow \mathsf{T}_S$ be the natural embedding. Then for any $s \in S'$ and any $T \in \mathsf{T}_f$, the preimage of s under the contraction map $|T| \to |T'|$ does not depend on the particular choice of the tree T', and defines a planar tree which we denote by $T(s)$. Sending T to the collection $\langle T(s) \rangle$ gives an order-preserving *factorization map*

$$\varphi_f : T_f \to \prod_{s \in S'} T_{f^{-1}(s)}.$$

In our combinatorial model, the maps ι_f and φ_f play the role of the corresponding maps for the configuration spaces D^S. Here is the combinatorial version of Definition 1.6 (the precise meaning of associativity is the same as in Definition 1.6).

Definition 1.9. A T-*algebra* $B^{\bullet}$ over k is a collection of the following data:

(i) for any finite set S, a complex $B^\bullet(S)$ of functors in $\operatorname{Fun}(\mathsf{T}_S, k)$,

(ii) for any surjective map $f : S \to S'$, a quasiisomorphism

$$B^\bullet(S) \cong \iota_f^* B^\bullet(S'),$$

subject to a natural associativity condition for any pair of composable maps,

(iii) for any surjective map $f : S \to S'$, a quasiisomorphism

$$j_f^* B^\bullet(S) \to \varphi_f^* \bigotimes_{s \in S'} B^\bullet(f^{-1}(s)), \tag{1.8}$$

again subject to a natural associativity condition, where $\otimes$ in the right-hand side stands for $\boxtimes$.

Moreover, say that a pair $T \geq T'$ of marked trees in some T_S is *internal* if the corresponding map $|T| \to |T'|$ does not glue together distinct marked points.

Definition 1.10. A T-algebra $B^\bullet$ is called *strict* if for any S and any internal pait $T \geq T'$ of trees in T_S, the natural map $B^\bullet(S)(T) \to B^\bullet(S)(T')$ is a quasiisomorphism.

1.7 The comparison theorem: step one.

As the reader might have guessed already, there is a comparison theorem which says that strict T-algebras "up to a quasiisomorphism" and factorization 2-algebras "up to a quasiisomorphism" are one and the same. To avoid dealing with model structures, I will not give a precise formulation of this, but I will sketch a proof. It goes in two steps.

The first step is purely formal. Assume given a topological space X equipped with a good stratification (to be precise, let us say that we have a Whitney stratification with a finite number of strata). Say that a finite stratification of the unit interval $I = [0,1]$ is *admissible* if all the closed strata are of the form $[b,1]$ for some $b \in [0,1]$. A continous map $\gamma : I \to X$ is *admissible* if so is the induced stratification of the interval I.

Definition 1.11. The *stratified fundamental groupoid* $\pi_1(X)$ is the category whose objects are points $x \in X$, and whose maps from x_0 to x_1 are homotopy classes of admissible maps $\gamma : I \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

This definition slightly abuses the terminology, since the stratified fundamental groupoid is usually not a groupoid: if the points $x_0, x_1 \in X$ lie in open strata X_0^o, X_1^o, then a map from x_1 to x_2 exists if and only if X_1^o lies in the closure of X_o^o. Nevertheless, if there is no stratification – that is, X has exactly one stratum – then $\pi_1(X)$ is the fundamental groupoid of X in the usual sense. In general, isomorphism classes of points $x \in \pi_1(X)$ correspond to strata: two points are isomorphic if and only if they lie in the same open stratum $X^o \subset X$.

Lemma 1.12. *In the assumptions above, there exists a natural equivalence of categories*

$$\operatorname{Shv}(X,k) \cong \operatorname{Fun}(\pi_1(X)^{opp}, k).$$

Sketch of a proof. In the case when X has no stratification, the claim is standard (locally constant sheaves are the same as representations of the fundamental groupoid). Next, consider the case $X = I$ with an admissible stratification. Then $\pi_1(X)$ is just the totally ordered set of strata considered as a category, and the claim is immediate. In the general case, for any $x_0, x_1 \in \pi_1(X)$

and a map from x_0 to x_1 represented by an admissible map $\gamma : I \to X$, any constructible sheaf $\mathcal{E} \in \mathrm{Shv}(X,k)$ gives by restrition a sheaf $\gamma^*\mathcal{E} \in \mathrm{Shv}(I,k)$. Applying the equivalence to I with the induced stratification, we obtain a map $\mathcal{E}_{x_1} \to \mathcal{E}_{x_0}$, where for any $x \in X$, $E_x = i_x^*\mathcal{E}$ is the pullback with respect to the embedding $i_x : \mathsf{pt} \to X$. This is obviously compatible with the compositions, and defines a comparison functor

$$\mathrm{Shv}(X,k) \to \mathrm{Fun}(\pi_1(X)^{opp}, k).$$

To prove that this is an equivalence, one argues by induction on the number of strata. □

At a first glance, the comparison functor in this Lemma uses the gluing maps (1.7), but in fact this is not quite true: the maps go in the wrong direction. However, if the open strata of the stratification are of homotopy type $K(\pi,1)$, we can work with the derived category $\mathcal{D}(\mathrm{Shv}(X,k)) \cong \mathcal{D}_c(X,k)$. Then one can apply Verdier duality, and obtain an equivalence

$$\mathcal{D}(\mathrm{Shv}(X,k)) \cong \mathcal{D}(\pi_1(X),k),$$

or actually, a stronger equivalence of the underlying DG categories. The comparison functor is similar, but a complex $\mathcal{E}_{\bullet}$ of constructible sheaves goes to a functor $\widetilde{\mathcal{E}}_{\bullet}$ which sends $x \in \pi_1(X)$ to $i_x^!\mathcal{E}_{\bullet}$ rather than $i_x^*\mathcal{E}_{\bullet}$ (this explains why we had to use $\iota_f^!$ rather than ι_f^* in Definition 1.6).

As we have noted already, D^S with the stratification by diagonals satisfies all the needed assumptions, so we obtain an equivalence of categories

$$\mathcal{D}(\mathrm{Shv}(D^S,k)) \cong \mathcal{D}(\mathsf{B}^S,k), \tag{1.9}$$

where $\mathsf{B}^S = \pi_1(D^S)$ is the stratified fundamental groupoid of the configuration space D^S.

All the natural maps ι_f, j_f, φ_f of Definition 1.6 are compatible with stratifications, thus induce corresponding maps on the fundamental groupoids B^S. Replacing $\mathrm{Shv}(D^S,k)$ with $\mathrm{Fun}(\mathsf{B}^S,k)$ in Definition 1.6, we obtain a notion of a B-*algebra*, and the equivalences (1.9) show that up to a quasiismorphism, factorization 2-algebras and B-algebras are one and the same.

1.8 The comparison theorem: step two.

The second step is a comparison between B-algebras and T-algebras. Now both sides are combinatorial, and a comparison functor is induced by a collection of functors

$$\mu_S : \mathsf{T}_S \to \mathsf{B}^S, \tag{1.10}$$

compatible with the maps ι_f, j_f and φ_f.

To obtain the functors μ_S, consider the following category $\widetilde{\mathsf{T}}_S$. Objects are stable marked trees $T \in \mathsf{T}_S$ together with an embedding $\sigma : |T| \to D$. Maps from $\sigma : |T| \to D$ to $\sigma' : |T'| \to D$ exist only if $T \geq T'$, and they are homotopy classes of continous maps $\gamma : |T| \times I \to D$ such that the restriction $\gamma : |T| \times \{x\} \to D$ is injective for any $x \in [0,1[$, the restriction $\gamma : |T| \times \{0\} \to D$ is equal to the map σ, and the restriction $\gamma : |T| \times \{1\} \to D$ is the composition of the natural map $|T| \to |T'|$ and the map $\sigma' : |T'| \to D$.

Then on one hand, we have a forgetful functor $\widetilde{\mu}_S : \widetilde{\mathsf{T}}_S \to \mathsf{B}^S$ which sends an embedded stable marked tree $\sigma : |T| \to D$ to the corresponding map $\sigma \circ \tau : S \to |T| \to D$, and forgets the rest; our definition of maps in $\widetilde{\mathsf{T}}_S$ insures that a map $\gamma : \sigma \to \sigma'$ gives an admissible path $\gamma : I \to D^S$. On the other hand, we have a forgetful functor $\widetilde{\mathsf{T}}_S \to \mathsf{T}_S$ which forgets the embedding, and since the space of embeddings is contractible, this is an equivalence of categories. Composing the inverse equivalence with the functor $\widetilde{\mu}_S$, we obtain the comparison functor μ_S of (1.10). By construction, these functors are obviously compatible with the maps ι_f, j_f and φ_f.

In general, the categories T_S and B^S are very far from each other. For example, it one restricts one's attention to the "maximal open strata" T_{id}, B_{id}, then the first is a partially ordered set, and the second is a groupoid. However, there is the following result.

Proposition 1.13. *The comparison functor* (1.10) *induces an equivalence*

$$\mathcal{D}(\mathsf{B}^S, k) \cong \mathcal{D}_{const}(\mathsf{T}_S, k),$$

where $\mathcal{D}_{const}(\mathsf{T}_S, k) \subset \mathcal{D}(\mathsf{T}_S, k)$ *is the full subcategory spanned by such* $E_{\bullet} \in \mathcal{D}(\mathsf{T}_S, k)$ *that the natural map* $E_{\bullet}(T) \to E_{\bullet}(T')$ *is a quasiisomorphism for any internal pair* $T \geq T'$*.*

Roughly speaking, B_S is obtained by inverting all internal arrows in T_S, and this localization procedure also works for the derived categories. In particular, the functor μ_S induces a homotopy equivalence of the classifying spaces $|\mathsf{T}_S|$ and $|\mathsf{B}^S|$.

I know of two direct proofs of Proposition 1.13. One is presented in my Tokyo lectures (in Lecture 10, to be precise). The argument is not particularly difficult. It proceeds by induction on the cardinality of S – removing a point $s \in S$ gives maps $\mathsf{T}_S \to \mathsf{T}_{S\setminus\{s\}}$, $\mathsf{B}^S \to \mathsf{B}^{S\setminus\{s\}}$, these map are fibrations in a suitable sense, and one is reduced to comparing the fibers, which is an easy computation.

Another and much earlier proof of a very similar statement appears in the paper by Kontsevich and Soibelman. What they prove is basically the statified homotopy equivalence $|\mathsf{T}_S| \cong D^S$, or rather, the equivalence $|\mathsf{T}_{\mathsf{id}}| \cong D_{\mathsf{id}} = D^{[n]}$ of the maximal open strata. To do this, they consider D as a complex variety, so that $D^{[n]}$ becomes a moduli space of n points on a disc. Then they parametrize the moduli by the technique of Strebel differentials (the same technique as the one used by Kontsevich in his proof of the Witten Conjecture). Remarkably, the trajectories of the relevant Strebel differentials are trees, not lines; this is how the unmarked points appear, and this is why they have valency at least 3. The end product is a regular cellular decomposition of $D^{[n]}$ with cells numbered by stable marked trees; by a standard topological lemma, this gives the result.

The difference between the two arguments is roughly speaking as follows: we use *all* the embeddings of a tree into a disc, and just say that the space of emebdding is contractible, while Kontsevich and Soibelman choose *a specific extremal point* in this contractible space. It would be very interesting to compare the constructions in more detail; unfortunately, the corresponding part of the Kontsevich-Soibleman paper is only one page long, and it only contains the constructions, without a hint of a proof that everything indeed works.

There are many other proofs of the homotopy equivalence $|\mathsf{T}_{\mathsf{id}}| \cong D^{[n]}$, or at least of the fact that the corresponding singular chain complexes are quasiisomorphic (for example, this is one of the crucial parts of the paper by McClure and Smith). However, all these proofs are indirect – people compare both sides by showing that they are equivalent to something else (usually a different and very nice combinatorial model for $D^{[n]}$ found by C. Berger).

Whatever proof of Proposition 1.13 one uses, it is important to notice that the comparison functors are compatible with the maps ι_f, j_f, φ_f of Definition 1.6 and Definition 1.9. Then the following is immediate.

Corollary 1.14. *The category of strict* T*-algebras up to a quasiisomorphism in the sense of Definition* 1.9 *is equivalent to the category of factorization* 2*-algebras up to a quasiisomorphism in the sense of Definition* 1.6. □

1.9 Hochschild cohomology as a 2-algebra.

With all the preliminary reductions that we have done, a 2-algebra structure on the Hochschild cohomology complex $CH^{\bullet}(A)$ of an associative algebra A can be constructed in a relatively straightforward way. What we will actually construct is a strict T-algebra in the sense of Definition 1.9 – this is the same as a 2-algebra by Corollary 1.14 and Conjecture 1.7. The construction is similar to the Morita-invariant construction of cyclic homology presented in the last lecture. Essentially, we just have to repeat everything with objects $[n] \in \Lambda$ of cyclic category replaced by marked trees $T \in \mathsf{T}_S$, in a way which is compatible with the maps ι_f, φ_f of Definition 1.9.

Recall that in the cyclic homology construction, the first step was to "lift Hochschild chains to the level of categories" – namely, instead of associating the k-module $A^{\otimes n}$ to an object $[n] \in \Lambda$, we consider the category $A^{\otimes n}\text{-}\mathsf{bimod}$ which serves as a version of the n-fold tensor product $(A\text{-}\mathsf{bimod})^{\otimes n}$. Then for any map $f : [n] \to [m]$, we construct a functor $f_* : A^{\otimes n}\text{-}\mathsf{bimod} \to A^{\otimes m}\text{-}\mathsf{bimod}$.

Hochschild cochains are the spaces $\mathsf{Hom}(A^{\otimes n}, A)$ of k-linear maps from $A^{\otimes n}$ to A, and their appropriate categorical replacement is the category

$$\operatorname{Fun}((A\text{-}\mathsf{bimod})^{\otimes n}, A\text{-}\mathsf{bimod}) \tag{1.11}$$

of functors from $(A\text{-}\mathsf{bimod})^{\otimes n} = A^{\otimes n}\text{-}\mathsf{bimod}$ to $A\text{-}\mathsf{bimod}$ which are in some sense "linear". The standard way to formalize this linearity is to say that a functor must be given by tensoring with a bimodule. Thus for any n, the natural category to consider is the category

$$(A^{opp \otimes n} \otimes A)\text{-}\mathsf{bimod}. \tag{1.12}$$

Note that if we have have two integers n, m, and we fix one of the n possible arguments in the "functor cochains" of (1.11), then we have a natural composition operation

$$\begin{aligned} \operatorname{Fun}((A\text{-}\mathsf{bimod})^{\otimes m}, A\text{-}\mathsf{bimod}) \times \operatorname{Fun}((A\text{-}\mathsf{bimod})^{\otimes n}, A\text{-}\mathsf{bimod}) \to \\ \to \operatorname{Fun}((A\text{-}\mathsf{bimod})^{\otimes n+m-1}, A\text{-}\mathsf{bimod}) \end{aligned} \tag{1.13}$$

which substitutes one cochain into the other. It is easy to rewrite this operation as a functor on the bimodule categories (1.12).

Assume now given a marked tree $\langle T, \tau : S \to V(T) \rangle$. Then for any vertex $v \in V(T)$ of valency $n+1$, we have one outgoing edge and n incoming edges. Denote $A_v = A^{opp \otimes n} \otimes A$, and let

$$A(T) = \bigotimes_{v \in \tau(S) \subset V(T)} A_T.$$

What we want to associate to the tree T is the category $A_T\text{-}\mathsf{bimod}$ of A_T-bimodules; roughly speaking, this is the product of all categories (1.12) associated to marked vertices in T.

Contracting an edge $e \in E(T)$ of the tree T gives a new tree T'. If the pair $T \geq T'$ is internal, then A_T is tautologically the same as $A_{T'}$. If not, the algebras are different, but we still have a natural functor

$$\mu_{T,T'} : A(T)\text{-}\mathsf{bimod} \to A(T')\text{-}\mathsf{bimod} \tag{1.14}$$

induced by (1.13). Composing these, we obtain a similar functor $\mu_{T,T'}$ for any pait $T \geq T'$, $T, T' \in \mathsf{T}_s$. All these functors are right-exact; if the pair $T \geq T'$ is internal, $\mu_{T,T'}$ is an equivalence.

Definition 1.15. Fix a finite set S. A T_S-*bimodule* M over the algebra A is a collection of

(i) an A_T-bimodule $M(T) \in A_T\text{-}\mathsf{bimod}$ for any $T \in \mathsf{T}_S$, and

(ii) a map $M_{T,T'} : \mu_{T,T'}(M(T)) \to M(T')$ for any pair $T \geq T'$, $T, T \in \mathsf{T}_S$,

subject to the obvious associativity condition, and such that the map $M_{T,T'}$ is an isomorphism for any internal pair $T \geq T'$.

For any S, T_S-bimodules form an abelian category which we denote $A\text{-}\mathsf{bimod}^S$. These categories are compatible with the maps ι_f, j_f, φ_f. For any S, we also have the "unit object" $A_\flat$ in the category $A\text{-}\mathsf{bimod}^S$ given by $A_\flat(T) = A(T)$ with the diagonal bimodule structure; these are obviously compatible with transition functors $\mu_{T,T'}$ and the maps ι_f, j_f, φ_f.

The second ingredient in the cyclic homology construction was the trace functor tr. This is replaced by the following. Note that for any n, we have a natural functor

$$\chi_n : (A^{opp\otimes n} \otimes A)\text{-}\mathsf{bimod} \to A\text{-}\mathsf{bimod}$$

which corresponds to evaluating a "functor cochain" $A^{\otimes n}\text{-}\mathsf{bimod} \to A\text{-}\mathsf{bimod}$ on the diagonal bimodule $A^{\otimes n}$. Composing these functors for all the marked vertices, we obtain a functor

$$\chi(T) : A_T\text{-}\mathsf{bimod} \to A^{\otimes\tau(S)}\text{-}\mathsf{bimod}$$

for any marked tree $\langle T, \tau : S \to V(T)\rangle$. We then let

$$\mathsf{hh}_T(M) = \mathsf{Hom}_{A^{\tau(S)\otimes}\text{-}\mathsf{bimod}}(A^{\tau(S)\otimes}, \chi(T)M)$$

for any $M \in A(T)\text{-}\mathsf{bimod}$. This is a left-exact functor. Moreover, if we are given an internal pair $T \geq T'$, then it commutes with the transition functor $\tau_{T,T'}$. In general, the functors do not commute exactly, but there is a natural map

$$\mathsf{hh}_T(M) \to \mathsf{hh}_{T'}(\mu_{T,T'}M).$$

Indeed, to see this map, it is enough to consider the situation when only one edge; in this case, the map is essentially induced by the natural product map

$$\mathsf{Hom}(A, M) \otimes \mathsf{Hom}(A, N) \to \mathsf{Hom}(A, M \otimes_A N)$$

for any two bimodules $M, N \in A\text{-}\mathsf{bimod}$. Thus all the functors hh_T together define a functor

$$\mathsf{hh} : A\text{-}\mathsf{bimod}^S \to \mathrm{Fun}(\mathsf{T}_S, k).$$

Taking its derived functor $R^\bullet\mathsf{hh}$ and applying it to the canonical T_S-bimodule $I \in A\text{-}\mathsf{bimod}^S$, we obtain an object $B^\bullet \in \mathcal{D}(\mathsf{T}_S, k)$. It is elementary to check that this is compatible with the maps ι_f, j_f and with the factorization maps φ_f, so that in the end, $B^\bullet$ is a strict T-algebra. This is our T-algebra: by definition, we have

$$B^\bullet(\mathsf{pt}) = \mathsf{RHom}^\bullet(A, A) \cong CH^\bullet(A).$$

1.10 Further questions.

I would like to finish the lecture with some open questions one might ask about Hochschild cohomology in general, and today's constructions in particular.

First of all, I have treated Hochschild homology and cohomology separately. But of course we all know that vector fields act on differential forms, we have the Cartan homotopy fomula and so on. Does this story have a non-commutative counterpart? For the most part, it does; the corresponding notion has been axiomatized by Tamarkin and Tsygan under the name of "non-commutative calculus". By this they understand all the higher structures present in the pair $\langle HH_\bullet(A), HH^\bullet(A)\rangle$: the Connes-Tsygan differential on $HH_\bullet(A)$, the 2-algebra structure on $HH^\bullet(A)$, and something else which encodes the interaction between the two. However, not much work has been done in this direction. In particular, I would like to have a development in the spirit similar to today's lecture

– combinatorially, this should correspond to *removing the choice of the root vertex* in our planar tree, and exploring the resulting additional cyclic symmetry.

A related question concerns the Calabi-Yau case: when the algebra A is equipped with a "non-degenerate volume form", one can identify $HH_\bullet(A)$ and $HH^\bullet(A)$, possibly with some shift. This results in a complex that has both the Connes-Tsygan differential and the 2-algebra structure. On the level of cohomology, the resulting theory is very well understood – the correct notion is that of a "Batalin-Vilkovisy algebra", which describes exactly the relation between the differential and the Gerstenhaber bracket. In terms of our factorization 2-algebras, this should correspond to considering factorization 2-algebras which are in addition *equivariant with respect to the natural $U(1)$-action on the disc D*, and one expects to have a statement analogous to the Deligne conjecture. This has not been done.

Another question concerns our combinatorial model for the configuration spaces. As we have seen, stable trees capture quite nicely the topology of $D^{[n]}$ and D^n. What about its complex geometry – can one see this too, in some way? Ultimately, the goal here would be to generalize our comparison theorem from construcitble sheaves to non-holonomic $\mathcal{D}$-modules, which conceivably might lead to new constructions of chiral algebras of Beilinson-Drinfeld. *A priori*, this seems quite far fetched. However, the Strebel differential construction of Kontsevich and Soibelman shows that trees do have some interpretation in terms of complex geometry, so perhaps it is possible to do something in this direction.

Another question raised by the Kontsevich-Soibelman construction is the correct level of generality for the whole business. First of all, stable marked planar trees give a model for the moduli space $\mathcal{M}_{0,n}$ of genus-0 curves with n marked points. There is a similar model for higher genera in terms of ribbon graphs (it is this model that Kontsevich has used in his proof of the Witten conjecture). One expects that at least in the Calabi-Yau case, there is also some higher-genus extension of the 2-algebra structure.

But on the other hand, the dual graph of a planar tree embedded into the disc gives a cellular decomposition of the disc of a very special kind. Can one generalize this to other cellular decompositions? In the approach of Kontsevich and Soibleman, the answer is "no": cells in the configuration space are parametrized by stable marked trees, and this is the end of the story. However, our construction is much softer: essentially, we only care about the homotopy type of the geometric realization of the category of trees. Thus there maybe be different versions of the story. At least one such does exist: as it happens, one can drop the stability condition on trees (the only change in the whole story is that non-stable trees form a category, not a partially ordered set). I don't know whether one can add other ribbon graphs corresponding to *all* cellular decomposition of the sphere, or whether one can go to the higher genus case.

Finally, a very intriguing question is raised by the conjectures I explained at the end of the last lecture – the ones relating cyclic homology and the moduli space of perfect objects. Is there a similar interpretation of Hochschild cohomology? Or more specifically – in the Calabi-Yau case when $HH_\bullet(A) \cong HH^\bullet(A)$, can we interpret the higher operations on Hochschild cohomology in terms of the moduli space $\mathcal{M}(A)$? For example, is there any cohomology computation similar to our computation of the homology of the Lie algebra $\mathfrak{gl}_\infty(A)$, where the relevant combinatorics is not that of points on a circle, but rather that of planar trees or ribbon graphs? This is not completely inconceivable, since ribbon graphs can be produced by invariant theory, and they do come up in the so-called "matrix models" in physics and related mathematics (for example, in Kontsevich's proof of the Witten Conjecture which I mentioned above). However, at present I have no idea where to look for such a computation.