Homological methods in Non-commutative Geometry

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These are lecture notes for a course I gave at the University of Tokyo in the winter term of 2007/2008. The title of the course was "Homological methods in Non-commutative geometry", by which I mean some assorted results about Hochschild and cyclic homology, on one hand, and Hochschild cohomology and Kontsevich's Formality Conjecture, on the other hand. The notes are essentially identical to the hand-outs given out during the lectures – I did not attempt any serious revision.

There were eleven lectures in total. Every lecture is preceeded by a brief abstract. The first seven deal with the homological part of the story (cyclic homology, its various definitions, various additional structures it possesses). Then there are four lecture centered around Hochschild cohomology and the formality theorem. The course ends rather abruptly, mostly because of the time constraints. One thing which I regret omitting is an introduction to the language of DG algebras and DG categories according to B. Toën, including his beautiful recent finiteness theorem arXiv:math/0611546. Further possible topics include, for instance, the homological structures associated to Calabi-Yau algebras and Calabi-Yau categories, where we can identify Hochschild homology and Hochschild cohomology, and consider the interplay between the Connes-Tsygan differential on the former, and the Gerstenhaber algebra structure on the later. The resulting notions – Batalin-Vilkovisky algebras, non-commutative calculus of Tsygan-Tamarkin, etc. – are very beautiful and important, but I don't feel ready to present them in introductory lectures. The same goes for the more advanced parts of formality (for instance, complete proof of Deligne conjecture, Etingof-Kazhdan quantization, recent work of Dolgushev-Tsygan-Tamarkin and Calaque-Van den Bergh on G_∞-formality) and for deformation theory of abelian categories.

So, in a nutshell, these lectures cover at most one third of a reasonable textbook on the subject, and I cannot really extend them because the other two thirds are still under active investigation by many mathematicians around the world.

In addition, there is no bibliography (some references are included in the text), and there are far fewer exact attributions than I would like (which is due to my ignorance, and most certainly should not be understood as claiming any original research).

Well, for what it's worth.

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Lecture 1.

The subject of Non-commutative geometry. Notions of a non-commutative geometry. Dictionary between notions from calculus and homological invariants. Hochschild Homology and Cohomology. Hochschild-Kostant-Rosenberg Theorem. Bar-resolution and the Hochschild complex. Cyclic homology (explicit definition).

1.1 The subject of Non-commutative Geometry.

It is an empirical fact that the idea of "non-commutative geometry", when seen for the first time, is met with deep scepticism (at least, this was my personal reaction for 10 years or so). Let me start these lectures with a short justification of the subject.

Back in the nineteenth century, and today in high school math, geometry was essentially set-theoretic: the subject of geometry was points, lines (sets of points of special types), and so on. This approach has been inherited by early algebraic geometry – instead of lines we maybe consider curves of higher degree, or higher-dimensional algebraic varieties, but we still think of them as sets of points with some additional structure.

However, starting from mid-twentieth century, and especially in the work of Grothendieck, a new viewpoint appeared, which can be loosely termed "categorical": one thinks of an algebraic variety simply as an object of the category of algebraic varieties. The precise "inner structure" of an algebraic variety is not so important anymore – what is important is how it behaves with respect to other varieties, what maps to or from other varieties does it admit, and so on. "Set of points" is just one functor on the category of algebraic varieties that we can use to study them; there are other important functors, such as, for instance, various cohomology theories.

These two "dual" approaches to algebraic geometry are not mutually exclusive, but rather complementary, and somewhat competing. To give you a non-trivial example, let us consider the Minimal Model Program. Here two methods of studying an algebraic variety X proved to be very successful. One is to study rational curves on X, their families, subvarieties they span etc. The other is to treat X as a whole and obtain results by considering its cohomology with various coefficients and using Vanishing Theorems. For example, the Cone Theorem claims that a certain part of the ample cone of X is polyhedral, with faces dual to certain classes in $H_2(X)$ called "extremal rays". If X is smooth, the Theorem can be proved by the "bend-and-break" techniques; extremal rays emerge as fundamental classes of certain rational curves on X. On the other hand, the Cone Theorem can be proved essentially by using consistently the Kawamata-Viehweg Vanishing Theorem; this only gives extremal rays as cohomology classes, with no generating rational curves, but it works in larger generality (for instance, for a singular X).

Now, the idea of "non-commutative" geometry is, in a nutshell, to try to replace the notion of an affine algebraic variety $X = \operatorname{Spec} A$ with something which would make sense for a non-commutative ring A. The desire to do so came originally from physics – one of the ways to interpret the formalism of quantum mechanic is to say that instead of the algebra of functions on a symplectic manifold M ("the phase space"), we should consider a certain non-commutative deformation of it. Mathematically, the procedure seems absurd. In order to define a spectrum $\operatorname{Spec} A$ of a ring A, you need A to be commutative, otherwise you cannot even define "points" of $\operatorname{Spec} A$ in any meaningful way. Thus the set-theoretic approach to non-commutative geometry quickly leads nowhere.

However, and this is somewhat surprising, the categorical approach does work: much more things can be generalized to the non-commutative setting than one had any right to expect beforehand. Let us list some of these things.

(i) Algebraic K-theory.

(ii) Differential forms and polyvector fields.

(iii) De Rham differential and de Rham cohomology, Lie bracket of vector fields, basic formalism of differential calculus.

(iv) Hodge theory (in its algebraic form given by Deligne).

(v) Cartier isomorphisms and Frobenius action on cristalline cohomology in positive characteristic.

Of these, the example of K-theory is the most obvious one: Quillen's definition of the K-theory of an algebraic variety $X = \operatorname{Spec} A$ involves only the abelian category $A\text{-}\mathsf{mod}$ of A-modules, and it works for a non-commutative ring A without any changes whatsoever. Before giving the non-commutative versions of the other notions on the list, however, we need to discuss more precisely what we mean by "non-commutative setting".

1.2 The notion of a non-commutative variety.

Actually, there are several levels of abstraction at which non-commutative geometry can be built. Namely, we can take as our definition of a "non-commutative variety" one of the following four.

(1) An associative ring A.

(2) A differential graded (DG) algebra $A^\bullet$.

(3) An abelian category $\mathcal{C}$.

(4) A triangulated category $\mathcal{D}$ "with some enhancement".

The relation between these levels is not linear, but rather as follows:

$$\begin{array}{ccc} (1) & \longrightarrow & (2) \\ \downarrow & & \downarrow \\ (3) & \longrightarrow & (4). \end{array} \tag{1.1}$$

Given an associative ring A, we can treat it as a DG algebra placed in degree 0 – this is the correspondence (1) $\Rightarrow$ (2). Or else, we can consider the category $A\text{-}\mathsf{mod}$ of left A-modules – this is the correspondence (1) $\Rightarrow$ (3). Given a DG algebra $A^\bullet$, we can construct the derived category $\mathcal{D}(A^\bullet)$ of left DG $A^\bullet$-modules, and given an abelian category $\mathcal{C}$, we can consider its derived category $\mathcal{D}(\mathcal{C})$ – this is (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4).

Of course, in any meaningful formalism, the usual notion of a (commutative) algebraic variety has to be included as a particular case. In the list above, (1) is the level of an affine algebraic variety $X = \operatorname{Spec} A$. Passing from (1) to (3) gives the category of A-modules, or, equivalently, the category of quasicoherent sheaves on X. This makes sense for an arbitrary, not necessarily affine scheme X – thus on level (3), we can work with any scheme X by replacing it with its category of quasicoherent sheaves. We can then pass to level (4), and take the derived category $\mathcal{D}(X)$.

What about (2)? As it turns out, an arbitrary scheme X also appears already on this level: the derived category $\mathcal{D}(X)$ of quasicoherent sheaves on X is equivalent to the derived category $\mathcal{D}(A^\bullet)$ of a certain (non-canonical) DG algebra $A^\bullet$. The rough slogan for this is that "every scheme is derived-affine".

Here are some other examples of non-commutative varieties that one would like to consider.

(i) Given a scheme X, one can consider a coherent sheaf $\mathcal{A}$ of algebras on X and the category of sheaves of $\mathcal{A}$. This is only "slightly" non-commutative, in the sense that we have an honest commutative scheme, and the non-commutative algebra sheaf is of finite rank over the commutative sheaf $\mathcal{O}_X$ (e.g. if $X = \operatorname{Spec} B$ is affine, then $\mathcal{A}$ comes from a non-commutative algebra which has B lying its center, and is of finite rank over this center). However, there are examples where this is useful. For instance, in the so-called *non-commutative resolutions* introduced by M. Van den Bergh, X is usually singular; generically over X, $\mathcal{A}$ is a sheaf of matrix algebras, so that its category of modules is equivalent to the category of coherent sheaves on X, but near the singular locus of X, $\mathcal{A}$ is no longer a matrix algebra, and it is "better behaved" than $\mathcal{O}_X$ – e.g. it has finite homological dimension.

(ii) Many interesting categories come from representation theory – representation of a Lie algebra, or of a quantum group, or versions of these in finite characteristic, and so on. These have appeared prominently, for examples, in the recent works of R. Rouquier.

(iii) In sympletic geometry, there is the so-called *Fukaya category* and its versions (e.g. the "Fukaya-Seidel category"). These only exist at level (4) above, and they are very hard to handle; still, the fully developed theory should apply to these categories, too.

Let us also mention that even if one is only interested in the usual schemes X, looking at them non-commutatively is still non-trivial, because *there are more maps between schemes X, X' when they are considered as non-commutative varieties.* E.g. on level (4), a map between triangulated categories is essentially a trinagulated functor, or maybe a pair of adjoint triangulated functors, depending on the specific formalism used – but in any approach, a Fourier-Mukai transform, for instance, gives a well-defined non-commutative map. Flips and flops in the Minimal Model Program are also expected to give non-commutative maps.

Passing to a higher level of abstraction in (1.1), we lose some information. A single abelian category can be equivalent to the category of modules for different rings A (this is known as Morita equivalence – e.g. a commutative algebra A is Morita-equivalent to its matrix algebra $M_n(A)$, for any $n \geq 2$). And a single triangulated category can appear as the derived category of quasicoherent sheaves on different schemes (e.g. related by the Fourier-Mukai transform) and the derived category of DG modules over different DG algebras (e.g. related by Koszul duality, the DG version of Morita equivalence). However, it seems that the information lost is inessential; especially if we think of various homological invariants of a non-commutative variety, they all are independent of the specifics lost when passing to (4). While this is not a self-evident first principle but rather an empirical observation, it seems to hold – again as a rough slogan, "non-commutative geometry is derived Morita-invariant". Thus it would be highly desirable to develop the theory directly on level (4) and not bother with inrelevant data.

However, at present it is not possible to do this. The reason is the well-known fact that the notion of triangulated category is "too weak". Here are some instances of this.

(i) "Cones are not functorial". Thus for a triangulated category $\mathcal{D}$, the category of functors $\operatorname{Fun}(I, \mathcal{D})$ for even the simplest diagrams I – e.g. the category of arrows in $\mathcal{D}$ – is not triangulated.

(ii) Triangulated categories do not patch together well. For instance, if we are given two triangulated categories $\mathcal{D}_1$, $\mathcal{D}_2$ equipped with triangulated functors to a triangulated category $\mathcal{D}$, the fibered product $\mathcal{D}_1 \times_{\mathcal{D}} \mathcal{D}_2$ is not triangulated.

(iii) Given two triangulated categories $\mathcal{D}_1$, $\mathcal{D}_2$, the category of triangulated functors $\operatorname{Fun}_{tr}(\mathcal{D}_1, \mathcal{D}_2)$ is not triangulated.

It is the consensus of all people working in the field that the correct notion is that of a triangulated category with some additional structure, called "enhancement"; however, there is no consensus as to what a convenient enhancement might be, exactly. Popular candidates are "DG-enhancement", "A_∞-enhancement" and "derivator enhancement". Within the framework of these lectures, let me just say that the only sufficiently developed notion of enhancement seems to be the DG approach, but using it is not much different from simply working in the context of DG algebras, that is, on our level (2).

Thus is the present course, we will not attempt to work in the full generality of (4) – we will start at (1), and then maybe go to (2) and/or (3).

However, it is important to keep in mind that (4) is the correct level. In particular, everything should and will be "derived-Morita-invariant" – DG algebras or abelian categories that have equivalent derived categories are indistinguishable from the non-commutative point of view.

1.3 A dictionary.

Let us now give a brief dictionary between some notions of algebraic geometry and their non-commutative counterparts. We will only do it in the affine case (level (1)). For convenience, we have summarized it in table form.

An affine scheme $X = \operatorname{Spec} A$	An associative algebra A
X is smooth	A has finite homological dimension
Differential forms $\Omega^\bullet(X)$	Hochschild homology classes $HH_\bullet(A)$
Polyvector fields $\Lambda^\bullet\mathcal{T}(X)$	Hochschild cohomology classes $HH^\bullet(A)$
De Rham differential d	Connes' differential B
De Rham cohomology $H^\bullet_{DR}(X)$	Cyclic homology $HC_\bullet(A)$, $HP_\bullet(A)$
Schouten bracket	Gerstenhaber bracket
Hodge-to-de Rham spectral sequence	Hochschild-to-cyclic spectral sequence
Cartier isomorphisms	A non-commutative version thereof

Here are some comments on the table.

(i) Polyvector fields are sections of the exterior algebra $\Lambda^\bullet\mathcal{T}(X)$ generated by the tangent bundle $\mathcal{T}(X)$, and Schouten bracket is a generalization of the Lie bracket of vector field to polyvector fields. It seems that in non-commutative geometry, it is not possible to just consider vector fields – all polyvector fields appear together as a package.

(ii) Similarly, *multiplication* in de Rham cohomology seems to be a purely commutative phenomenon – in the general non-commutative setting, it does not exist.

(iii) The first line corresponds to a famous theorem of Serre which claims that the category of coherent sheaves on a scheme X has finite homological dimension if and only if X is regular. In the literature, some alternative notions of smoothness for non-commutative varieties are discussed; however, we will not use them.

(iv) The last line takes place in positive characteristic, that is, for schemes and algebras defined over a field k with $p = \operatorname{char} k > 0$.

All the items in the left column are probably very familiar (expect for maybe the last line, which we will explain in due course). The notions in the right column probably are not familiar. In the first few lectures of this course, we will explain them. We start with Hochschild Homology and Cohomology.

1.4 Hochschild Homology and Cohomology.

Assume given an associative unital algebra A over a field k.

Definition 1.1. *Hochschild homology* $HH_\bullet(A)$ of the algebra A is given by

$$HH_\bullet(A) = \operatorname{Tor}_\bullet^{A^{opp}\otimes A}(A, A). \tag{1.2}$$

Hochschild cohomology $HH^\bullet(A)$ of the algebra A is given by

$$HH^\bullet(A) = \operatorname{Ext}^\bullet_{A^{opp}\otimes A}(A, A). \tag{1.3}$$

Here A^{opp} is the opposite algebra to A – the same algebra with multiplication written in the opposite direction (if A is commutative, then $A^{opp} \cong A$, but in general they might be different). Left modules over $A^{opp} \otimes A$ are the same as bimodules over A, and A has a natural structure of A-bimodule, called *the diagonal bimodule* – this is the meaning of A in (1.3) and in the right-hand side of $\operatorname{Tor}_\bullet(-,-)$ in (1.2). However, A also has a natural structure of a *right* module over $A^{opp} \otimes A$ – and this is what we use in the left-hand side of $\operatorname{Tor}_\bullet(-,-)$ in (1.2).

We note that by definition $HH^\bullet(A)$ is an algebra (take the composition of $\operatorname{Ext}^\bullet$-s), and $HH_\bullet(A)$ has a natural structure of a right module over $HH^\bullet(A)$. In general, neither of them has a structure of an A-module.

Given an A-bimodule M, we can also define Hochschild homology and cohomology with coefficients in M by setting

$$HH_\bullet(A, M) = \operatorname{Tor}_\bullet^{A^{opp}\otimes A}(A, M), \qquad HH^\bullet(A, M) = \operatorname{Ext}^\bullet_{A^{opp}\otimes A}(A, M).$$

In particular, $HH_\bullet(A, -)$ is the derived functor of the left-exact functor $A\text{-}\mathsf{bimod} \to k\text{-Vect}$ from A-bimodules to k-vector spaces given by $M \mapsto A \otimes_{A^{opp}\otimes A} M$. Equivalently, this functor can be defined as follows:

$$M \mapsto M/\{am - ma \mid a \in A, m \in M\}.$$

The reason Hochschild homology and cohomology is interesting – and indeed, the starting point for the whole brand of non-commutative geometry which we discuss in these lecture – is the following classic theorem.

Theorem 1.2 (Hochschild-Kostant-Rosenberg, 1962). *Assume that A is commutative, and that $X = \operatorname{Spec} A$ is a smooth algebraic variety of finite type over k. Then there exist isomorphisms*

$$HH_\bullet(A) \cong \Omega^\bullet(X), \qquad HH^\bullet(A) \cong \Lambda^\bullet\mathcal{T}(X),$$

where $\Omega^\bullet(A)$ are the spaces of differential forms on the affine variery X, and $\Lambda^\bullet\mathcal{T}(A)$ are the spaces of polyvector fields – the sections of the exterior powers of the tangent sheaf $\mathcal{T}(X)$.

Proof. To compute $HH_\bullet(A)$ and $HH^\bullet(A)$, we need to find a convenient projective resolution of the diagonal bimodule A. Since A is commutative, we can identify A and A^{opp}, so that A-bimodules are the same as $A \otimes A$-modules. Let $I \subset A \otimes A$ be the kernel of the natural surjective map $m : A \otimes A \to A$, $m(a_1 \otimes a_2) = a_1 a_2$. Then I is an ideal in $A \otimes A$, and by definition, the module $\Omega^1(A)$ of 1-forms on A is equal to the quotient I/I^2. Thus we have a canonical surjective map

$$\eta : I \to \Omega^1(A).$$

Since $X = \operatorname{Spec} A$ is smooth of finite type, $\Omega^1(A)$ is a projective A-module. Therefore, if consider the A-bimodule I as an A-module by restriction to one of the factors in $A \otimes A$ – say the second one – then the map η admits a splitting map $\Omega^1(A) \to I$, which extends to a map

$$s : A \otimes \Omega^1(A) \to I$$

of A-bimodules. But the A-bimodule $A \otimes \Omega^1(A)$ is projective; thus we can let $P_0 = A \otimes A$, $P_1 = A \otimes \Omega^1(A)$, and we have a start of a projective resolution

$$P_1 \xrightarrow{s} P_0 \xrightarrow{m} A$$

of the diagonal bimodule A. Extend it to a "Koszul complex" $P_\bullet$ by setting $P_i = \Lambda^i_{A\otimes A}(P_1)$, $i \geq 0$, and extending s to a derivation $d : P_{\bullet+1} \to P_\bullet$ of this exterior algebra. This gives a certain complex $P_\bullet$, and it well-know that

> $P_\bullet$ is a resolution of A outside of a certain Zariski-closed subset $Z \subset X \times X$ which does not intersect the diagonal.

Therefore the complex $P_\bullet$ can be used to compute $HH_\bullet(A)$ and $HH^\bullet(A)$; doing this gives the desired isomorphism. □

Exercise 1.1. *Show that the isomorphisms in Theorem* 1.2 *are canonical.*

We note that this proof does not need any assumptions on characteristic (the original proof of Hochschild-Kostant-Rosenberg was slightly different, and it only worked in characteristic 0).

1.5 The bar resolution and the Hochschild complex.

The Koszul resolution is very convenient, but it only exists for a smooth commutative algebra A. We will now introduce another resolution for the diagonal bimodule called *the bar resulution* which is much bigger, but exists in full generality. This gives a certain large but canonical complex for computing $HH_\bullet(A)$ and $HH^\bullet(A)$.

The bar resolution $C_\bullet(A)$ starts with the same free A-bimodule $C_0(A) = A \otimes A$ as the Koszul resolution. Since we want the resolution to exist for any A, there is not much we can build upon to proceed to higher degrees – we have to use A itself. Thus for any $n \geq 1$, we let

$$C_n(A) = A^{\otimes(n+2)} = A \otimes A^{\otimes n} \otimes A,$$

the free A-bimodule generated by the k-vector space A. The differential $C_{n+1}(A) \to C_n(A)$ is denoted b' for historical reasons, and it is given by

$$b' = \sum_{i=1}^{n+2} (-1)^i \, \mathsf{id}^{\otimes i} \otimes m \otimes \mathsf{id}^{\otimes n+2-i}, \tag{1.4}$$

where, as before, $m : A \otimes A \to A$ is the multiplication map. We note that b' is obviously an A-bimodule map.

There is also a version with coefficients: assume given an A-bimodule M, and denote the A-action maps $A \otimes M \to M$, $M \otimes A \to M$ by the same letter m. Then we let $C_n(A, M) = A^{\otimes(n+1)} \otimes M$, $n \geq 0$, and we define the map $b' : C_{n+1}(A, M) \to C_n(A, M)$ by the same formula (1.4).

Lemma 1.3. *For any A, M, the complex $\langle C_\bullet(A, M), b' \rangle$ is a resolution of the bimodule M.*

Proof. The fact that b' squares to 0 is a standard computation which we leave as an exersize (it also has an explanation in terms of simplicial sets which we will give later). To prove that $C_\bullet(A, M)$ is a resolution, extend it to a complex $C'_\bullet(A, M)$ by shifting the degree by 1 and adding the term A – that is, we let

$$C'_n(A, M) = A^{\otimes n} \otimes M$$

for $n \geq 0$, with the differential b' given by the same formula (1.4). Then we have to prove that $C'_{\bullet}(A, M)$ is acyclic. But indeed, the map $h : C'_{\bullet}(A, M) \to C'_{\bullet+1}(A, M)$ given by

$$h(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n,$$

obviously satisfies $h \circ b' + b' \circ h = \mathsf{id}$, thus gives a contracting homotopy for $C'_{\bullet}(A, M)$. □

Exercise 1.2. *Show that for any A-bimodule M, the bimodule $A \otimes M$ is acyclic for the Hochschild homology functor (that is, $HH_i(A, A \otimes M) = 0$ for $i \geq 1$). Hint: compute $HH_i(A, A \otimes M)$ by using the bar resolution for the right $A^{opp} \otimes A$-module A in the left-hand side of $\mathrm{Tor}_{\bullet}^{A^{opp} \otimes A}(A, A \otimes M)$.*

By virtue of Exercise 1.2, the resolution $C_{\bullet}(A, M)$ can be used for the computation of the Hochschild homology groups $HH_{\bullet}(A, M)$. This gives a complex whose terms are also given by $A^{\otimes n} \otimes M$, $n \geq 0$, but the differential is given by

$$b = b' + (-1)^{n+1} t, \tag{1.5}$$

with the correction term t being equal to

$$t(a_0 \otimes \cdots \otimes a_{n+1} \otimes m) = a_1 \otimes \cdots \otimes a_{n+1} \otimes m a_0$$

for any $a_0, \dots, a_{n+1} \in A$, $m \in M$. This is known as the *Hochschild homology complex.*

Geometrically, one can think of the components $a_0, \dots, a_{n-1}, m$ of some tensor in $A^{\otimes n} \otimes M$ as having been placed at $n + 1$ points on the unit interval $[0, 1]$, including the egde points $0, 1 \in [0, 1]$; then each of the terms in the differential b' corresponds to contracting an interval between two neighboring points and multiplying the components sitting at its endpoints. To visualize the differential b in a similar way, one has to take $n + 1$ points placed on the unit circle S^1 instead of the unit interval, including the point $1 \in S^1$, where we put the component m.

1.6 Cyclic homology – explicit definition.

In the case $M = A$, the terms in the Hochschild homology complex are just $A^{\otimes n+1}$, $n \geq 0$, and they acquire an additional symmetry: we let $\tau : A^{\otimes n+1} \to A^{\otimes n+1}$ to be the cyclic permutation multiplied by $(-1)^n$. Note that in spite of the sign change, we have $\tau^{n+1} = \mathsf{id}$, so that it generates an action of the cyclic group $\mathbb{Z}/(n + 1)\mathbb{Z}$ on every $A^{\otimes n+1}$. The fundamental fact here is the following.

Lemma 1.4. *For any n, we have*

$$\begin{aligned}(\mathsf{id} - \tau) \circ b' &= -b \circ (\mathsf{id} - \tau),\\ (\mathsf{id} + \tau + \cdots + \tau^{n-1}) \circ b &= -b' \circ (\mathsf{id} + \tau + \cdots + \tau^n)\end{aligned}$$

as maps from $A^{\otimes n+1}$ to $A^{\otimes n}$.

Proof. Denote $m_i = \mathsf{id}^i \otimes m \otimes \mathsf{id}^{n-i} : A^{\otimes n+1} \to A^{\otimes n}$, $0 \leq i \leq n - 1$, so that $b' = m_0 - m_1 + \cdots + (-1)^{n-1} m_{n-1}$, and let $m_n = t = (-1)^n (b - b')$. Then we obviously have

$$m_{i+1} \circ \tau = \tau \circ m_i$$

for $0 \leq i \leq n - 1$, and $m_0 \circ \tau = (-1)^n m_n$. Formally applying these identities, we conclude that

$$\begin{aligned}\sum_{0 \leq i \leq n} (-1)^i m_i \circ (\mathsf{id} - \tau) &= \sum_{0 \leq i \leq n} (-1)^i m_i - m_0 - \sum_{1 \leq i \leq n} (-1)^i \tau \circ m_{i-1}\\ &= -(\mathsf{id} - \tau) \circ \sum_{0 \leq i \leq n-1} (-1)^i m_i,\end{aligned} \tag{1.6}$$

$$
\begin{aligned}
b' \circ (\mathsf{id} + \tau + \dots + \tau^n) &= \sum_{0 \le i \le n-1} \sum_{0 \le j \le n} (-1)^i m_i \circ \tau^j \\
&= \sum_{0 \le j \le i \le n-1} (-1)^i \tau^j \circ m_{i-j} + \sum_{1 \le i \le j \le n} (-1)^{i+n} \tau^{j-1} \circ m_{n+i-j} \\
&= -(\mathsf{id} + \tau + \dots + \tau^{n-1}) \circ b,
\end{aligned} \tag{1.7}
$$

which proves the claim. □

As a corollary, the following diagram is in fact a bicomplex.

$$
\begin{array}{ccccccc}
\dots \longrightarrow & A & \xrightarrow{\mathsf{id}} & A & \xrightarrow{0} & A \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots \longrightarrow & A \otimes A & \xrightarrow{\mathsf{id} + \tau} & A \otimes A & \xrightarrow{\mathsf{id} - \tau} & A \otimes A \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots & \dots & & \dots & & \dots \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots \longrightarrow & A^{\otimes n} & \xrightarrow{\mathsf{id} + \tau + \dots + \tau^{n-1}} & A^{\otimes n} & \xrightarrow{\mathsf{id} - \tau} & A^{\otimes n} \\
& \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b}
\end{array} \tag{1.8}
$$

Here it is understood that the whole thing extends indefinitely to the left, all the even-numbered columns are the same, all odd-numbered columns are the same, and the bicomplex is invariant with respect to the horizontal shift by 2 columns.

Definition 1.5. The total homology of the bicomplex (1.8) is called the *cyclic homology* of the algebra A, and denoted by $HC_\bullet(A)$.

We see right away that the first, the third, and so on column when counting from the right is the Hochschild homology complex computing $HH_\bullet(A)$, and the second, the fourth, and so on column is the acyclic complex $C'_\bullet(A)$. (the top term is A, and the rest is the bar resolution for A). Thus the spectral sequence for this bicomplex has the form

$$HH_\bullet(A)[u^{-1}] \Rightarrow HC_\bullet(A), \tag{1.9}$$

where u is a formal parameter of cohomological degree 2, and $HH_\bullet(A)[u^{-1}]$ is shorthand for "polynomials in u^{-1} with coefficients in $HH_\bullet(A)$". This is known as *Hochschild-to-cyclic*, or *Hodge-to-de Rham* spectral sequence (we will see in the next lecture that it reduces to the usual Hodge-to-de Rham spectral sequence in the smooth commutative case).

Shifting (1.8) to the right by 2 columns gives the *periodicity map* $u : HC_{\bullet+2}(A) \to HC_\bullet(A)$, which fits into an exact triangle

$$HH_{\bullet+2} \longrightarrow HC_{\bullet+2}(A) \longrightarrow HC_\bullet(A) \longrightarrow , \tag{1.10}$$

known as the *Connes' exact sequence*. One can also invert the periodicity map – in other words, extend the bicomplex (1.8) not only to the left, but also to the right. This gives the *periodic cyclic homology* $HP_\bullet(A)$. Since the bicomplex for $HP_\bullet(A)$ is infinite in both directions, there is a choice involved in taking the total complex: we can take either the product, or the sum of the terms. We take the product.

Remark 1.6. The n-th row of the complex (1.8) is the standard complex which computes the homology $H_\bullet(\mathbb{Z}/n\mathbb{Z}, A^{\otimes n})$ of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. In the periodic version, we have the so-called *Tate homology* instead of the usual homology. It is known that, $Z/n\mathbb{Z}$ being finite, Tate homology is always trivial over a base field of characteristic 0. Were we to take the sum of terms of the periodic bicomplex instead of the product in the definition of $HP_\bullet(A)$, the corresponding spectral sequence would have converged, and the resulting total complex would have been *acyclic*. This is the first instance of an important feature of the theory of cyclic homology: convergence or non-convergence of various spectral sequences is often not automatic, and, far from being just a technical nuissance, has a real meaning.

Lecture 2.

Second bicomplex for cyclic homology. Connes' differential. Cyclic homology and the de Rham cohomology in the HKR case. Homology of small categories. Simplicial vector spaces and homology of the category Δ^{opp}.

2.1 Second bicomplex for cyclic homology.

Recall that in the end of the last lecture, we have defined cyclic homology $HC_\bullet(A)$ of an associative unital algebra A over a field k as the homology of the total complex of a certain explicit bicomplex (1.8) constructed from A and its tensor powers[1]. This definition is very *ad hoc*. Historically, it was arrived at as a result of a certain computation of the homology of Lie algebras of matrices over A; it is not clear at all what is the invariant meaning of this explicit bicomplex. Next several lectures will be devoted mostly to various alternative definitions of cyclic homology. Unfortunately, all of them are *ad hoc* to some degree, and none is completely satisfactory and should be regarded as final. No really good explanation of what is going on exists to this day. But we can at least do computations.

The first thing to do is to notice that not only we know that the even-numbered columns $C'_\bullet(A)$ of the cyclic bicomplex (1.8) are acyclic, but we actually have a contracting homotopy h for them given by $h(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n$. This can be used to remove these acyclic columns entirely. The result is the *second bicomplex for cyclic homology* which has the form

$$
\begin{array}{ccccccc}
 & & & & & & A \\
 & & & & & & \uparrow b \\
 & & & & A & \xrightarrow{B} & A^{\otimes 2} \\
 & & & & \uparrow b & & \uparrow b \\
 & & A & \xrightarrow{B} & A^{\otimes 2} & \xrightarrow{B} & A^{\otimes 3} \\
 & & \uparrow b & & \uparrow b & & \uparrow b \\
A & \xrightarrow{B} & A^{\otimes 2} & \xrightarrow{B} & A^{\otimes 3} & \xrightarrow{B} & A^{\otimes 4}, \\
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b
\end{array}
\tag{2.1}
$$

with the horizontal differential $B : A^{\otimes n} \to A^{\otimes (n+1)}$ given by

$$B = (\mathsf{id} - \tau) \circ h \circ (\mathsf{id} + \tau + \cdots + \tau^{n-1}).$$

This differential B is known as the *Connes' differential*, or the *Connes-Tsygan differential*, or the *Rinehart differential*. In the commutative case, it was discovered by G. Rinehart back in the 1960es; then it was forgotten, and rediscovered independently by A. Connes and B. Tsygan in about 1982 (in the general associative case).

Lemma 2.1. *The diagram* (2.1) *is a bicomplex whose total complex is quasiisomorphic to the total complex of* (1.8).

[1] By the way, a good reference for everything related to cyclic homology is J.-L. Loday's book *Cyclic homology*, Springer, 1998. Personally, I find also very useful an old overview article B. Feigin, B. Tsygan, *Additive K-theory*, in Lecture Notes in Math, vol. 1289.

Proof. This is a general fact from linear algebra which has nothing to do with the specifics of the situation. Assume given a bicomplex $K_{\bullet,\bullet}$ with differentials $d_{1,0}$, $d_{0,1}$, and assume given a contracting homotopy h for the complex $\langle K_{i,\bullet}, d_{0,1}\rangle$ for every odd $i \geq 1$. Define the diagram $\langle K'_{\bullet,\bullet}, d'_{1,0}, d'_{0,1}\rangle$ by

$$K'_{i,j} = K_{2i,j-i}, \quad d'_{0,1} = d_{0,1}, \quad d'_{1,0} = d_{1,0} \circ h \circ d_{1,0}.$$

Then $d'_{1,0} \circ d'_{1,0} = d_{1,0} \circ h \circ d^2_{1,0} \circ h \circ d_{1,0} = 0$, and

$$\begin{aligned} d'_{1,0} \circ d'_{0,1} + d'_{0,1} \circ d'_{1,0} &= d_{1,0} \circ h \circ d_{1,0} \circ d_{0,1} + d_{0,1} \circ d_{1,0} \circ h \circ d_{1,0} \\ &= -d_{1,0} \circ h \circ d_{0,1} \circ d_{1,0} - d_{1,0} \circ d_{0,1} \circ h \circ d_{1,0} \\ &= -d_{1,0} \circ (h \circ d_{0,1} + d_{0,1} \circ h) \circ d_{1,0} = -d_{1,0} \circ d_{1,0} = 0, \end{aligned}$$

so that $K'_{\bullet,\bullet}$ is indeed a bicomplex, and one checks easily that the map

$$\bigoplus_i (-1)^i \operatorname{\mathsf{id}} \oplus (-1)^{i+1}(h \circ d_{1,0}) : \bigoplus_i K'_{i,\bullet-i} = \bigoplus_i K_{2i,\bullet-2i} \to \bigoplus_i K_{i,\bullet-i}$$

is a chain homotopy equivalence between the total complexes of $K_{\bullet,\bullet}$ and $K'_{\bullet,\bullet}$. □

Exercise 2.1. *Check this.*

2.2 Comparison with de Rham cohomology.

The main advantage of the complex (2.1) with respect to (1.8) is that it allows the comparison with the usual de Rham cohomology in the commutative case.

Proposition 2.2. *In the assumptions of the Hochschild-Kostant-Rosenberg Theorem, denote* $n = \operatorname{\mathsf{dim}} \operatorname{Spec} A$*, and assume that* $n!$ *is invertible in the base field* k *(thus either* $\operatorname{\mathsf{char}} k = 0$*, or* $\operatorname{\mathsf{char}} k > n$*). Then the HKR isomorphism* $HH_\bullet(A) \cong \Omega^\bullet_A$ *extends to a quasiisomorphism between the bicomplex* (2.1) *and the bicomplex*

$$\begin{array}{ccccccc} & & & & & & A \\ & & & & & & \uparrow{\scriptstyle 0} \\ & & & & A & \xrightarrow{d} & \Omega^2_A \\ & & & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \\ & & A & \xrightarrow{d} & \Omega^2_A & \xrightarrow{d} & \Omega^3_A \\ & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \\ A & \xrightarrow{d} & \Omega^2_A & \xrightarrow{d} & \Omega^3_A & \xrightarrow{d} & \Omega^4_A, \\ \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \end{array}$$

where the vertical differential is 0*, and the horizontal differential is the de Rham differential* d.

Proof. First we show that under the additional assumption of the Proposition, the HKR isomorphism extends to a canonical quasiisomorphism P between the Hochschild complex and the complex $\langle \Omega^\bullet_A, 0\rangle$. This quasiisomorphism P is given by

$$P(a_0 \otimes a_1 \otimes \dots \otimes a_i) = \frac{1}{i!} a_0 da_1 \wedge \dots \wedge a_i.$$

This is obviously a map of complexes: indeed, since $d(a_1a_2) = a_1da_2 + a_2da_1$ by the Leibnitz rule, the expression for $P(b(a_0 \otimes \dots \otimes a_i))$ consists of terms of the form

$$a_0a_jda_1 \wedge \dots \wedge da_{j-1} \wedge da_{j+1} \wedge \dots \wedge da_i,$$

every such term appears exactly twice, and with opposite signs. Thus P induces a map $p : HH_\bullet(A) \to \Omega^\bullet_A$. By HKR, both sides are isomorphic flat finitely generated A-modules; by Nakayama Lemma, to prove that p an isomorphism, it suffices to prove that it is surjective. This is clear – since A is commutative, the alternating sum

$$\sum_\sigma \mathsf{sgn}(\sigma)a_0 \otimes \sigma(a_1 \otimes \dots \otimes a_i)$$

over all the permutations σ of the indices $1, \dots, i$ is a Hochschild cycle for any $a_0, \dots, a_i \in A$, and we have

$$P\left(\sum_\sigma \mathsf{sgn}(\sigma)a_0 \otimes \sigma(a_1 \otimes \dots \otimes a_i)\right) = a_0da_1 \wedge \dots \wedge a_i.$$

So, p is an isomorphism, and P is indeed a quasiisomorphism. It remains to prove that it sends the Connes-Tsygan differential B to the de Rham differential d – that is, we have $P \circ B = d \circ P$. This is also very easy to see. Indeed, every term in $B(a_0 \otimes \dots a_i)$ contains 1 as one of the factors. Since 1 is annihilated by the de Rham differential d, the only non-trivial contribution to $P(B(a_0 \otimes \dots a_i))$ comes from the terms which contain 1 as the first factor, so that we have

$$\begin{aligned} P(B(a_0 \otimes \dots a_i)) &= \sum_{j=0}^{i-1} P(h(\tau^j(a_0 \otimes \dots \otimes a_i))) = \frac{1}{i!}\sum_{j=0}^{i-1} \tau^j(da_0 \wedge \dots \wedge da_i) \\ &= \frac{1}{(i-1)!} da_0 \wedge \dots \wedge da_i, \end{aligned}$$

which is exactly $d(P(a_0 \otimes \dots \otimes a_i))$. □

Corollary 2.3. *In the assumptions of Proposition* 2.2*, we have a natural isomorphism*

$$HP_\bullet(A) \cong H^\bullet_{DR}(\operatorname{Spec} A)((u)).$$

Proof. Clear. □

Remark 2.4. For example, the Connes-Tsygan differential B in the lowest degree, $B : A \to A^{\otimes 2}$, is given by

$$B(a) = 1 \otimes a + a \otimes 1,$$

which is very close to the formula $a \otimes 1 - 1 \otimes a$ which gives the universal differential $A \to \Omega^1(A)$ into the module of Kähler differentials $\Omega^1(A)$ for a commutative algebra A. The difference in the sign is irrelevant because of the HKR identification of $HH_1(A)$ and $\Omega^1(A)$ – if one works out explicitly the identification given in Lecture 1, one checks that $1 \otimes a$ goes to 0, so that it does not matter with which sign we take it. The comparison map P in the lowest degree just sends $a \otimes b$ to adb, so that $P(B(a)) = da$.

2.3 Generalities on small categories.

Our next goal is to give a slightly less *ad hoc* definition of cyclic homology also introduced by A. Connes. This is based on the techniques of the so-called homology of small categories. Let us describe it.

For any small category Γ and any base field k, the category $\mathrm{Fun}(\Gamma, k)$ of functors from Γ to k-vector spaces is an abelian category, and the direct limit functor $\lim_{\overrightarrow{\Gamma}}$ is right-exact. Its derived functors are called *homology functors* of the category Γ and denoted by $H_\bullet(\Gamma, E)$ for any $E \in \mathrm{Fun}(\Gamma, k)$. For instance, if Γ is a groupoid with one object with automorphism group G, then $\mathrm{Fun}(\Gamma, k)$ is the category of k-representations of the group G; the homology $H_\bullet(\Gamma, -)$ is then tautologically the same as the group homology $H_\bullet(G, -)$. Analogously, the inverse limit functor $\lim_{\overleftarrow{\Gamma}}$ is left-exact, and its derived functors $H^\bullet(\Gamma, -)$ are the cohomology functors of the category Γ. In the group case, this corresponds to the usual cohomology of the group. By definition of the inverse limit, we have

$$H^\bullet(\Gamma, E) = \mathrm{Ext}^\bullet(k^\Gamma, E),$$

where k^Γ denotes the constant functor from Γ to k-Vect. In particular, $H^\bullet(\Gamma, k^\Gamma) = \mathrm{Ext}^\bullet(k^\Gamma, k^\Gamma)$ is an algebra, and the homology $H_\bullet(\Gamma, k^\Gamma)$ with constant coefficients is a module over this algebra.

In general, it is not easy to compute the homology of a small category Γ with arbitrary coefficients $E \in \mathrm{Fun}(\Gamma, k)$. One way to do it is to use resolutions by the *representable functors* $k_{[a]}$, $[a] \in \Gamma$ – these are by definition given by

$$k_{[a]}([b]) = k[\Gamma([a], [b])]$$

for any $[b] \in \Gamma$, where $\Gamma([a], [b])$ is the set of maps from $[a]$ to $[b]$ in Γ, and $k[-]$ denotes the k-linear span. By Yoneda Lemma, we have $\mathrm{Hom}(k_{[a]}, E') = E'([a])$ for any $E' \in \mathrm{Fun}(\Gamma, k)$; therefore $k_{[a]}$ is a projective object in $\mathrm{Fun}(\Gamma, k)$, higher homology groups $H_i(\Gamma, k_{[a]})$, $i \geq 1$ vanish, and again by Yoneda Lemma, we have

$$\mathrm{Hom}(\lim_{\overrightarrow{\Gamma}} k_{[a]}, k) \cong \mathrm{Hom}(k_{[a]}, k^\Gamma) \cong k^\Gamma([a]) = k, \tag{2.2}$$

so that $H_0(\Gamma, k_{[a]}) = k$. Every functor $E \in \mathrm{Fun}(\Gamma, k)$ admits a resolution by sums of representable functors — for example, we have a natural adjunction map

$$\bigoplus_{[a] \in \Gamma} E([a]) \otimes k_{[a]} \to E,$$

and this map is obviously surjective. Analogously, for cohomology, we can use *co-representable functors* $k^{[a]}$ given by

$$k^{[a]}([b]) = k[\Gamma([b], [a])]^*;$$

they are injective, $H^0(\Gamma, k^{[a]}) \cong k$, and every $E \in \mathrm{Fun}(\Gamma, k)$ has a resolution by products of functors of this type.

One can also think of functors in $\mathrm{Fun}(\Gamma, k)$ as "presheaves of k-vector spaces on Γ^{opp}". This is of course a very complicated name for a very simple thing, but it is useful because it brings to mind familiar facts about sheaves on topological spaces or étale sheaves on schemes. Most of these facts hold for functor categories as well, and the proofs are actually much easier. Specifically, it is convenient to use a version of Grothendieck's "formalism of six functors". Namely, if we are given two small categories Γ, Γ', and a functor $\gamma : \Gamma \to \Gamma'$, then we have an obvious restriction functor $\gamma^* : \mathrm{Fun}(\Gamma', k) \to \mathrm{Fun}(\Gamma, k)$. This functor has a left-adjoint $\gamma_!$ and a right-adjoint f_*, called the left and right *Kan extensions*. (If you cannot remember which is left and which is right, but are familiar with sheaves, then the notation $\gamma_!$, γ_* will be helpful.)

The direct and inverse limit over a small category Γ are special cases of this construction – they are Kan extensions with respect to the projection $\Gamma \to \mathsf{pt}$ onto the point category pt. The representable and co-representable functors $k_{[a]}$, $k^{[a]}$ are obtained by Kan extensions with respect to the embedding $\mathsf{pt} \to \Gamma$ of the object $[a] \in \Gamma$. Given three categories Γ, Γ', Γ'', and two functors $\gamma : \Gamma \to \Gamma'$, $\gamma' : \Gamma' \to \Gamma''$, we obviously have $(\gamma' \circ \gamma)^* \cong \gamma^* \circ {\gamma'}^*$, which implies by adjunction $\gamma'_! \circ \gamma_! \cong (\gamma' \circ \gamma)_!$ and $\gamma'_* \circ \gamma_* \cong (\gamma' \circ \gamma)_*$. In general, the Kan extensions $\gamma_!$, γ_* have derived functors $L^{\bullet} f_!$, $R^{\bullet} f_*$; just as in the case of homology and cohomology, one can compute them by using resolutions by representable resp. corepresentable functors.

2.4 Homology of the category Δ^{opp}.

Probably the first useful fact about homology of small categories is a description of the homology of the category Δ^{opp}, the opposite to the category Δ of finite non-empty totally ordered sets. We denote by $[n] \in \Delta^{opp}$ the set of cardinality n. Objects $E \in \operatorname{Fun}(\Delta^{opp}, k)$ are known as *simplicial k-vector spaces*. Explicitly, such an object is given by k-vector spaces $E([n])$, $n \geq 1$, and various maps between them, among which one traditionally distinguishes the *face maps* $d_n^i : E([n+1]) \to E_{[n]}$, $0 \leq i \leq n$ – the face map d_n^i corresponds to the injective map $[n] \to [n+1]$ whose image does not contain the $(i+1)$-st element in $[n+1]$.

Lemma 2.5. *For any simplicial vector space $E \in \operatorname{Fun}(\Delta^{opp}, k)$, the homology $H_{\bullet}(\Delta^{opp}, E)$ can be computed by the* standard complex $E_{\bullet}$ *given by $E_n = E([n+1])$, $n \geq 0$, with differential $d : E_n \to E_{n-1}$, $n \geq 1$, equal to*

$$d = \sum_{0 \leq i \leq n} (-1)^i d_n^i.$$

Proof. By definition, we have a map $E_0 = E([1]) \to H_0(\Delta^{opp}, E)$, which obviously factors through the cokernel of the differential d, and this is functorial in E.

Denote by $H'_{\bullet}(\Delta^{opp}, E)$ the homology groups of the standard complex $E_{\bullet}$. Then every short exact sequence of simplicial vector spaces induces a long exact sequence of $H'_{\bullet}(\Delta^{opp}, -)$, so that $H'_{\bullet}(\Delta^{opp}, -)$ form a δ-functor. Moreover, $H'_0(\Delta^{opp}, E)$ is by definition the cokernel of the map $d = d_1^0 - d_1^1 : E([2]) \to E([1])$. This is the same as the direct limit of the diagram

$$E([2]) \underset{d_1^1}{\overset{d_1^0}{\rightrightarrows}} E([1])$$

of two k-vector spaces $E([2])$, $E([1])$ and two maps d_1^0, d_1^1 between them (a direct limit of this type is called a *coequalizer*). Since this diagram has an obvious map to Δ^{opp}, we have a natural map

$$H'_0(\Delta^{opp}, E) \to \varinjlim_{\Delta^{opp}} E = H_0(\Delta^{opp}, E),$$

and by the universal property of derived functors, it extends to a canonical map

$$H'_{\bullet}(\Delta^{opp}, E) \to H_{\bullet}(\Delta^{opp}, E) \tag{2.3}$$

of δ-functors. We have to prove that it is an isomorphism. Since every $E \in \operatorname{Fun}(\Delta^{opp}, k)$ admits a resolution by sums of representable functors $k_{[n]}$, $[n] \in \Delta^{opp}$, it suffices to prove that the map (2.3) is an isomorphism for all $E = k_{[n]}$ (this is known as *the method of acyclic models*). This is clear: $H_i(\Delta^{opp}, k_{[n]})$ is k for $i = 0$ and 0 otherwise, and the left-hand side of (2.3) is the homology of the standard complex of an n-simplex, which is also k in degree 0 and 0 in higher degrees. □

Exercise 2.2. *Compute the cohomology $H^{\bullet}(\Delta^{opp}, E)$. Hint: compute $k^{[1]}$.*

Lecture 3.

Connes' cyclic category Λ. Cyclic homology as homology of the category Λ. Yet another bicomplex, and a definition of cyclic homology using arbitrary resolutions.

3.1 Connes' category Λ.

For applications to cyclic homology, A. Connes introduced a special small category known as *the cyclic category* and denoted by Λ. Objects $[n]$ of Λ are indexed by positive integers n, just as for Δ^{opp}. Maps between $[n]$ and $[m]$ can be defined in various equivalent ways; we give two of them.

Topological description. The object $[n]$ is thought of as a "wheel" – the circle S^1 with n distinct marked points, called *vertices*. A continuous map $f : [n] \to [m]$ is *good* if it sends marked points to marked points, has degree 1, and is *monotonous* in the following sense: for any connected interval $[a, b] \subset S^1$, the preimage $f^{-1}([a, b]) \subset S^1$ is connected. Morphisms from $[n]$ to $[m]$ in the category Λ are homotopy classes of good maps $f : [n] \to [m]$.

Combinatorial description. Consider the category Cycl of linearly ordered sets equipped with an order-preserving endomorphism σ. Let $[n] \in$ Cycl be the set $\mathbb{Z}$ with the natural linear order and endomorphism $\sigma : \mathbb{Z} \to \mathbb{Z}$, $\sigma(a) = a + n$. Let $\Lambda_\infty \subset$ Cycl be the full subcategory spanned by $[n]$, $n \geq 1$ – in other words, for any n, m, let $\Lambda_\infty([n], [m])$ be the set of all maps $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a) \leq f(b) \quad \text{whenever } a \leq b, \qquad f(a + n) = f(a) + m, \tag{3.1}$$

for any $a, b \in \mathbb{Z}$. For any $[n], [m] \in \Lambda_\infty$, the set $\Lambda_\infty([n], [m])$ is acted upon by the endomorphism σ (on the left, or on the right, by definition it does not matter). We define the set of maps $\Lambda([n], [m])$ in the category Λ by $\Lambda([n], [m]) = \Lambda_\infty([n], [m])/\sigma$.

Here is the correspondence between the two definitions. First of all, we note that homotopy classes of continuous monotonous maps from $\mathbb{R}$ to itself whcih send integral points into integral points are obviously in one-to-one correspondence with non-descreaing maps from $\mathbb{Z}$ to itself. Now, in the topological description above, we may assume that if we consider S^1 as the unit disc in the complex plane $\mathbb{C}$, then the marked points are placed at the roots of unity. Then the universal cover of S^1 is $\mathbb{R}$, and after rescaling, we may assume that exactly the integral points are marked. Thus any good map $f : S^1 \to S^1$ induces a map $\mathbb{R} \to \mathbb{R}$ which sends integral points into integral points, or in other words, a non-decreasing map from $\mathbb{Z}$ to itself. Such a map $\mathbb{R} \to \mathbb{R}$ comes from a map $S^1 \to S^1$ if and only if the corresponding map $\mathbb{Z} \to \mathbb{Z}$ commutes with σ.

There is also an explicit description of maps in Λ by generators and relations which we will not need; an interested reader can find it, for instance, in Chapter 6 of Loday's book.

Given an object $[n] \in \Lambda$, it will be convenient to denote by $V([n])$ the set of vertices of the wheel $[n]$ (in the topological description), and it will be also convenient to denote by $E([n])$ the set of *edges* of the wheel – that is, the clock-wise intervals $(s, s') \subset S^1$ between the two neighboring vertices $s, s' \in V([n])$.

Lemma 3.1. *The category Λ is self-dual: we have $\Lambda \cong \Lambda^{opp}$.*

Proof. In the combinatorial description, define a map $\Lambda_\infty([m], [n]) \to \Lambda_\infty([n], [m])$ by $f \mapsto f^o$, $f^o : \mathbb{Z} \to \mathbb{Z}$ given by

$$f^o(a) = \min\{b \in \mathbb{Z} | f(b) \geq a\}.$$

This is obviously compatible with compositions and bijective, so that we get an isomorphism $\Lambda_\infty \cong \Lambda_\infty^{opp}$. Being compatible with σ, it descends to Λ.

In the topological description, note that for any map $f : [n] \to [n]$ and any edge $e = (s, s') \in E([m])$, the preimage $f^{-1}(e) \subset S^1$ with respect to the corresponding good map $f : S^1 \to S^1$ lies entirely within a single edge $e' \in E([m'])$, Thus we get a natural map $f^o : E([m]) \to E([n])$. We leave it to the reader to check that this extends to a duality functor $\Lambda \to \Lambda^o$ which interchanges $V([n])$ and $E([n])$. □

If we only consider those maps in (3.1) which send $0 \in \mathbb{Z}$ to 0, then the resulting subcategory in Λ_∞ is equivalent to Δ^{opp}.

Exercise 3.1. *Check this. Hint: use the duality* $\Lambda \cong \Lambda^o$.

This gives a canonical embedding $j : \Delta^{opp} \to \Lambda_\infty$, and consequently, an embedding $j : \Delta^{opp} \to \Lambda$ (this is injective on maps). Functors in $\mathrm{Fun}(\Lambda, k)$ are called *cyclic k-vector spaces*. Any cyclic k-vector space E defines by restriction a simplicial k-vector space $j^*E \in \mathrm{Fun}(\Delta^{opp}, k)$.

3.2 Homology of the category Λ.

The category Λ conveniently encodes the maps m_i and τ between various tensor powers $A^{\otimes n}$ used in the complex (1.8): m_i corresponds to the map $f \in \Lambda([n+1], [n])$ given by

$$f(a(n+1)+b) = \begin{cases} an+b, & b \leq i, \\ an+b-1, & b > i, \end{cases}$$

where $0 \leq b \leq n$, and τ is the map $a \mapsto a+1$, twisted by the sign (alternatively, one can say that m_i are obtained from face maps in Δ^{opp} under the embedding $\Delta^{opp} \subset \Lambda_p$). The relations $m_{i+1} \circ \tau = \tau \circ m_i$, $0 \leq i \leq n-1$, and $m_0 \circ \tau = (-1)^n m_n$ between these maps which we used in the proof of Lemma 1.4 are encoded in the composition laws of the category Λ. Thus for any object $E \in \mathrm{Fun}(\Lambda, k)$ – they are called *cyclic vector spaces* – one can form the bicomplex of the type (1.8):

$$\begin{array}{ccccccc}
\dots & \longrightarrow & E([1]) & \xrightarrow{\mathsf{id}} & E([1]) & \xrightarrow{\mathsf{id}-\tau} & E([1]) \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots & \longrightarrow & E([2]) & \xrightarrow{\mathsf{id}+\tau} & E([2]) & \xrightarrow{\mathsf{id}-\tau} & E([2]) \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots & & \dots & & \dots & & \dots \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} \\
\dots & \longrightarrow & E([n]) & \xrightarrow{\mathsf{id}+\tau+\dots+\tau^{n-1}} & E([n]) & \xrightarrow{\mathsf{id}-\tau} & E([n]) \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b}
\end{array} \tag{3.2}$$

(where b and b' are obtained from m_i and τ by the same formulas as in (1.8)). We also have the periodic version, the Connes' exact sequence and the Hodge-to-de Rham spectral sequence (where the role of Hochschild homology is played by the homology $H_\bullet(\Delta^{opp}, j^*E)$).

Lemma 3.2. *For any $E \in \mathrm{Fun}(\Lambda, k)$, the homology $H_\bullet(\Lambda, E)$ can be computed by the bicomplex* (1.8).

Proof. As in Lemma 2.5, we use the method of acyclic models. We denote by $H'_\bullet(\Lambda, E)$ the homology of the total complex of the bicomplex (3.2). Just as in Lemma 2.5, we have a natural map $H'_0(\Lambda, E) \to H_0(\Lambda, E)$, we obtain an induced functorial map

$$H'_{\bullet}(\Lambda, E) \to H_{\bullet}(\Lambda, E),$$

and we have to prove that it is an isomorphism for $E = k_{[n]}$, $[n] \in \Lambda$. We know that for such E, in the right-hand side we have k in degree 0 and 0 in higher degrees. On the other hand, the action of the cyclic group $\mathbb{Z}/m\mathbb{Z}$ generated by $\tau \in \Lambda([m],[m])$ on $\Lambda([n],[m])$ is obviously free, and we have

$$\Lambda([n],[m])/\tau \cong \Delta^{opp}([n],[m])$$

– every $f : \mathbb{Z} \to \mathbb{Z}$ can be uniquely decomposed as $f = \tau^j \circ f_0$, where $0 \leq j < m$, and f_0 sends 0 to 0. The rows of the complex (1.8) compute

$$H_{\bullet}(\mathbb{Z}/m\mathbb{Z}, k_{[n]}([m])) \cong k\left[\Delta^{opp}([n],[m])\right],$$

and the first term in the corresponding spectral sequence is the standard complex for the simplicial vector space $k^{\Delta}_{[n]} \in \operatorname{Fun}(\Delta^{opp}, k)$ represented by $[n] \in \Delta^{opp}$. Therefore this complex computes $H_{\bullet}(\Delta^{opp}, k^{\Delta}_{[n]})$, and we are done by Lemma 2.5. □

There is one useful special case where the computation of $H_{\bullet}(\Lambda, E)$ is even easier.

Definition 3.3. A cyclic vector space $E \in \operatorname{Fun}(\Lambda, k)$ is *clean* if for any $[n] \in \Lambda$, the homology $H_i(\mathbb{Z}/n\mathbb{Z}, E([n]))$ with respect to the $\mathbb{Z}/n\mathbb{Z}$-action on $E([n])$ given by τ is trivial for all $i \geq 1$.

In practice, a cyclic vector space can be clean for two reasons. First, $E([n])$ might be a free $k[\mathbb{Z}/n\mathbb{Z}]$-module for any n. Second, the base field k might have characteristic 0, so that finite groups have no higher homology with any coeffients. In any case, for a clean $E \in \operatorname{Fun}(\Lambda, k)$, computing the homology of the rows of the bicomplex (3.2) reduces to taking the coinvariants $E([n])_{\tau}$ with respect to the autmorphism τ, and the whole (3.2) reduces to a complex

$$\dots \xrightarrow{b} E([n])_{\tau} \xrightarrow{b} \dots \xrightarrow{b} E([2])_{\tau} \xrightarrow{b} E([1])_{\tau}, \tag{3.3}$$

with the differential induced by the differential b of (3.2). We note that the coinvariants $E([n])_{\tau}$, $n \geq 1$, do not form a simplicial vector space; nevertheless, the differential b is well-defined.

3.3 The small category definition of cyclic homology.

Assume now again given an associative unital algebra A over a field k. To define cyclic homology $HC_{\bullet}(A)$ as homology of the cyclic category Λ, one constructs a cyclic k-vector space $A_{\#}$ in the following way: for any $[n] \in \Lambda$, $A_{\#}([n]) = A^{\otimes n}$, where we think of the factors A in the tensor product as being numbered by vertices of the wheel $[n]$, and for any map $f : [n] \to [m]$, the corresponding map $A_{\#}(f) : A^{\otimes n} \to A^{\otimes m}$ is given by

$$A_{\#}(f)\left(\bigotimes_{i \in V([n])} a_i\right) = \bigotimes_{j \in V([m])} \prod_{i \in f^{-1}(j)} a_i, \tag{3.4}$$

where $V([n])$, $V([m])$ are the sets of vertices of the wheel $[n],[m] \in \Lambda$. We note that for any $j \in V([m])$, the finite set $f^{-1}(j)$ has a natural total order given by the clockwise order on the circle S^1. Thus, although A need not be commutative, the product in the right-hand side is well-defined. If $f^{-1}(j)$ is empty for some $j \in V([m])$, then the right-hand side involves a product numbered by the empty set; this is defined to be the unity element $1 \in A$.

As an immediate corollary of Lemma 3.2, we obtain the following.

Proposition 3.4. *We have a natural isomorphism $HC_\bullet(A) \cong H_\bullet(\Lambda, A_\#)$.* □

This isomorphism is also obviously compatible with the periodicity, the Connes' exact sequence, and the Hodge-to-de Rham spectral sequence. In particular, the standard complex for the simplicial k-vector space $j^*A_\#$ is precisely the Hochschild homology complex, so that we have $HH_\bullet(A) = H_\bullet(\Delta^{opp}, j^*A_\#)$.

Exercise 3.2. *Show that the Hochschild homology complex which computes $HH_\bullet(A, M)$ for an A-bimodule M also is the standard complex for a simplicial k-vector space. Does it extend to a cyclic vector space?*

3.4 Example: yet another bicomplex for cyclic homology.

The definition of cyclic homology using small categories may seem too abstract at first, but this is actually a very convenient technical tool: it allows to control the combinatorics of various complexes in a quite efficient way. As an illustration of this, let me sketch, in $\operatorname{char} 0$, yet one more description of cyclic homology by an explicit complex (this definition has certain advantages explained in the next subsection).

For any k-vector space V equipped with a non-zero covector $\eta \in V^*$, $\eta : V \to k$, contraction with η defines a differential $\delta : \Lambda^{\bullet+1}V \to \Lambda^\bullet V$ on the exterior algebra $\Lambda^\bullet V$, and the complex $\langle \Lambda^\bullet V, \delta \rangle$ is acyclic, so that $\Lambda^{\geq 1}V$ is a resolution of $k = \Lambda^0 V$. This construction depends functorially on the pair $\langle V, \eta \rangle$, so that it can be applied poinwise to the representable functor $k_{[1]} \in \operatorname{Fun}(\Lambda, k)$ equipped with the natural map $\eta : k_{[1]} \to k^\Lambda$. The result is a resolution $\Lambda^\bullet k_{[1]}$ of the constant cyclic vector space $k^\Lambda \in \operatorname{Fun}(\Lambda, k)$.

Here is another description of the exterior powers $\Lambda^\bullet k_{[1]}$. Consider a representable functor $k_{[i]}$ for some $i \geq 1$, and let $\overline{k}_{[i]} \in \operatorname{Fun}(\Lambda, k)$ be its quotient given by

$$\overline{k}_{[i]}([n]) = k\left[\Lambda([i],[n])\right] / \{f \in \Lambda([i],[n]) \mid f \text{ not injective}\};$$

in other words, $\overline{k}_{[i]}([n])$ is spanned by injective maps from $[i]$ to $[n]$. Then $k_{[i]}$ is acted upon by the cyclic group $\mathbb{Z}/i\mathbb{Z}$ of automorphisms of $[i] \in \Lambda$, this action descends to the quotient $\overline{k}_{[i]}$, and we have

$$\Lambda^i k_{[i]} = \left(\overline{k}_{[i]}\right)_\tau,$$

where $\tau : \overline{k}_{[i]} \to \overline{k}_{[i]}$ is the generator of $\mathbb{Z}/i\mathbb{Z}$ twisted by $(-1)^{i+1}$. The differential δ lifts to a differential $\delta : \overline{k}_{[i]} \to \overline{k}_{[i-1]}$ given by the alternating sum of the maps $\overline{k}_{[i]} \to \overline{k}_{[i-1]}$ induced by the i injective maps $[i-1] \to [i]$. We note, however, that the complex $\overline{k}_{[\bullet]}$ is no longer a resolution of k^Λ.

Lemma 3.5. *For any $i \geq 1$, we have $H_j(\Lambda, \overline{k}_{[i]}) = 0$ if $j \neq i-1$, and k if $j = i-1$. The $\mathbb{Z}/i\mathbb{Z}$-action on $k = H_{i-1}(\Lambda, \overline{k}_{[i]})$ by the $\mathbb{Z}/i\mathbb{Z}$-action on $\overline{k}_{[i]}$ is given by the sign representation. Moreover, for any $E \in \operatorname{Fun}(\Lambda, k)$, we have $H_\bullet(\Lambda, \overline{k}_{[i]} \otimes E) \cong H_{\bullet+i-1}(\Delta^{opp}, E)$, with the sign action of $\mathbb{Z}/i\mathbb{Z}$.*

Proof. The cyclic object $\overline{k}_{[i]} \in \operatorname{Fun}(\Lambda, k)$ is clean, and the corresponding complex (3.3) is the quotient of the standard complex of the elementary $(i-1)$-simplex by the subcomplex spanned by all faces of dimension less than $i-1$. In other words, $H_\bullet(\Lambda, \overline{k}_{[i]})$ is the reduced homology of the $(i-1)$-sphere. This proves the first claim. The second claim is obvious: the term of degree $i-1$ in the complex (3.3) is isomorphic to $(k[\mathbb{Z}/i\mathbb{Z}])_\tau$, and this is the sign representation by the definition of τ. The third claim now follows immediately from the well-known Künneth formula, which says that for any simplicial vector spaces $V, W \in \operatorname{Fun}(\Delta^{opp}, k)$, the standard complex of the product $V \otimes W$ is naturally quasiisomorphic to the product of the standard complexes for V and W. □

Now, for any cyclic k-vector space $E \in \operatorname{Fun}(\Lambda, k)$, and any $[i] \in \Lambda$, the product $E \otimes \overline{k}_{[i]} \in \operatorname{Fun}(\Lambda, k)$ is clean, so that it makes sense to consider the complex (3.3). Then the differential $\delta : \overline{k}_{[i]} \to \overline{k}[i-1]$ induces a map between these complexes, and we can form a bicomplex $K_{\bullet,\bullet}(E)$ given by

$$K_{i,j}(E) = \left(\overline{k}_i([j+1]) \otimes E([j+1])\right)_\tau, \tag{3.5}$$

where the horizontal differential $K_{\bullet+1,\bullet}(E) \to K_{\bullet,\bullet}(E)$, henceforth denoted by $\widetilde{B}$, is induced by δ, and the vertical differential is the Hochschild differential b, as in (3.3).

Lemma 3.6. *Assume that* $\operatorname{char} k = 0$. *Then for any* $E \in \operatorname{Fun}(\Lambda, k)$*, the total complex of the bicomplex* (3.5) *computes the homology* $H_\bullet(\Lambda, k)$*.*

Proof. Since $\operatorname{char} k = 0$, every cyclic vector space is clean, and we can compute cyclic homology by using the complex (3.3). Since $\langle \Lambda^{\geq 1} k_{[1]}, \delta \rangle$ is a resolution of the constant cyclic vector space k^Λ, we have

$$H_\bullet(\Lambda, E) \cong \mathbb{H}_\bullet(\Lambda, K_\bullet \otimes E),$$

where $K_\bullet \cong \Lambda^{\bullet+1} k_{[1]}$, and the differential in $K_\bullet \otimes E$ is induced by δ. Applying (3.3) to the right-hand side *almost* gives the bicomplex (3.5) – the difference is that we take $K_i = (\overline{k}_{[i+1]})_\tau$ instead of $\overline{k}_{[i+1]}$. Thus it suffices to prove that the natural map

$$H_\bullet(\Lambda, \overline{k}_{[i]} \otimes E) \to H_\bullet\left(\Lambda, \left(\overline{k}_{[i]}\right)_\tau \otimes E\right)$$

is an isomorphism for any $[i] \in \Lambda$. But since $\operatorname{char} k = 0$, the cyclic groups have no homology, so that the right-hand side is isomorphic to

$$H_\bullet(\Lambda, \overline{k}_{[i]} \otimes E)_\tau.$$

And by Lemma 3.5, τ on $H_\bullet(\Lambda, \overline{k}_{[i]} \otimes E)$ is the identity map. □

Assume now that $E = A_\#$ for some associative unital A-algebra A. Then the bicomplex (3.5) is similar to the second bicomplex (2.1) for cyclic homology in the following sense: for any $i \geq 0$, the column $K_{i,\bullet}(A_\#)$ of (3.5) computes the Hochschild homology $HH_\bullet(A)$, with the same degree shift as in (2.1).

What happens is the following. Recall that to obtain the Hochschild homology complex, one uses the bar resolution $C_\bullet(A)$. However, to compute the Hochschild homology $HH_\bullet(A)$, any other resolution would do. In particular, we can take any integer $n \geq 2$, and consider the n-fold tensor product

$$C^n_\bullet(A) = C_\bullet(A) \otimes_A C_\bullet(A) \otimes_A \cdots \otimes_A C_\bullet(A).$$

This is obviously a complex of free A-bimodules, and it is quasiisomorphic to $A \otimes_A A \otimes_A \cdots \otimes_A A \cong A$, so that it is a good resolution. Using this resolution to compute $HH_\bullet(A)$, we obtain a complex $CH^n_\bullet(A)$ whose l-th term $CH^n_l(A)$ is the sum of several copies of $A^{\otimes(n+l)}$, and these copies are numbered by elements in the set

$$M^n_l = \Lambda_{inj}([n], [l+n])/\tau$$

of injective maps $[n] \to [l+n]$ considered modulo the action of the cyclic permutation $\tau : [l+n] \to [l+n]$. In other words, the terms of the complex $CH^n_\bullet(A)$ are numbered by wheels $[n+m]$, $m \geq 0$, with n marked points considered modulo cyclic permutation. These n points cut the wheel into n intervals of lengths $l_1, l_2, \dots, l_n$ with $l_1 + l_2 + \cdots + l_n = m + n$, and the corresponding term in $CH^n_l(A)$ computes the summand

$$A \otimes_{A^{opp} \otimes A} (C_{l_1 - 1}(A) \otimes_A C_{l_2 - 1}(A) \otimes_A \cdots \otimes_A C_{l_1 - 1}(A))$$

in

$$CH^n_\bullet(A) = A \otimes_{A^{opp} \otimes A} (C_\bullet(A) \otimes_A \cdots \otimes_A C_\bullet(A)).$$

The differential $\widetilde{b} : CH^n_{\bullet+1}(A) \to CH^n_\bullet(A)$ restricted to the term which corresponds to some injective $f : [n] \to [n+l+1]$ is the alternating sum of the maps m_i corresponding to surjective maps $[n+l+1] \to [n+l]$ such that the composition $[n] \to [n+l+1] \to [n+l]$ is still injective – in other words, we allow to contract edges of the marked wheel $[n+l+1]$ *unless an edge connects two marked points*. Of course, $CH^1_\bullet(A)$ is the usual Hochschild homology complex, and $\widetilde{b} = b$ is the usual Hochschild differential (since there is only one marked point, every edge can be contracted).

We leave it to the reader to check that the complex $CH^n_\bullet(A)$ is precisely isomorphic to the complex $K_{n,\bullet+n}(A_\#)$.

One can also show that the periodicity in $HC_\bullet(A)$ corresponds to shifting the bicomplex (3.5) by one column to the left, just as in (2.1), so that the Hodge filtration on $HC_\bullet(A)$ is also induced by the stupid filtration on (3.5) in the horizontal direction. Thus *a postriori*, (3.5) and (2.1) are even quasiisomorphic as bicomplexes, and the horizontal differential $\widetilde{B}$ in (3.5) can be identified with the Connes-Tsygan differential B. However, this is not at all easy to see by a direct computation.

3.5 Cyclic homology computed by arbitrary resolution.

To show why (3.5) is useful, let me show how it can be modified so that the bar resolution $C_\bullet(A)$ is replaced with an arbitrary projective resolution $P_\bullet$ of the diagonal bimodule S (I follow the exposition in my paper *Cyclic homology with coefficients*, `math.KT/0702068`, which is based on ideas of B. Tsygan).

For simplicity, I will only explain how to do this for the first two columns of (3.5). This gives a resolution-independent description of the Connes-Tsygan differential $B = \widetilde{B}$, but says nothing about possible higher differentials in the Hodge-to-de Rham spectral sequence.

Fix a projective resolution $P_\bullet$ with the augmentation map $r : P_\bullet \to A$. Consider the resolution $P^2_\bullet = P_\bullet \otimes_A P_\bullet$ of the same diagonal bimodule A. Note that the augmentation map r induces *two* quasiisomorphisms $r_0, r_1 : P^2_\bullet \to P_\bullet$ given by

$$r_0 = r \otimes_A \mathsf{id}, \qquad r_1 = \mathsf{id} \otimes_A r.$$

In general, there is no reason why these two maps should be equal. However, being two maps of projective resolutions of A which induce the same identity map on A itself, they should be chain-homotopic. Choose a chain homotopy $\iota : P^2_\bullet \to P_{\bullet+1}$.

Now consider the complexes

$$\overline{P}_\bullet = A \otimes_{A^{opp} \otimes A} P_\bullet, \quad \overline{P}^2_\bullet = A \otimes_{A^{opp} \otimes A} P^2_\bullet$$

which compute $HH_\bullet(A)$, and the maps $\overline{r}_0$, $\overline{r}_0$, $\overline{\iota}$ between them induces by r_0, r_1 and ι. Notice that the complex $\overline{P}^2_\bullet$ has another description: we have

$$\overline{P}^2_\bullet = \bigoplus_{l,\bullet-l} A \otimes_{A^{opp} \otimes A} P_l \otimes_A P_{\bullet-l},$$

and for any two A-bimodules M, N, we have

$$A \otimes_{A^{opp} \otimes A} (M \otimes_A N) = M \otimes N / \{ma \otimes n - m \otimes an, am \otimes n - m \otimes na \mid a \in A, m \in M, n \in N\},$$

which is manifestly symmetric in m and n. Thus we have a natural involution $\tau : \overline{P}^2_\bullet \to \overline{P}^2_\bullet$. This involution obviously interchanges $\overline{r}_0$ and $\overline{r}_1$, but there is no reason why it should be in any way compatible with the map $\overline{\iota}$ – all we can say is that $\tau \circ \overline{\iota}$ is another chain homotopy between $\overline{r}_0$ and $\overline{r}_1$. Thus the map

$$\widetilde{B} = \overline{\iota} - \tau \circ \overline{\iota} : \overline{P}^2_{\bullet} \to \overline{P}_{\bullet+1}$$

commutes with the differentials.

Lemma 3.7. *The map $\widetilde{B}$ induces the same map on the Hochschild homology $HH_{\bullet}(A)$ as the Connes-Tsygan differential B.*

Sketch of a proof. One checks that the map we need to describe does not depend on choices: neither of a projective resolution $P_{\bullet}$, since any two such resolutions are chain-homotopy equivalent, nor of the map ι, since any two such are chain-homotopic to each other. Thus to compute it, we can take any $P_{\bullet}$ and any ι. If we take $P_{\bullet} = C_{\bullet}(A)$, the bar-resolution, and let ι be the sum of tautological maps $A^{\otimes l} \otimes A^{\otimes l'} \to A^{\otimes l+l'}$, then $\widetilde{B}$ is precisely the same as in the bicomplex (3.5). □

Remark 3.8. In the assumptions of the Hochschild-Kostant-Rosenberg Theorem, it would be very interesting to try to work out explicitly the map $\widetilde{B}$ for the Koszul resolution.

Lecture 4.

Combinatorics of the category Λ: cohomology of Λ and $\Lambda_{\leq n}$, periodicity, classifying spaces. Fibrations and cofibrations of small categories; Λ_∞ as a fibered category over Λ.

4.1 Cohomology of the category Λ and periodicity.

In the last lecture, we have shown that the homology $H_\bullet(\Lambda, E)$ with coefficients in some cyclic vector space $E \in \operatorname{Fun}(\Lambda, E)$ can be computed by the standard complex (3.2); in particular, we have the periodicity map $u : H_{\bullet+2}(\Lambda, E) \to H_\bullet(\Lambda, E)$ and the Connes' exact triangle

$$H_\bullet(\Delta^{opp}, j^*E) \longrightarrow H_\bullet(\Lambda, E) \xrightarrow{u} H_{\bullet-2}(\Lambda, E) \longrightarrow ,$$

where $j : \Delta^{opp} \to \Lambda$ is the embedding defined in the last lecture. Today, we want to give a more invariant description of the periodicity map. That such a description should exist is more-or-less clear. Indeed, homology $H_\bullet(\Lambda, -)$ — or rather, hyperhomology $\mathbb{H}(\Lambda, -)$ — is a functor from the derived category $\mathcal{D}(\Lambda, k)$ of the abelian category $\operatorname{Fun}(\Lambda, k)$ to the derived category $\mathcal{D}(k\text{-Vect})$. By definition, this functor is adjoint to the tautological embedding $\mathcal{D}(k\text{-Vect}) \to \mathcal{D}(\Lambda, k)$, $k \mapsto k^\Lambda$, so that by Yoneda Lemma, every natural transformation $H_{\bullet+2}(\Lambda, -) \to H_\bullet(\Lambda, -)$ should be induced by an element in

$$\operatorname{Ext}^2(k^\Lambda, k^\Lambda) = H^2(\Lambda, k).$$

Thus to describe periodicity, we have to compute the cohomology $H^\bullet(\Lambda, k)$ of the category Λ with constant coefficients $k = k^\Lambda \in \operatorname{Fun}(\Lambda, k)$.

The computation itself is not difficult: since the category Λ is self-dual, the complex (3.2) has an obvious dualization, and exactly the same argument as in the proof of Lemma 3.2 shows that dualized complex computes $H^\bullet(\Lambda, E)$ for any $E \in \operatorname{Fun}(\Lambda, k)$. For the constant functor k, this gives

$$H^\bullet(\Lambda, k) \cong k[u], \tag{4.1}$$

where, as before, $k[u]$ means "the space of polynomials in one formal variable u of degree 2". It is only slightly more difficult to see that the isomorphism (4.1) is an algebra isomorphism, and the action of the generator $u \in H^2(\Lambda, k)$ on homology $H_\bullet(\Lambda, -)$ is the periodicity map. One can argue, for instance, as follows. The same operation of "shifting the bicomplex by two columns" induces a periodicity map $H^\bullet(\Lambda, E) \to H^{\bullet+2}(\Lambda, E)$; this map is functorial, thus (1) induced by an element in $H^2(\Lambda, k)$, and obviously the same one, and (2) compatible with the algebra structure on

$$H^\bullet(\Lambda, k) = \operatorname{Ext}^\bullet(k^\Lambda, k^\Lambda),$$

so that $H^\bullet(\Lambda, k)$ must be a unital algebra over the polynomial algebra $k[u]$ generated by the periodicity map. Since by (4.1), it is isomorphic to $k[u]$ as a $k[u]$-module, it must also be isomorphic to $k[u]$ as an algebra.

However, it will be useful to have a more explicit description of the generator $u \in H^2(\Lambda, k)$.

To obtain such a description, we use the topological interpretation of the category Λ — in other words, we treat $[n] \in \Lambda$ as a wheel formed by marking n points on the circle S^1. Note that this defines a cellular decomposition of the circle: its 0-cells are vertices $v \in V([n])$, and its 1-cells are edges $e \in E([n])$. Denote by $C_\bullet([n])$ the corresponding complex of length 2 which computes the homology $H_\bullet(S^1, k)$. Any map $f \in \Lambda([n], [m])$ induces a cellular map $S^1 \to S^1$, or at any rate, a map which sends 0-skeleton into 0-skeleton, and thus induces a map $C_\bullet([n]) \to C_\bullet([m])$. In this way, $C_\bullet([n])$ becomes a length-2 complex of cyclic vector spaces. Since the homology of the circle $H_i(S^1, k)$ is equal to k for $i = 0, 1$ and 0 otherwise, and does not depend on the cellular decomposition, the homology of the complex $C_\bullet \in \mathrm{Fun}(\Lambda, k)$ is k^Λ in degree 0 and 1, and 0 in other degrees. Thus we have an exact sequence

$$0 \longrightarrow k^\Lambda \longrightarrow C_1 \longrightarrow C_0 \longrightarrow k \longrightarrow 0 \tag{4.2}$$

of cyclic vector spaces. Explicitly, $V([n]) \cong \Lambda([1],[n])$, so that $C_0([n]) = k[V([n])] = k[\Lambda([1],[n])]$, and C_0 is canonically isomorphic to the representable functor $k_{[1]}$. As for C_1, we have by definition

$$C_1([n]) = k[E([n])] = k[\Lambda([n],[1])],$$

and the map $C_1(f) : C_1([n]) \to C_1([m])$ corresponding to a map $f : [n] \to [m]$ is given by

$$C_1(f)(e) = \sum_{e' \in f^{o-1}(e)} e' \in k[E([m])] \tag{4.3}$$

for any edge $e \in E([n])$, so that C_1 is canonically identified with the corepresentable functor $k^{[1]}$. All in all, the exact sequence (4.2) can be rewritten as

$$0 \longrightarrow k^\Lambda \longrightarrow k^{[1]} \longrightarrow k_{[1]} \longrightarrow k^\Lambda \longrightarrow 0. \tag{4.4}$$

This represents by Yoneda a certain class in $H^2(\Lambda, k) = \mathrm{Ext}^2(k^\Lambda, k^\Lambda)$.

Lemma 4.1. *The class $u' \in H^2(\Lambda, k)$ represented by* (4.4) *is equal to the periodicity generator u.*

Proof. Let us first prove the equality up to an invertible constant. To do this, it suffices to prove that the cone of the map $H_{\bullet+2}(\Lambda, k) \to H_\bullet(\Lambda, k)$ induced by u' is isomorphic to k in degree 0 and trivial in other degrees. This cone is the hyperhomology $\mathbb{H}(\Lambda, C_\bullet)$. Since $C_0 = k_{[1]}$ is representable, it already has all the homology we want from the cone, so that we have to prove that

$$H_\bullet(\Lambda, C_1) = H_\bullet(\Lambda, k^{[1]}) = 0$$

(in all degrees). Denote by M the kernel of the natural map $k_{[1]} \to k^\Lambda$, so that we have short exact sequences

$$0 \longrightarrow M \longrightarrow k_{[1]} \longrightarrow k^\Lambda \longrightarrow 0,$$
$$0 \longrightarrow k^\Lambda \longrightarrow k^{[1]} \longrightarrow M \longrightarrow 0.$$

Computing the homology long exact sequence for the first of these exact sequences, we see that the boundary differential $\delta_1 : H_i(\Lambda, M) \to H_{i+1}(\Lambda, k)$ is non-trivial, so that the first short exact sequence is not split, and that in fact δ_1 is an isomorphism for all $i \geq 0$. To prove the claim, it suffices to check that the boundary differential $\delta_2 : H_{i+1}(\Lambda, M) \to H_i(\Lambda, k)$ in the second long exact sequence also is an isomorphism for all i. Since everything is compatible with with $k[u']$-action, it suffices to prove it for $i = 0$ – in other words, we have to prove that the generator of $H_0(\Lambda, k) = k$ goes to 0 under the map $k^\Lambda \to k^{[1]}$. But if not, this means by definition that the second short exact sequence is split. This is not possible: the duality $\Lambda \cong \Lambda^{opp}$ together with the usual duality $k\text{-Vect}^{opp} \to k\text{-Vect}$, $V \mapsto V^*$ induce a fully faithfull duality functor $\mathrm{Fun}(\Lambda, k)^o \to \mathrm{Fun}(\Lambda, k)$, and this functor sends our short exact sequences into each other.

As for the constant, we note that it obviously must be universal, thus invertible in any field, thus either 1 or -1. On the other hand, in the definition of (4.4) there is a choice: we have to choose an orientation of the cirle S^1. Switching the orientation changes the sign of u', so that we can always achieve $u = u'$. We leave it at that. □

4.2 Canonical resolution.

We can extend the exact sequence (4.4) to a resolution of the constant functor k^Λ by iterating it – the result is a complex of the form

$$\dots \longrightarrow k^{[1]} \longrightarrow k_{[1]} \longrightarrow k^{[1]} \longrightarrow k_{[1]},$$

where the maps $k^{[1]} \to k_{[1]}$ are as in (4.4), and the maps $k_{[1]} \to k^{[1]}$ are the composition maps $k_{[1]} \to k^\Lambda \to k^{[1]}$. Moreover, for any cyclic vector space $E \in \operatorname{Fun}(\Lambda, k)$, we have a canonical resolution

$$\dots \longrightarrow k^{[1]} \otimes E \longrightarrow k_{[1]} \otimes E \longrightarrow k^{[1]} \otimes E \longrightarrow k_{[1]} \otimes E. \tag{4.5}$$

The periodicity map for E is induced by $\mathsf{id} \otimes u \in \operatorname{Ext}^2(E, E)$, and it can be represented explicitly by the obvious periodicity endomorphism of (4.5) which shift everything to the left by two terms.

It is instructive to see what happens if compute $H_\bullet(\Lambda, E)$ by replacing E with (4.5), as in Lemma 3.6 in the last Lecture. Both $k_{[1]} \otimes E$ and $k^{[1]} \otimes E$ are clean in the sense of Definition 3.3, so that we can compute $H_\bullet(\Lambda, -)$ by the complex (3.3). Applying it to (4.5) gives a double complex $M_{i,j}(E)$ with terms

$$M_{i,j}(E) = \begin{cases} (k_{[1]}([j+1]) \otimes E([j+1]))_\tau, & i \text{ even}, \\ (k^{[1]}([j+1]) \otimes E([j+1]))_\tau, & i \text{ odd}. \end{cases}$$

To identify further $M_{0,j}(E) = E([j+1])$, we need to choose a vertex $v \in V([j+1])$ (for instance, we may fix the embedding $j : \Delta^{opp} \to \Lambda$), and to identify $M_{1,j}(E) = E([j+1])$, we need to to choose an edge $e \in E([j+1])$ (for instance, since choosing $v \in V([j+1])$ cuts the wheel and defines a total order on $E([j+1])$, we can take the last edge with respect to this order). To compute the differential $b : M_{i,j}(E) \to M_{i,j-1}(E)$, we note that for any contraction $[j+1] \to [j]$ of an edge $e' \in E([j+1])$, the corresponding face map $m_e : k_{[1]}([j+1]) \to k_{[1]}([j])$ sends the chosen vertex $v \in k[V([j+1])] = k_{[1]}([j+1])$ to the chosen vertex $v \in k[V([j])]$. On the other hand, it immediately follows from (4.3) that the face map $m'_{e'} : k^{[1]}([j+1]) \to k^{[1]}([j])$ sends the chosen last edge $e \in k[E([j+1])]$ to $e \in k[E([j])]$ if $e \neq e'$, and to 0 otherwise. Thus the diferential $b : M_{i,j}(E) \to M_{i,j-1}(E)$ is given by

$$b = \sum_{0 \leq l \leq j} (-1)^j r_l m_l,$$

where $r_l = 0$ if i is odd and $l = j$, and $r_l = 1$ otherwise. Thus $M_{\bullet,\bullet}(E)$ becomes exactly isomorphic to the original bicomplex (3.2) for the cyclic vector space E. We also have $H_\bullet(\Lambda, E \otimes k^{[1]}) = 0$, and $H_\bullet(\Lambda, E \otimes k_{[1]}) = H_\bullet(\Delta^{opp}, j^*E)$.

4.3 Nerves and geometric realizations.

To anyone who studied algebraic topology, the cohomology algebra $H^\bullet(\Lambda, k) = k[u]$ of the category Λ will seem familiar: the same algebra appears as the cohomology algebra $H^\bullet(\mathbb{C}P^\infty, k)$ of the infinite-dimensional complex projective space $\mathbb{C}P^\infty$, the classifying space $BU(1)$ for the unit circle group $U(1) = S^1$. This is not a simple coincidence. The relation between Λ and $\mathbb{C}P^\infty$ has been one of the recurring themes of the whole theory of cyclic homology from its very beginning.

The relation occurs at various levels, and while the most advanced ones are not properly understood even today, we do understand the picture up to a certain point. The next level after the cohomology isomorphism is that of the so-called *geometric realizations*.

Unfortunately, we do not have time to present the notion of the geometric realization in full detail (it is easily available in the literature; my personal favourite is the exposition in Chapter I of Gelfand-Manin's book, also Quillen has a nice and concise exposition in his paper on higher K-theory in Lecture Notes in Math., vol. 341). Let us just briefly remind the reader that to any small category Γ, one associated a simplicial set $N(\Gamma)$ called *the nerve* of the category Γ. By definition, 0-simplices in $N(\Gamma)$ are objects of Γ, 1-simplices are morphisms, 2-simplices are pairs of composable morphisms $[a_1] \to [a_2] \to [a_3]$, and so on – n-simplices in $N(\Gamma)$ are functors to Γ from the totally ordered set $[n+1]$ considered as a category in the usual way. Given a simplicial set $X \in \operatorname{Fun}(\Delta^{opp}, \mathrm{Sets})$, one forms a topological space $|X|$ called the *geometric realization* of X by gluing together the elementary simplices Δ^n, one for each n-simplex in $X([n+1])$. Given a small category Γ, we will call $|N(\Gamma)|$ its geometric realization, and we will denote it simply by $|\Gamma|$.

Here are some simple properties of the geometric realization.

(i) We have $|\Gamma| \cong |\Gamma^{opp}|$.

(ii) A functor $\gamma : \Gamma \to \Gamma'$ induces a map $|\gamma| : |\Gamma| \to |\Gamma'|$, and a map $\gamma_1 \to \gamma_2$ between functors γ_1, γ_2 induces a homotopy between $|\gamma_1|$ and $|\gamma_2|$.

(iii) Consequently, if a functor $\gamma : \Gamma \to \Gamma'$ has an adjoint, then $|\gamma|$ is a homotopy equivalence. In particular, if Γ has a final, or an initial object, then $|\Gamma|$ is contractible.

(iv) If Γ is a connected groupoid, and an object $[a] \in \Gamma$ has automorphism group is G, then up to homotopy, $|\Gamma|$ is the classifying space BG.

To any functor $E \in \operatorname{Fun}(\Gamma, k)$, one associates a constructible sheaf $\mathcal{E}$ of k-vector spaces on $|\Gamma|$ by the following rule: for any n-simplex $[a_0] \to \dots \to [a_n]$ of $N(\Gamma)$, the restriction of $\mathcal{E}$ to the corresponding simplex $\Delta^n \subset |\Gamma|$ is the constant sheaf with fiber $E([a_0])$, and the gluing maps are either identical or induced by the action of morphisms in Γ. The gives an exact comparison functor $\operatorname{Fun}(\Gamma, k) \to \operatorname{Shv}(|\Gamma|, k)$. This functor is fully faithful, and it is even fully faithful on the level of derived categories: for any $E, E' \in \operatorname{Fun}(\Gamma, k)$ with corresponding sheaves $\mathcal{E}, \mathcal{E}' \in \operatorname{Shv}(|\Gamma|, k)$, the natural map

$$\operatorname{Ext}^{\bullet}(E, E') \to \operatorname{Ext}^{\bullet}(\mathcal{E}, \mathcal{E}')$$

is an isomorphism in all degrees (to prove it, one can, for instanse, use the Godement resolution of $E \in \operatorname{Fun}(\Gamma, k)$ by representable sheaves, as in Lecture 2). Of course, the comparison functor is not an equivalence: in general, the category $\operatorname{Shv}(|\Gamma|, k)$ is much larger. However, we have the following obvious fact.

Definition 4.2. A functor $E \in \operatorname{Fun}(\Gamma, k)$ is *locally constant* if for any morphism $f : [a] \to [a']$ in Γ, the corresponding map $E([a]) \to E([a'])$ is invertible.

Lemma 4.3. *The comparison functor induces an equivalence between the derived category* $\mathcal{D}_{lc}(\Gamma, k)$ *of complexes with locally constant homology and the derived category* $\operatorname{Shv}_{lc}(|\Gamma|, k)$ *of complexes of sheaves on* $|\Gamma|$ *whose homology sheaves are locally constant.* □

Corollary 4.4. *Assume that for any field k and for any locally constant* $E \in \operatorname{Fun}(\Gamma, k)$*, we have* $H_{\bullet}(\Gamma, E) = E([a])$*, where* $[a] \in \Gamma$ *is s fixed object. Then* $|\Gamma|$ *is contractible.*

Proof. By the well-known Whitehead Theorem, a map $f : X \to Y$ of CW-complexes is a homotopy equivalence if for any local systems A on Y, B on X, the induced maps $H_{\bullet}(X, f^*A) \to H_{\bullet}(Y, A)$, $H_{\bullet}(X, B) \to H_{\bullet}(Y, f_*B)$ are isomorphisms. □

Going back to the cyclic category Λ: our goal is to prove that $|\Lambda|$ is homotopically equivalent to $\mathbb{C}P^\infty$. We will do it indirectly, in two steps: first, we prove that the realization $|\Lambda_\infty|$ of the category Λ_∞ is contractible, then we prove that the projection functor $\Lambda_\infty \to \Lambda$ induces a fibration $|\Lambda_\infty| \to |\Lambda|$ whose fiber is the circle $S^1 = U(1)$ – thus $|\Lambda_\infty|$ can be taken as the contractible space $EU(1)$, and $|\Lambda|$ is homotopy equivalent to the classifying space $EU(1)/U(1) = BU(1) \cong \mathbb{C}P^\infty$. For the first step, we only need Corollary 4.4, but for the second step, we need to develop some machinery of fibrations for small categories.

4.4 Fibrations and cofibration of small categories.

The notion of a fibered and cofibered category was introduced by Grothendieck in SGA1, Ch.VI, which is perhaps still the best reference for those who can read French; nowadays, this machinery is usually called *Grothendieck construction*. Let me give the basic definitions.

Assume given a functor $\gamma : \Gamma' \to \Gamma$ between small categories Γ, Γ'. By the *fiber* $\Gamma'_{[a]}$ over an object $[a] \in \Gamma$ we understand the subcategory $\Gamma'_{[a]} \to \Gamma'$ of objects $[a'] \subset \Gamma'$ such that $\gamma([a']) = [a]$, and morphisms f such that $\gamma(f) = \mathsf{id}$. A morphism $f : [a] \to [b]$ in Γ' is called *Cartesian* if it has the following universal property:

- any morphism $f' : [a'] \to [b]$ such that $\gamma(f') = \gamma(f)$ factors through f by means of a unique map $[a'] \to [a]$ in $\Gamma'_{\gamma([a])}$.

Definition 4.5. A functor $\gamma : \Gamma' \to \Gamma$ is called a *fibration* if

(i) for any $f : [a] \to [b]$ in Γ, and any $b' \in \Gamma'_{[b]}$, there exists a Cartesian morphism $f' : [a'] \to [b']$ such that $\gamma(f') = f$, and

(ii) the composition of two Cartesian morphisms is Cartesian.

Condition (i) here mimics the "covering homotopy" condition in the definition of a fibration in algebraic topology, but it is in fact much more precise — indeed, the Cartesian covering morphism f', having the universal property, is uniquely defined. Grothendieck also introduced "cofibrations" as functors $\gamma : \Gamma' \to \Gamma$ such that $\gamma^{opp} : \Gamma'^{opp} \to \Gamma^{opp}$ is a fibration. This terminology is slightly unfortunate because the topological analogy is still a fibration – "cofibration" in topology means something completely different. For this reason, now the term "op-fibration" is sometimes used. However, we will stick to Grothendieck's original terminology.

Assume given a fibration $\gamma : \Gamma' \to \Gamma$ and a morphism $f : [a] \to [b]$ in Γ. Then for any $[b'] \in \Gamma'_{[b]}$, we by definition have a Cartesian morphism $f' : [a'] \to [b']$, and using the universal property of the Cartesian morphism, one checks that the correspondence $[b'] \mapsto [a']$ is functorial: we have a functor $f^* : \Gamma'_{[b]} \to \Gamma'_{[a]}$, $[b'] \mapsto [a']$. Using condition (ii) of Definition 4.5, one checks that for any composable pair of maps f, g, we have a natural isomorphism $(f \circ g)^* \cong g^* \circ f^*$, and there is a compatibility constraint for these isomorphisms when we are given a composable triple f, g, h. All in all, the correspondence $[a] \mapsto \Gamma'_{[a]}$, $f \mapsto f^*$ defines a contravriant "weak functor" from Γ to the category of small categories. Conversely, every such "weak functor", appropriately defined, arises in this way. This was the main reason for Grothendieck's definition of a fibration – it gives a nice and short replacement for the cumbersome notion of a weak functor, with all its higher isomorphisms and compatibility constraints.

Today, we will only need one basic fact about fibrations, and we will use it without a proof.

Definition 4.6. A fibration $\gamma : \Gamma' \to \Gamma$ is *locally constant* if for any f in Γ, the functor f^* is an equivalence.

Proposition 4.7. *Assume given a connected small category* Γ *and a locally constant fibration* $\gamma : \Gamma' \to \Gamma$*. Then the homotopy fiber of the induced map* $|\gamma| : |\Gamma'| \to |\Gamma|$ *is naturally homotopy equivalent to the realization* $|\Gamma'_{[a]}|$ *of the fiber over any object* $[a] \in \Gamma$*.* □

4.5 Computation of $|\Lambda|$.

We can now prove that the realization $|\Lambda|$ is equivalent to $\mathbb{C}P^\infty$. We start with the following.

Lemma 4.8. *The realization* $|\Lambda_\infty|$ *is contractible.*

Proof. By Corollary 4.4, it suffices to prove that $H_\bullet(\Lambda_\infty, E) \cong E([1])$ for any locally constant $E \in \operatorname{Fun}(\Lambda_\infty, k)$. The homology of the category Λ_∞ can be computed by a complex similar to (3.2): we take (3.2) and remove everything except for the two right-most columns. We leave it to the reader to check that this indeed computes $H_\bullet(\Lambda_\infty, E)$ (while the rows of the complex now have only length 2, they still compute the homology of the infinite cyclic group $\mathbb{Z} = \mathsf{Aut}([n])$, and the same proof as in Lemma 3.2 works). Since we now only have two columns, and one of them is contractible, the Connes' exact sequence reduces to an isomorphism

$$H_\bullet(\Delta^{opp}, j^*E) \cong H_\bullet(\Lambda_\infty, E).$$

Since j^*E is obviously locally constant, it suffices to check that the realization $|\Delta^{opp}|$ of the category Δ^{opp} is contractible. This is clear — Δ^{opp} has an initial object. □

Lemma 4.9. *The natural functor* $\Lambda_\infty \to \Lambda$ *is a locally constant fibration whose fiber is the groupoid* $\mathsf{pt}_\mathbb{Z}$ *with one object whose automorphisms group is* $\mathbb{Z}$*.*

Proof. We use the combinatorial description of the category Λ. Then for any $[n], [m] \in \Lambda$, the map $\Lambda_\infty([n], [m]) \to \Lambda([n], [m])$ is surjective by definition, and one checks easily that *any* map $f \in \Lambda_\infty([n], [m])$ is Cartesian. The fiber, again by definition, has one object, and its automorphism group is freely generated by the automorphism σ. □

Proposition 4.10. *We have a homotopy equivalence* $|\Lambda| \cong \mathbb{C}P^\infty \cong BU(1)$*.*

Proof. By Proposition 4.7 and Lemma 4.9, the homotopy fiber of the map $|\Lambda_\infty| \to |\Lambda|$ is homotopy equivalent to $|\mathsf{pt}_\mathbb{Z}|$, and since $|\Lambda_\infty|$ is contractible, this implies that $|\mathsf{pt}_\mathbb{Z}|$ is homotopy equivalent to the loop space of $|\Lambda|$. But $\mathsf{pt}_\mathbb{Z}$ is a groupoid, so that $|\mathsf{pt}_\mathbb{Z}|$ is equivalent to the classifying space $B\mathbb{Z} \cong S^1$. This means that $|\Lambda|$ has only one non-trivial homotopy group, namely $\pi_2(|\Lambda|) = \mathbb{Z}$, so that it must be the Eilenberg-MacLane space $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. □

As a corollary, we see that the derived category $\mathcal{D}_{lc}(\Lambda, k)$ of complexes of cyclic objects with locally constant homology objects is equivalent to the derived category of complexes of sheaves of $\mathbb{C}P^\infty$ with locally constant homology sheaves. The objects in this latter category are also know as $U(1)$-equivariant constructible sheaves on the point pt.

Lecture 5.

The structure of $\mathcal{D}_{lc}(\Lambda, k)$ and $\operatorname{Fun}(\Lambda, k)$; Dold-Kan equivalence, mixed complexes. Cyclic bimodules. Cyclic homology as a derived functor.

5.1 The structure of the category $\operatorname{Fun}(\Lambda, k)$.

In the last lecture, we have proved that the geometric realization $|\Lambda|$ of the Connes' cyclic category is homotopy equivalent to the infinite projective space $\mathbb{C}P^\infty$. In particular, we have an equivalence

$$\mathcal{D}_{lc}(\Lambda, k) \cong \mathcal{D}_{lc}(\operatorname{Shv}(\mathbb{C}P^\infty, k)),$$

where $\mathcal{D}_{lc}$ means "the full subcategory in the derived category $\mathcal{D}(\Lambda, k)$ spanned by complexes with locally constant homology", and similarly in the right-hand side. The category in the right-hand side is also equivalent to the derived category of S^1-equivariant sheaves on a point. Besides these topological descriptions, there is also the following very simple combinatorial description.

Let $\mathcal{D}^{per}(k\text{-Vect})$ be the periodic derived category of the category k-Vect – namely, $\mathcal{D}^{per}(k)$ is the triangulated category obtained by considering the category of quadruples $\langle V_+, V_-, d_+, d_- \rangle$ of two vector spaces V_+, V_- and two maps $d_+ : V_+ \to V_-$, $d_- : V_- \to V_+$ such that $d_+ \circ d_- = d_- \circ d_+ = 0$, and inverting quasiisomorphisms. Equivalently, $\mathcal{D}^{per}(k)$ is the homotopy category of 2-periodic complexes $V_\bullet$ of k-vector spaces (with $V_+ = V_{2\bullet}$, $V_- = V_{2\bullet+1}$, and d_+, d_- being the components of the differential). Just as the usual derived category $\mathcal{D}(k\text{-Vect})$ has filtered version $\mathcal{D}\mathsf{F}(k\text{-Vect})$, we define the filtered periodic category $\mathcal{D}\mathsf{F}^{per}(k\text{-Vect})$ by considering 2-periodic filtered complexes $F^\bullet V_\bullet$ such that $F^\bullet V_\bullet \cong F^{\bullet+1} V_{\bullet+2}$ – note the shift in the filtration! Then for any cyclic vector space $E \in \operatorname{Fun}(\Lambda, k)$, the periodic cyclic homology $HP_\bullet(E)$ equipped with the Hodge filtration is an object in $\mathcal{D}\mathsf{F}^{per}(k\text{-Vect})$, so that we have a natural functor

$$HP_\bullet(-) : \mathcal{D}(\Lambda, k) \to \mathcal{D}\mathsf{F}^{per}(k\text{-Vect}).$$

Exercise 5.1. *Show that the induced functor* $\mathcal{D}_{lc}(\Lambda, k) \to \mathcal{D}\mathsf{F}^{per}(k\text{-Vect})$ *is an equivalence of categories. Hint: both categories are generated by* k*, so that it suffices to compare* $\operatorname{Ext}^\bullet(k, k)$.

Thus an object $\mathcal{D}_{lc}(\Lambda, k)$, when compared to its periodic cyclic homology equipped with the Hodge filtration, contains exactly the same amount of information, we lose nothing by taking $HP_\bullet(-)$. What can be said about non-constant cyclic vector spaces — in other words, how complicated is the category $\operatorname{Fun}(\Lambda, k)$? Unfortunately, the answer is "very complicated".

This might not seem surprising, because the category Λ contains so many maps. However, so does the category Δ. Nevertheless, there is the following surprising fact, discovered about 50 years ago independently by A. Dold and D. Kan.

Theorem 5.1 (Dold,Kan). *The abelian category* $\operatorname{Fun}(\Delta^{opp}, k)$ *of simplicial* k*-vector spaces is equivalent to the category* $C^{\leq 0}(k)$ *of complexes of* k*-vector spaces cocentrated in non-positive degrees.*

Proof. There are many proofs, but they all involve either non-trivial computations, or non-trivial combinatorics. We will not give any of them, but we will indicate what the equivalence is. Given a simplicial vector space $E \in \operatorname{Fun}(\Delta^{opp}, k)$, we take its stadrd complex $E_\bullet$, and we replace it with its *normalized quotient* $N(E)_\bullet$ given by

$$N(E)_i = E_i / \sum \operatorname{\mathsf{Im}} s_j,$$

where $s_j : E_{i-1} \to E_i$ are the degeneration maps (induced by surjective maps $[i] \to [i-1]$). □

There exist also various generalizations of the Dold-Kan equivalence. First, the category $\operatorname{Fun}(\Delta, k)$ of co-simplicial vector spaces is equivalent to the category $C^{\geq 0}(k)$ of complexes concentrated in non-negative degrees (this is not surprising, since $\operatorname{Fun}(\Delta, k)$ and $\operatorname{Fun}(\Delta^{opp}, k)$ are more-or-less dual to each other). One can also consider the subcategory $\Delta_+ \subset \Delta$ with the same objects, and only those maps $[n] \to [m]$ which send the first element to the first element. Then $\operatorname{Fun}(\Delta_+^{opp}, k)$ is equivalent to the category of k-vector spaces graded by non-positive integers (restriction to $\Delta_+^{opp} \subset \Delta^{opp}$ corresponds to forgetting the differential in the complex). Finally, if one "truncates" Δ and considers the full subcategory $\Delta_{\leq n} \subset \Delta$ spanned by objects $[1], \dots, [n]$, then $\operatorname{Fun}(\Delta_{\leq n}^{opp}, k)$ is equivalent to the category $C^{[1-n,0]}(k)$ of complexes concentrated in degrees from $1 - n$ to 0, and similarly for $\operatorname{Fun}(\Delta_{\leq n}, k)$ and for Δ_+.

Now, we have a natural embedding $\Delta^{opp} \subset \Lambda$, so that we have a flag of subcategories $\Delta_+^{opp} \subset \Delta^{opp} \subset \Lambda$. We know that Λ is self-dual, $\Lambda \cong \Lambda^{opp}$. One checks easily that Δ_+^{opp} is preserved by this self-duality — we have $\Delta_+^{opp} \cong \Delta_+$. The intermediate category $\Delta^{opp} \subset \Lambda$ is not preserved, so that by duality, we get an embedding $\Delta \subset \Lambda$. All in all, we have the following diagram.

$$\begin{array}{ccc} \Delta_+ \cong \Delta_+^{opp} & \longrightarrow & \Delta \\ \big\downarrow & & \big\downarrow \\ \Delta^{opp} & \longrightarrow & \Lambda \cong \Lambda^{opp}. \end{array}$$

Applying restrictions and the Dold-Kan equivalence, we associate to any cyclic vector space $E \in \operatorname{Fun}(\Lambda, k)$ a complex $E_\bullet \in C^{\leq 0}(k)$ and a complex $E^\bullet \in C^{\geq 0}(k)$, and since the diagram of categories commutes, we also have natural identifications $E_i \cong E^i$ as k-vector spaces. In other words, we have a collection E_i, $i \geq 0$ of k-vector spaces and *two* differentials $b : E_i \to E_{i-1}$, $B : E_i \to E_{i+1}$. One can check that these differentials anti-commute, $bB + Bb = 0$. The result is what is known in the literature as a *mixed complex.*

Definition 5.2. A *mixed complex* $E_\bullet$ is a collection E_i, $i \geq 0$ of k-vector spaces and two maps $b : E_i \to E_{i-1}$, $B : E_i \to E_{i+1}$ such that $b^2 = B^2 = bB + Bb = 0$.

Mixed complexes form a nice abelian category $M^{\leq 0}(k)$ which is not much more complicated than the category of complexes $C^{\leq 0}(k)$, and we have a comparison functor $\operatorname{Fun}(\Lambda, k) \to M(k)$. But the obvious analog of the Dold-Kan Theorem is *wrong* — the comparison functor is not an equivalence.

The only fact which is true is the following: define the derived category $\mathcal{D}(M(k))$ of mixed complexes by inverting the maps which are quasiisomorphisms with respect to the differential b. Then the comparison functor $\mathcal{D}_{lc}(\Lambda, k) \to \mathcal{D}(M(k))$ is an equivalence (and $\mathcal{D}(M(k))$ is equivalent to $\mathcal{D}\mathsf{F}^{per}(k)$ – this is an instance of the so-called *Koszul*, or $S - \Lambda$ *duality*). However, even when we pass to the restricted categories $\operatorname{Fun}(\Lambda_{\leq n}, k)$, $M^{[1-n,0]}(k)$, with the obvious notation, the comparison functor probably is an equivalence only if k has characteristic 0. And for non-locally constant functors, things only get worse.

To sum up: while in the literature on cyclic homology people often use mixed complexes as a basic object, especially in characteristic 0, in reality, cyclic vector spaces contain strictly more information. And we will see later at least one example where this extra information is crucially important.

5.2 Projecting to $\mathcal{D}_{lc}(\Lambda, k)$.

One moral of the above story is that it is much preferable to work with the locally constant subcategory $\mathcal{D}_{lc}(\Lambda, k)$ rather then with the whole category $\mathcal{D}(\Lambda, k)$. An immediate problem is that the cyclic vector space $A_\# \in \operatorname{Fun}(\Lambda, k)$ defined for an associative unital k-algebra A is not locally constant unless $A = k$. However, we can force it to be locally constant. Namely, the embedding $\mathcal{D}_{lc}(\Lambda, k) \subset \mathcal{D}(\Lambda, k)$ admits a left-adjoint functor $\mathsf{lc} : \mathcal{D}(\Lambda, k) \to \mathcal{D}_{lc}(\Lambda, k)$. Since $\mathcal{D}_{lc}(\Lambda, k) \subset \mathcal{D}(\Lambda, k)$ is a full subcategory, lc is identical on $\mathcal{D}(\Lambda, k)$, and since it contains the constant cyclic vector space k^Λ which corepresents the homology functor, the homology functor factors through lc, so that for any $E \in \mathcal{D}(\Lambda, k)$, we have a canonical isomorphism $H_\bullet(\Lambda, E) \cong H_\bullet(\Lambda, \mathsf{lc}(E))$.

The existence of the adjoint functor lc is easy to prove by general nonsense, but it is perhaps more interesting to use the following explicit construction.

Consider the category $\Lambda^{opp} \times \Lambda$, and consider the functor $\mathsf{I} \in \operatorname{Fun}(\Lambda^{opp} \times \Lambda, k)$ spanned by the Hom-functor: we set

$$\mathsf{I}([n] \times [m]) = k\left[\Lambda([m], [n])\right].$$

Denote by π, π^o the natural projections $\pi : \Lambda^{opp} \times \Lambda \to \Lambda$, $\pi : \Lambda^{opp} \times \Lambda \to \Lambda^{opp}$. We claim that for any $E \in \operatorname{Fun}(\Lambda, k)$, we have a natural isomorphism

$$H_\bullet(\Lambda^{opp} \times \Lambda, \mathsf{I} \otimes \pi^* E) \cong H_\bullet(\Lambda, k). \tag{5.1}$$

Indeed, by an obvious version of the Künneth formula, we can compute the homology in the left-hand side first along Λ^{opp}, and then along Λ. Then it suffices to show that for any $[n] \in \Lambda$, we have a functorial isomorphism

$$H_\bullet(\Lambda^{opp}, E([n]) \otimes \mathsf{I}|_{\Lambda^{opp} \times [n]}) \cong E([n]).$$

But here we can take $E([n])$ out of the brackets, so that it suffices to consider the case $E([n]) = k$, and the restriction $\mathsf{I}_{\Lambda^{opp} \times [n]}$ is nothing but the representable functor $k_{[n]}^{\Lambda^{opp}}$, so that its homology is indeed isomorphic to k concentrated in degree 0.

But on the other hand, we can compute the left-hand side of (5.1) by first using the projection π^o. By general nonsense, we have

$$H_\bullet(\Lambda^{opp} \times \Lambda, \mathsf{I} \otimes \pi^* E) \cong H_\bullet(\Lambda^{opp}, L^\bullet \pi^o_!(\mathsf{I} \otimes \pi^* E)),$$

and since $\Lambda \cong \Lambda^{opp}$, we can define $\mathsf{lc}(E) = L^\bullet \pi^o_!(\mathsf{I} \otimes \pi^* E))$. All we have to do is to prove that it is locally constant. Indeed, by the Künneth formula, for any $[n] \in \Lambda^{opp} \cong \Lambda$ we have

$$\mathsf{lc}(E)([n]) = H_\bullet(\Lambda, k_{[n]} \otimes E),$$

where the representable functor $k_{[n]}$ is the restriction of I to $[n] \times \Lambda \subset \Lambda^{opp} \times \Lambda$. But $k_{[n]}$ is clean in the sense of Definition 3.3, so that

$$H_\bullet(\Lambda, k_{[n]} \otimes E) \cong H_\bullet(\Delta^{opp}, k_{[n]}^{\Delta^{opp}} \otimes j^* E).$$

By the well-known Künneth formula for simplicial vector spaces, the right-hand side is canonically isomorphic to

$$H_\bullet(\Delta^{opp}, k_{[n]}^{\Delta^{opp}}) \cong H_\bullet(\Delta^{opp}, E) \otimes H_\bullet(\Delta^{opp}, E),$$

which is manifestly independent of $[n]$.

5.3 Cyclic bimodules.

If one writes down explicitly $\mathsf{lc}(E)$ by using the complex (3.3), the result is very similar to the "third bicomplex" (3.5) for cyclic homology which we defined in Lecture 3. One can also clearly see why that construction only worked in characteristic 0. The columns in (3.5) are naturally assembled into a cyclic object, not in a simplicial one; when we simply imposed the differential $\widetilde{B}$ on them, we in effect forgot the cyclic structure and only considered the underlying simplicial structure. In $\mathsf{char}\, 0$, this did not matter – the cyclic group action on each column is actually trivial, so that we we compute $H_\bullet(\Lambda, \mathsf{lc}(E))$ by (3.3), taking coinvariants with respect to τ can be omitted. In the general case, we do need to compute honestly the cyclic homology $H_\bullet(\Lambda, \mathsf{lc}(E))$.

However, the bicomplex (3.5), although it only worked in $\mathsf{char}\, 0$, was very interesting for the computation of the cyclic homology $HC_\bullet(A)$ of an associative algebra A, because it had a version where the bar resolution $C_\bullet(A)$ of the diagonal A-bimodule could be replaced with an arbitrary resolution $P_\bullet$ (at least for the two rightmost columns). Now that we know the full truth, can we perhaps give a version of that construction which is valid in any characteristic and for all columns, not only the two rightmost ones?

It turns out that we can do even better — it is possible to obtain the whole $\mathsf{lc}_\bullet(A_\#)$ as an object of $\mathcal{D}_{lc}(\Lambda, k)$ completely canonically, without any explicit choice at all, neither of a resolution $P_\bullet$, nor of the homotopy ι, as in Lecture 3, part 3.5. Or rather, the choices do occur, but they are all packed into a single choice of a projective resolution in some appropriate abelian category, and cyclic homology is obtained as a derived functor on this abelian category (just as Hochschild homology is the derived functor on the abelian category A-bimod of A-bimodules).

To construct this new category, which we call the category of *cyclic A-bimodules*, we use the technique of fibered and cofibered categories explained in Lecture 4.

Assume given a small category Γ and a category $\mathcal{C}$ equipped with a cofibration $\pi : \mathcal{C} \to \Gamma$. Thus for any $[a] \in \Gamma$, we have the fiber $\mathcal{C}_{[a]}$, and for any map $f : [a] \to [b]$, we have a transition functor $f_! : \mathcal{C}_{[a]} \to \mathcal{C}_{[b]}$. Denote by $\mathrm{Sec}(\mathcal{C})$ the category of sections $\Gamma \to \mathcal{C}$ of the projection $\pi : \mathcal{C} \to \Gamma$. Explicitly, an object $M \in \mathrm{Sec}(\mathcal{C})$ is given by a collection of objects $M_{[a]} \in \mathcal{C}_{[a]}$ for all $[a] \in \Gamma$, and of transtion maps $\iota_f : f_! M_{[a]} \to M_{[b]}$ for all $f : [a] \to [b]$, subject ot natural compatibilities.

Proposition 5.3. *Assume that all the fibers $\mathcal{C}_{[a]}$ of the cofibration $\pi : \mathcal{C} \to \Gamma$ are abelian, and all the transition functors $f_! : \mathcal{C}_{[a]} \to \mathcal{C}_{[b]}$ are left-exact. Then the category $\mathrm{Sec}(\mathcal{C})$ is abelian.*

Sketch of a proof. To prove that an additive category is abelian, one has to show that it has kernels and cokernels, and they satisfy some additional conditions (such as "the cokernel of the kernel is isomorphic to the kernel of the cokernel). The kernel and cokernel of a map $\varphi : M \to N$ in $\mathrm{Sec}(\mathcal{C})$ are taken fiberwise, $\mathsf{Coker}\, \varphi_{[a]} = \mathsf{Coker}\, \varphi_{[a]}$, $\mathsf{Ker}\, \varphi_{[a]} = \mathsf{Ker}\, \varphi_{[a]}$. The transition maps of the kernel are induced from those of M, and to construct the transtion maps for the cokernel, one uses the fact that the transition functors $f_!$ are right-exact. All the extra conditions can be checked fiberwise, where they follow from the assumption that all fibers are abelian. $\square$

As we can see from its explicit description, the category $\mathrm{Sec}(\mathcal{C})$ is rather large. One can define a smaller subcategory by only considering those sections that are *Cartesian* – that is, any $f : [a] \to [b]$ goes to a Cartesian map in $\mathcal{C}$. Equivalently, in the explicit description above, all the transition maps $\iota_f : f_! M_{[a]} \to M_{[b]}$ must be isomorphisms. This is often a much smaller category, but it need not be abelian (unless all the transition functors are exact, not just right-exact, which rarely happens in practice). A reasonable thing to do is to consider the derived category $\mathcal{D}(\mathrm{Sec}(\mathcal{C}))$ and the full subcategory $\mathcal{D}_{cart}(\mathrm{Sec}(\mathcal{C})) \subset \mathcal{D}(\mathrm{Sec}(\mathcal{C}))$ of complexes with Cartesian cohomology.

Assume now given an associative unital algebra A, and consider the category A-bimod of A-bimodules. This is a unital tensor category: we have the (non-symmetric) associative tensor product functor $m : A\text{-}\mathsf{bimod} \times A\text{-}\mathsf{bimod} \to A\text{-}\mathsf{bimod}$, $M_1 \times M_2 \mapsto M_1 \otimes_A M_2$. Moreover, we can also consider the category $A^{\otimes 2}$-bimod of $A^{\otimes 2}$-bimodules, and the exterior product functor $A\text{-}\mathsf{bimod} \times \mathcal{A}\text{-}\mathsf{bimod} \to A^{\otimes 2}\text{-}\mathsf{bimod}$, $M_1 \otimes M_2 \mapsto M_1 \boxtimes M_2$ is a fully faithful embedding. The tensor product functor then obviously extends to a right-exact functor $m : A^{\otimes 2}\text{-}\mathsf{bimod} \to A\text{-}\mathsf{bimod}$. Since the tensor product on A-bimod is associative, we can iterate this and obtain the right-exact tensor product functors $m_n : \mathcal{A}^{\otimes n}\text{-}\mathsf{bimod} \to A\text{-}\mathsf{bimod}$ for any $n \geq 1$. For $n = 0$, we take $A^{\otimes 0}$ to be k, and $m_0 : k\text{-Vect} \to A\text{-}\mathsf{bimod}$ is the functor which sends k to the unit object of A-bimod.

What we want to do now is to take the construction of the cyclic vector space $A_\#$, and replace the unital associative algebra A with the unital associative tensor category A-bimod. The result is a category cofibered over Λ which we denote by A-$\mathsf{bimod}_\#$. The fibers are given by

$$A\text{-}\mathsf{bimod}_\#([n]) = A^{\otimes n}\text{-}\mathsf{bimod},$$

and the transition functors $f_!$ are induced by the multiplication functors m_n by the same formula (3.4) as in the definition of the cyclic vector space $A_\#$.

Definition 5.4. A *cyclic A-bimodule M* is a Cartesian section of the cofibration A-$\mathsf{bimod}_\# \to \Lambda$.

Explicitly, a cyclic A-bimodule M is given a collection of $M_{[n]} \in A^{\otimes n}$-$\mathsf{bimod}$, $n \geq 1$, and transition maps between them. However, because all transition maps are isomorphisms, the bimodules $M_{[n]}$, $n \geq 2$ can be computed from the first bimodule $M_1 = M_{[1]}$ — it suffices to apply the transition functor $f_!$ for some map $f : [1] \to [n]$. Since such a map is not unique, extending a given $M_1 \in A$-bimod to a cyclic bimodule requires extra data. It is enough, for instance, to specify an $A^{\otimes 2}$-bimodule isomorphism $\tau : A \boxtimes M \to M \boxtimes A$ such that the induced maps $\tau_{23} : A \boxtimes A \boxtimes M \to A \boxtimes M \boxtimes A$, $\tau_{12} : A \boxtimes M \boxtimes A \to M \boxtimes A \boxtimes A$, $\tau_{23} : M \boxtimes A \boxtimes A \to A \boxtimes A \boxtimes M$ satisfy

$$\tau_{31} \circ \tau_{12} \circ \tau_{23} = \mathsf{id}\,. \tag{5.2}$$

The category of cyclic A-bimodules is abelian, but this is an accident: the category that must be abelian for general reasons is the category $\operatorname{Sec}(A\text{-}\mathsf{bimod}_\#)$ of all sections of the cofibration A-$\mathsf{bimod}_\# \to A$. Thus we consider the derived category $\mathcal{D}\Lambda(A\text{-}\mathsf{bimod}) = \mathcal{D}(\operatorname{Sec}(A\text{-}\mathsf{bimod}_\#))$, and we define the *derived category of cyclic bimodules* $\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod})$ as the full subcategory

$$\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod}) = \mathcal{D}_{cart}(\operatorname{Sec}(A\text{-}\mathsf{bimod}_\#)) \subset \mathcal{D}(\operatorname{Sec}(A\text{-}\mathsf{bimod}_\#)) = \mathcal{D}\Lambda(A\text{-}\mathsf{bimod})$$

of complexes with Cartesian cohomology.

We note that even though the category $\operatorname{Sec}_{cart}(A\text{-}\mathsf{bimod}_\#)$ of cyclic bimodules *per se* happens to be abelian, its derived category is smaller than $\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod})$. For instance, if $A = k$, so that $A^{\otimes n} = k$ for any $n \geq 0$, with identical transition funcotrs, then $\operatorname{Sec}(A\text{-}\mathsf{bimod})$ is exactly equivalent to $\operatorname{Fun}(\Lambda, k)$, and the Cartesian sections correspond to locally constant functors. But every locally constant cyclic vector space is constant, while $\mathcal{D}\Lambda_{lc}(k\text{-}\mathsf{bimod}) \cong \mathcal{D}_{lc}(\Lambda, k)$ is a non-trivial category.

5.4 Cyclic homology as a derived functor.

We now recall that the Hochschild homology $H_\bullet(A, M)$ with coefficients in an A-bimodule M is by definition the derived functor of the functor $M \mapsto A \otimes_{A^{opp} \otimes A} M$, which can be equivalently described as the following right-exact *trace functor*

$$\mathsf{tr}(M) = M/\{am - ma \mid a \in A, m \in M\}.$$

We prefer this description because it clearly has the following "trace property": for any two A-bimodules M, N, there exists a canonical isomorphism $\mathsf{tr}(M \otimes_A N) \cong \mathsf{tr}(N \otimes M)$. Even more generally, for any $A^{\otimes n}$-module M_n, we can define

$$\mathsf{tr}(M_n) = M/\{am - m\sigma(a) \mid a \in A^{\otimes n}, m \in M_n\},$$

where $\sigma : A^{\otimes n} \to A^{\otimes n}$ is the cyclic permutation. These trace functors obviously commute with the transition functors of the cofibered category A-$\mathsf{bimod}_\#$, so that tr extends to a functor $\mathsf{tr} : A\text{-}\mathsf{bimod}_\# \to k\text{-}\mathrm{Vect}$ which sends every Cartesian map to an isomorphism of vector spaces.

We can now apply the trace functor tr fiberwise, to obtain a Cartesian functor $\mathsf{tr} : A\text{-}\mathsf{bimod}_\# \to k\text{-Vect}^\Lambda$, where $k\text{-Vect}^\Lambda = k\text{-Vect} \times \Lambda$ is the constant cofibration with fiber k-Vect. This induces a right-exact functor

$$\mathsf{tr} : \operatorname{Sec}(A\text{-}\mathsf{bimod}_\#) \to \operatorname{Fun}(\Lambda, k),$$

and since $\mathsf{tr} : A\text{-}\mathsf{bimod}_\# \to k\text{-Vect}^\Lambda$ is Cartesian, the derived functor $L^\bullet \mathsf{tr} : \mathcal{D}\Lambda(A\text{-}\mathsf{bimod}) \to \mathcal{D}(\Lambda, k)$ sends $\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod})$ into $\mathcal{D}_{lc}(\Lambda, k)$.

Definition 5.5. The *cyclic homology* $HC_\bullet(A, M)$ of the algebra A with coefficients in some $M \in \mathcal{D}\Lambda(A\text{-}\mathsf{bimod})$ is given by

$$HC_\bullet(A, M) = H_\bullet(\Lambda, L^\bullet \mathsf{tr}(M)).$$

In general, it is not easy to construct cyclic bimodules. However, one cyclic bimodule manifestly exists for any algebra A — this is $A_\#$, with the diagonal $A^{\otimes n}$-bimodule structure on every $A_\#([n]) = A^{\otimes n}$.

Proposition 5.6. *For any algebra A, we have $HC_\bullet(A, A_\#) \cong HC_\bullet(A)$.*

Proof. Notice that we can define a simpler notion of cyclic homology with coefficients in some $M \in \mathcal{D}\Lambda(A\text{-}\mathsf{bimod})$ – we can forget the $A^{\otimes n}$-bimodule structure on $M([n])$, and treat M simply as a complex of cyclic vector spaces. Denote $H_\bullet(\Lambda, M)$ by $HC'_\bullet(A, M)$. We have obvious projection maps $M([n]) \to \mathsf{tr}(M([n]))$ which induce a functorial map

$$HC'_\bullet(A, M) \to HC_\bullet(A, M). \tag{5.3}$$

We have to show that this map is an isomorphism for $M = A_\#$. It suffices to prove that it is an isomorphism for any $M \in \mathcal{D}\Lambda(A\text{-}\mathsf{bimod})$, or even for any $M \in \operatorname{Sec}(A\text{-}\mathsf{bimod}_\#)$. We note that the evaluation at $[n] \in \Lambda$ induces a functor $\operatorname{Sec}(A\text{-}\mathsf{bimod}) \to A^{\otimes n}\text{-}\mathsf{bimod}$, which has a left-adjoint $i_!^{[n]} : A^{\otimes n} \to \operatorname{Sec}(A\text{-}\mathsf{bimod})$. Explicitly, for any $A^{\otimes n}$-bimodule P, we have

$$i_!^{[n]} P([m]) = \bigoplus_{f:[n]\to[m]} f_! P. \tag{5.4}$$

If P is projective, then $i_!^{[n]} P$ is projective in $\operatorname{Sec}(A\text{-}\mathsf{bimod})$ by adjunction, and $\operatorname{Sec}(A\text{-}\mathsf{bimod})$ obviously has enough projectives of this type, so it is enough to prove that (5.3) is an isomorphism for $M = i_!^{[n]} P$. Even further, it is enough to consider objects $P^n \in \operatorname{Sec}(A\text{-}\mathsf{bimod})$ given by

$$P^n = i_!^{[n]} A^{\otimes n} \otimes A^{\otimes n},$$

where on the right-hand side we have the free $A^{\otimes n}$-bimodule with one generator.

Since P^n is projective, we have $L^i \mathsf{tr}(P^n) = 0$ for $i \geq 1$, and $\mathsf{tr}(P^n) \in \operatorname{Fun}(\Lambda, k)$ is isomorphic to $A^{\otimes n} \otimes k_{[n]}$; thus the right-hand side of (5.3) with $M = P^n$ is canonically isomorphic to $A^{\otimes n}$ in degree 0, and trivial otherwise. As for the left-hand side, we see from (5.4) that

$$P^n([m]) = \bigoplus_{f:[n]\to[m]} A^{\otimes(n+m)}.$$

In particular, it is clean, so that $H_\bullet(\Lambda, P^n)$ can be computed by the complex (3.3). We leave it to the reader to check that the resulting complex $P^n_\bullet$ can be described as follows: if we take the augmented bar resolution $C_\bullet(A)$, $C_i(A) = A^{\otimes i+1}$ and consider the n-fold tensor power $C^n_\bullet = C_\bullet(A)^{\otimes n}$, then

$$P^n_i = C^n_{i+1}$$

for any $i \geq 0$. Since the whole complex $C^n_\bullet$, being the n-fold tensor power of the acyclic complex $C_\bullet(A)$, is itself acyclic, the complex $P^n_\bullet$ is a resolution for the 0-th term C^n_0, which is again $A^{\otimes n}$. □

Lecture 6.

Cyclic homology for general tensor categories. Morita-invariance. Example: cyclic homology of a group algebra. Regulator map.

6.1 Cyclic homology for general tensor categories.

In the last lecture, we have constructed the derived category $\mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod})$ of cyclic bimodules over an associative algebra A, and we have re-defined cyclic homology by means of a trace functor $\mathsf{tr} : \mathcal{D}\Lambda_{lc}(A\text{-}\mathsf{bimod}) \to \mathcal{D}_{lc}(\Lambda, k)$. The algebra A itself essentially only appeared in the construction though the tensor category A-bimod of A-bimodules. A natural question is, can we do the same construction for a more general tensor category $\mathcal{C}$?

To start with, we need to construct a category $\mathcal{C}_\#$ cofibered over Λ. Here there is one problem: there is no well-defined tensor product for general abelian categories. Namely, we can introduce the following.

Definition 6.1. Assume given two abelian k-linear categories $\mathcal{C}_1$, $\mathcal{C}_2$. The *tensor product* $\mathcal{C}_1 \otimes \mathcal{C}_2$ is a k-linear abelian category equipped with a functor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2$ which is k-linear and right-exact in each variable, and has the following universal property:

- for any k-linear abelian category $\mathcal{C}'$, any functor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}'$ which is k-linear and right-exact in each variable factors through $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2$, and the facrorization is unique up to an isomorphism.

The problem is, while the tensor product in this sense is obviously unique up to an equivalence, it does not always exist. However, it does exist for categories of modules or bimodules: one can show that for any k-algebras A, B, we have $A\text{-}\mathsf{mod} \otimes B\text{-}\mathsf{mod} \cong (A \otimes B)\text{-}\mathsf{mod}$, $A\text{-}\mathsf{bimod} \otimes B\text{-}\mathsf{bimod} \cong (A \otimes B)\text{-}\mathsf{bimod}$ – thus the category $A^{\otimes n}\text{-}\mathsf{bimod}$ which we used in the last lecture is actually $A\text{-}\mathsf{bimod}^{\otimes n}$ in the sense of Definition 6.2. There are other interesting cases, too. Thus we simply impose this as an assumption.

Definition 6.2. A k-linear abelian tensor category $\mathcal{C}$ is *good* if it has arbitrary sums, the tensor product functor is right-exact in each variable, and for any n, there exists a tensor product $\mathcal{C}^{\otimes n}$ in the sense of Definition 6.2.

Remark 6.3. Sometimes in the representation-theoretic literature, "tensor category" means "symmetric tensor category" – that is, the tensor product is not only bilinear, but also symmetric – and tensor categories in the normal sense are called "monoidal". The reason for this is completely unclear to me, and this is bad terminology – in the standard language of category theory, "monoidal" does not imply that the tensor product is a bilinear functor.

Given a good k-linear unital tensor category $\mathcal{C}$, we can literally repeat the construction of the last lecture and obtain a category $\mathcal{C}_\#$ which is cofibered over Λ – the fiber $(\mathcal{C}_\#)_{[n]}$ is the category $\mathcal{C}^{\otimes n}$, and the transition functors $f_!$ are obtained from the tensor product functors $m_n : \mathcal{C}^{\otimes n} \to \mathcal{C}$ (for $n = 0$, we take $\mathcal{C}^{\otimes n} = k\text{-Vect}$, and $m_0 : k\text{-Vect} \to \mathcal{C}$ is the functor which sends k to the unit object in $\mathcal{C}$). Again, the category $\operatorname{Sec}(\mathcal{C}_\#)$ of sections of the cofibration $\mathcal{C}_\# \to \Lambda$ is abelian by Proposition 5.3, so that we can consider the derived category $\mathcal{D}\Lambda(\mathcal{C}) = \mathcal{D}(\operatorname{Sec}(\mathcal{C}_\#))$ and the full triangulated subcategory $\mathcal{D}\Lambda_{lc}(\mathcal{C}) = \mathcal{D}_{cart}(\operatorname{Sec}(\mathcal{C}_\#))$ spanned by Cartesian sections. We will call $\mathcal{D}\Lambda_{lc}(\mathcal{C}_\#)$ the *cyclic envelope* of $\mathcal{C}_\#$.

Cyclic envelope only depends on the tensor category $\mathcal{C}$. However, already to define Hochschild homology $HH_\bullet(\mathcal{C})$ of the category $\mathcal{C}$, we need an extra datum – a right-exact "trace functor".

Definition 6.4. Assume given a good k-linear tensor category $\mathcal{C}$. A *trace functor* on $\mathcal{C}$ is a functor $\mathsf{tr} : \mathcal{C} \to k\text{-Vect}$ which is extended to a functor $\mathsf{tr} : \mathcal{C}_\# \to k\text{-Vect}$ in such a way that $\mathsf{tr}(f)$ is invertible for any Cartesian map f in $\mathcal{C}_\#/\Lambda$.

Explicitly, a trace functor is given by a functor $\mathsf{tr} : \mathcal{C} \to k\text{-Vect}$ and an isomorphism

$$\tau : \mathsf{tr}(M \otimes N) \cong \mathsf{tr}(N \otimes M) \tag{6.1}$$

for any two objects $M, N \in \mathcal{C}$. The isomorphism τ should be functorial in both M and N, and satisfy the condition $\tau_{31} \circ \tau_{12} \circ \tau_{23} = \mathsf{id}$, as in (5.2). We leave it to the reader to check that such an isomorphism τ uniquely defines an extension of tr to the whole category $\mathcal{C}_\#$.

Given a good k-linear tensor category $\mathcal{C}$ equipped with a trace functor tr, we can repeat the construction of the last lecture: we extend tr to a functor $\mathsf{tr} : \mathrm{Sec}(\mathcal{C}_\#) \to \mathrm{Fun}(\Lambda, k)$, and consider the corresponding dervied functor $L^\bullet \mathsf{tr} : \mathcal{D}\Lambda(\mathcal{C}) \to \mathcal{D}(\Lambda, k)$. As before, it sends $\mathcal{D}\Lambda_{lc}(\mathcal{C}) \subset \mathcal{D}\Lambda(\mathcal{C})$ into $\mathcal{D}_{lc}(\Lambda, k)$.

Definition 6.5. *Hochschild homology* $HH_\bullet(\mathcal{C}, \mathsf{tr})$ of the pair $\langle \mathcal{C}, \mathsf{tr} \rangle$ is given by

$$HH_\bullet(\mathcal{C}, \mathsf{tr}) = L^\bullet \mathsf{tr}(\mathsf{I}),$$

where $\mathsf{I} \subset \mathcal{C}$ is the unit object. *Cyclic homology* $HC_\bullet(\mathcal{C}, \mathsf{tr})$ of the pair $\langle \mathcal{C}, \mathsf{tr} \rangle$ is given by

$$HC_\bullet(\mathcal{C}, \mathsf{tr}) = H_\bullet(\Lambda, L^\bullet \mathsf{tr}\, \mathsf{I}_\#),$$

where $\mathsf{I}_\# \in \mathrm{Sec}_{cart}(\mathcal{C}_\#)$ is the Cartesian section of $\mathcal{C}_\# \to \Lambda$ which sends an object $[n] \in \Lambda$ to $\mathsf{I}^{\boxtimes n} \in \mathcal{C}^{\otimes n}$, the n-th power of the unit object $\mathsf{I} \in \mathcal{C}$.

Of course, in the case $\mathcal{C} = A\text{-}\mathsf{bimod}$, tr as in the last lecture, we have $HC_\bullet(A\text{-}\mathsf{bimod}, \mathsf{tr}) = HC_\bullet(A, A_\#) = HC_\bullet(A)$ by virtue of Proposition 5.6.

6.2 Morita-invariance of cyclic homology.

As an application of the general formalism developed above, we prove that Hochschild and cyclic homology of an associative algebra A only depends on the category $A\text{-}\mathsf{mod}$ of left A-modules. This is known as *Morita invariance.*

A typical situation is the following. Assume given two k-algebras A, B, and a k-linear functor $F : A\text{-}\mathsf{mod} \to B\text{-}\mathsf{mod}$. Assume that F is right-exact and commutes with infinite direct sums. Consider the B-module $P = F(A)$. Since $\mathrm{End}_A(A) = A^{opp}$, P is not only a left B-module, but also a right A-module – in other words, an $A - B$-bimodule. By definition, we have $F(A) = A \otimes_A P$; since F is right-exact and commutes with arbitrary sums, the same is true for any $M \in A\text{-}\mathsf{mod}$ – the bimodule P represents the functor F in the sense that we have a functorial isomorphism

$$F(M) \cong M \otimes_A P.$$

If F is an equivalence of categories, then the inverse equivalence F^{-1} is of course also right-exact and commutes with sums; thus we have a $B - A$-bimodule P^o representing F^{-1}, and since $F \circ F^{-1} \cong \mathsf{Id}$, $F^{-1} \circ F \cong \mathsf{Id}$, we have isomorphisms

$$A \cong P \otimes_B P^o \in A\text{-}\mathsf{bimod}, \qquad B \cong P^o \otimes_A P \in B\text{-}\mathsf{bimod}. \tag{6.2}$$

Proposition 6.6. *Assume given two associative k-algebras A, B, and an equivalence $A\text{-}\mathsf{mod} \cong B\text{-}\mathsf{mod}$. Then there exist natural isomorphisms $HH_\bullet(A) \cong HH_\bullet(B)$, $HC_\bullet(A) \cong HC_\bullet(B)$.*

Proof. As we have already proved, every right-exact k-linear functor $G : A\text{-}\mathsf{mod} \to A\text{-}\mathsf{mod}$ which commutes with sums is represented by an A-bimodule Q. Conversely, every $Q \in A\text{-}\mathsf{bimod}$ represents such a functor. Tensor product of bimodules corresponds to the composition of functor. Therefore the k-linear tensor category $A\text{-}\mathsf{bimod}$ only depends on the k-linear abelian category $A\text{-}\mathsf{mod}$, and can be recovered as the category of endofunctors of $A\text{-}\mathsf{mod}$ of a certain kind (k-linear, right-exact, preserving sums). Thus in our situation, we have a natural equivalence $F : A\text{-}\mathsf{bimod} \cong B\text{-}\mathsf{bimod}$ of k-linear abelian tensor categories. It induces an equivalence of the corresponding categories of cyclic bimodules. To finish the proof, it suffices to prove that the equivalence $A\text{-}\mathsf{bimod} \cong B\text{-}\mathsf{bimod}$ is compatible with the natural trace functors on both side. This is obvious: for any $M \in A\text{-}\mathsf{bimod}$, we have

$$\mathsf{tr}(M) = A \otimes_{A^{opp} \otimes A} M \cong B \otimes_{B^{opp} \otimes B} (P \otimes P^o) \otimes_{A^{opp} \otimes A} M \cong B \otimes_{B^{opp} \otimes B} F(M) = \mathsf{tr}(F(M)),$$

where P and P^o are as in (6.2). □

6.3 Example: group algebras

Traditionally, in every exposition of cyclic homology, the authors devote some time to one very special case, that of a group algebra. I don't really know why — whether it's because this is needed to construct the regulator map from higher algebraic K-theory, or because there are interesting new things special for the group algebra case, or for some other reason. But let me follow the tradition. This will also give us an example where the general theory of cyclic homology for tensor categories is applied to a tensor category which is not a category of bimodules.

Assume given a group G, and consider the group algebra $k[G]$. This is an associative unital algebra, so it has Hochschild and cyclic homology, and the category of $k[G]$-bimodules is a tensor category. However, since G is a group, the category $G\text{-}\mathsf{mod} = k[G]\text{-}\mathsf{mod}$ of representation of G a.k.a. left $k[G]$-modules is a tensor category in its own right. Moreover, there is an obvious functor $\gamma : G\text{-}\mathsf{mod} \to k[G]\text{-}\mathsf{bimod}$ which sends a representation $V \in G\text{-}\mathsf{mod}$ to a functor $G\text{-}\mathsf{bimod} \to G\text{-}\mathsf{bimod}$ given by $M \mapsto M \otimes V$ (here we use the interpretation of $k[G]$-bimodules as endofunctors of the category $G\text{-}\mathsf{mod}$). This functor is obviously exact and obviously tensor. Explicitly, it is given by

$$\gamma(V) = V \otimes R,$$

where we denote $R = k[G]$, the left $k[G]$-action on $V \otimes R$ is through V and R, and the right action is through R: we have $g_1(v \otimes g)g_2 = g_1 v \otimes g_1 g g_2$. If we have two representations $V_1, V_2 \in G\text{-}\mathsf{mod}$, the natural isomorphism $\gamma(V_1 \otimes V_2) \cong \gamma(V_1) \otimes_{k[G]} \gamma(V_2)$ is given by the map

$$V_1 \otimes V_2 \otimes R \to (V_1 \otimes R) \otimes_{k[G]} (V_2 \otimes R) \tag{6.3}$$

which sends $v_1 \otimes v_2 \otimes g$ to $(v_1 \otimes 1) \otimes (v_2 \otimes g)$, where $1 \in G$ is the unity element.

Since the functor γ is tensor, the usual trace functor tr on $k[G]\text{-}\mathsf{bimod}$ gives by restriction a trace functor $\mathsf{tr}^R = \mathsf{tr} \circ \gamma$ on $G\text{-}\mathsf{mod}$. Explicitly, it is given by

$$\mathsf{tr}^R(V) = (V \otimes R)/\{g_1 v \otimes g_1 g - v \otimes g g_1 \mid g, g_1 \in G, v \in V\},$$

and since the quotient is over all g and all g_1, we might as well replace g with $g g_1^{-1}$. Then we have

$$\mathsf{tr}^R(V) = (V \otimes R)/\{g_1 v \otimes g_1 g g_1^{-1} - v \otimes g\} = (V \otimes R)_G,$$

the G-coinvariants in the G-representation $V \otimes R$, where R is equipped with the adjoint G-action. One can also check, and this is important, that the identification survives on the level of derived functors — the natural map

$$L^\bullet \mathsf{tr}(\gamma(V)) \to L^\bullet \mathsf{tr}^R(V) = H_\bullet(G, V \otimes R) \tag{6.4}$$

is an isomorphism in all degrees. For example, for the trivial representation $V = k$, we obtain an isomorphism $HH_\bullet(k[G]) \cong H_\bullet(G, R)$. The isomorphism $\tau^R : \mathsf{tr}^R(V_1 \otimes V_2) \cong \mathsf{tr}^R(V_2 \otimes V_1)$ of (6.1) is induced by the usual symmetry isomorphism $V_1 \otimes V_2 \to V_2 \otimes V_1$ and the isomorphism (6.3); explicitly, τ^R is the map on the spaces of coinvariants induced by the map

$$\widetilde{\tau}^R : V_1 \otimes V_2 \otimes R \to V_2 \otimes V_1 \otimes R, \qquad \widetilde{\tau}^R(v_1 \otimes v_2 \otimes g) = gv_2 \otimes v_1 \otimes g. \tag{6.5}$$

One easily checks that the map $\widetilde{\tau}^R$ defined in this way is actually a map of G-representations.

Applying the general theory of cyclic homology with coefficients, we extend this isomorphism to an isomorphism

$$HC_\bullet(k[G]) \cong HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^R).$$

We now note that the adjoint representation $R = k[G]$ canonically splits into a direct sum $R = \bigoplus_{\langle g\rangle} R^g$ over the conjugacy classes $\langle g\rangle \subset G$, $R^g = k[\langle g\rangle]$, and this induces a canonical direct sum decomposition

$$\mathsf{tr}^R = \bigoplus_{\langle g\rangle} \mathsf{tr}^g \tag{6.6}$$

of the trace functor tr^R: we set $\mathsf{tr}^g(V) = (V \otimes R^g)_G$, and since the isomorphism $\widetilde{\mathsf{tr}}^R$ of (6.5) obviously respects the direct sum decomposition, the isomorphism τ^R induces isomorphisms (6.1) for every component tr^g. Therefore we actually have a canonical direct sum decomposition of cyclic homology:

$$HC_\bullet(k[G]) = \bigoplus_{\langle g\rangle} HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^g), \tag{6.7}$$

and a corresponding decomposition for $HH_\bullet(k[G])$.

However, we can say more. Consider the component tr^1 in the decomposition (6.6) which corresponds to the unity element $1 \in G$. Then we have $\mathsf{tr}^1(V) = V_G$, the space of G-coinvariants, and

$$HH_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1) \cong H_\bullet(G, k).$$

What can we say about the cyclic homology $HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1)$? Looking at (6.5), we see that the isomorphism $\mathsf{tr}^1(V_1 \otimes V_2) \cong \mathsf{tr}^1(V_2 \otimes V_1)$ for the trace functor tr^1 is induced by the symmetry isomorphism $v_1 \otimes v_2 \to v_2 \otimes v_1$. We can rephrase this in the following way: since the tensor category G-mod is symmetric, *any* right-exact functor $F : G\text{-}\mathsf{mod} \to k\text{-Vect}$ canonically extends to a trace functor $F_\# : G\text{-}\mathsf{mod}_\# \to k\text{-Vect}$, and it is this trace functor structure that tr^1 has — we have $\mathsf{tr}^1 \cong \mathsf{Coinv}_\#$, where $\mathsf{Coinv} : G\text{-}\mathsf{mod} \to k\text{-Vect}$ is the coinvariants functor, $V \mapsto V_G$.

In other words, the identity functor $\mathsf{Id} : G\text{-}\mathsf{mod} \to G\text{-}\mathsf{mod}$ *can also be considered as a trace functor*, albeit with values in G-mod rather than k-Vect, so that we have a functor

$$L^\bullet \mathsf{Id} : \mathcal{D}\Lambda(G\text{-}\mathsf{mod}) \to \mathcal{D}(\Lambda, G\text{-}\mathsf{mod}) = \mathcal{D}(\Lambda \times \mathsf{pt}_G, k),$$

where pt_G is the category with one object with automorphism group G, and the trace functor $L^\bullet \mathsf{tr}^1$ factors through $L^\bullet \mathsf{Id}$, so that we have

$$HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1) = H_\bullet(\Lambda \times \mathsf{pt}_G, L^\bullet \mathsf{Id}(\mathsf{I}_\#)).$$

Moreover, Id is exact, so that there is no need to take its derived functor, and we simply have $L^\bullet \mathsf{Id}(\mathsf{I}_\#) = \mathsf{Id}(\mathsf{I}_\#) = k^{\Lambda \times \mathsf{pt}_G}$, the constant cyclic k-vector space with the trivial action of G. Thus by the Künneth formule, we have

$$HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1) \cong H_\bullet(\Lambda \times \mathsf{pt}_G), k) = H_\bullet(\Lambda, k) \otimes H_\bullet(\mathsf{pt}_G, k).$$

Since $H_\bullet(\mathsf{pt}_G, k) = H_\bullet(G, k) = HH_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1)$, we conclude that *the Hodge-to-de Rham spectral sequence for the cyclic homology $HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1)$ canonically degenerates*: we have a canonical isomorphism

$$HC_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1) \cong HH_\bullet(G\text{-}\mathsf{mod}, \mathsf{tr}^1)[u^{-1}] \tag{6.8}$$

for the unity component in the direct sum decomposition (6.7).

6.4 The regulator map

To finish today's lecture, let me give the standard application of the above computation of groups algebras: I will construct the *higher regulator* a.k.a. *higher Chern character map* from Quillen's higher K-theory to cyclic homology.

Recall that to define higher K-theory of an algebra A, one considers the group $GL_\infty(A) = \lim_\to GL_N(A)$ of infinite matrices over A and its classifying space $BGL_\infty(A)$. This is of course an Eilenberg-MacLane space of type $K(\pi, 1)$. However, Quillen defined a certain very non-trivial operation called *the plus-construction* with replaces a topological space X with another topological space X^+ so that the homology is the same, $H_\bullet(X, \mathbb{Z}) \cong H_\bullet(X^+, \mathbb{Z})$, but X^+ has an abelian fundamental group. Then by definition, higher K-groups of A are given by

$$K^\bullet(A) = \pi_\bullet(BGL_\infty(A)^+),$$

the homotopy groups of the plus-construction $BGL_\infty(A)$.

These groups are very hard to compute (not surprisingly, since homotopy groups in general are hard to compute). Fortunately, to construct the regulator, we do not need to do it. Namely, for any topological space X, there exists a canonical Hurewitz map $\pi_\bullet(X) \to H_\bullet(X)$. The regulator map factors through the Hurewitz map for BGL_∞^+, so that the source of the map we will construct is actually the homology $H_\bullet(BGL_\infty^+)$. At this point, we can also get rid of the plus-construction: by its very definition, $H_\bullet(X) = H_\bullet(X^+)$, so that $H_\bullet(BGL_\infty^+) = H_\bullet(BGL_\infty) = H_\bullet(GL_\infty, \mathbb{Z})$, the homology of the group $GL_\infty(A)$ with trivial coefficients. In fact, our map will further factor through $H_\bullet(GL_\infty(A), k)$.

What is the natural target of the regulator map? Comparison with the Chern character map in algebraic geometry suggests at first that this should the de Rham cohomology groups $H^\bullet_{DR}(-)$ — in our situation, these correspond to the periodic cyclic homology groups $HP_\bullet(A)$. However, it is known that the Chern character actually behaves nicely with respect to the Hodge filtration — the Chern character map $K_0(X) \to \bigoplus_i H^{2i}_{DR}(X)$ for an smooth algebraic variety X actually factors through $\bigoplus_i F^i H^{2i}_{DR}(X)$. In the non-commutative situation, this corresponds to taking the 0-th graded piece of the Hodge filtration on $HP_\bullet(A)$. This has its own name.

Definition 6.7. The *negative cyclic homology* $HC^-_\bullet(A)$ of an algebra A is the 0-th term $F^0 HP_\bullet(A)$ of the Hodge filtration on $HP_\bullet(A)$.

If we compute $HP_\bullet(A)$ by the standard periodic bicomplex, then computing $HC^-_\bullet(A)$ amount to removing all the columns *to the left* of the 0-th one — as opposed to the usual $HC_\bullet(A)$, where we remove everything to the right. This explains the adjective "negative".

So, what we want to do is to construct a map $H_\bullet(GL_\infty(A), k) \to HC^-_\bullet(A)$. This is done in three steps.

First, fix some integer N, and consider the group algebra $k[GL_N(A)]$. This has a natural map into the algebra $\mathrm{Mat}_N(A)$ of $N \times N$-matrices in A – an element $g \in GL_N(A)$ goes to itself considered as an element in $\mathrm{Mat}_N(A)$. The map of algebras induces a map of negative cyclic homology; passing to the limit, we obtain a map

$$\lim_\to HC^-_\bullet(k[GL_N(A)]) \to \lim_\to HC^-_\bullet(\mathrm{Mat}_N(A)).$$

Second, we observe that by Morita-invariance of cyclic homology, the directed system in the right-hand side is actually constant — we have $HC_\bullet^-(\mathrm{Mat}_N(A)) \cong HC_\bullet^-(A)$ for any N. Thus we have constructed a map

$$\lim_{\to} HC_\bullet^-(k[GL_N(A)]) \to HC_\bullet^-(A).$$

Finally, we use the direct sum decomposition (6.7) — we take the graded piece of (6.7) corresponding to the unity element $1 \in GL_N(A)$, and apply the canonical Hodge-to-de Rham degeneration (6.8). This gives a canonical map

$$H_\bullet(GL_\infty(A), k) = \lim_{\to} H_\bullet(GL_N(A), k) \to \lim_{\to} HC_\bullet^-(k[GL_N(A)]).$$

Composing the two maps above, and plugging in the Hurewitz map, we obtain the higher regulator map $K_\bullet(A) \to HC_\bullet^-(A)$.

Lecture 7.

Cartier isomorphism in the commutative case. The categories Λ_p. Frobenius and quasi-Frobenius maps. Non-commutative case: the Cartier isomorphism for algebras with a quasi-Frobenius map. Remarks on the general case.

7.1 Cartier isomorphism in the commutative case.

The goal of this lecture is to explain the construction of the so-called *Cartier isomorphism* for algebras over a finite field k. We start by recalling what happens in the commutative case.

Fix a finite field k of characteristic $p = \operatorname{char} k$, and consider a smooth affine variety $X = \operatorname{Spec} A$ over k. Assume that $p > \dim X$, and consider the de Rham complex $\Omega^{\bullet}_X$. This complex behaves very differently from what we have in characteristic 0. For instance, in characteristic 0, a function f is closed with respect to the de Rham differential if and only if it is locally constant. In our situation, however, the p-th power a^p of any $a \in A$ is closed: we have $df^p = pf^{p-1}df = 0$. About fifty years ago, P. Cartier has shown that this gives all the closed functions, and moreover, the situation in higher degrees is similar — for any $n \geq 0$, there exists a canonical *Cartier isomorphism*

$$C : H^n_{DR}(X) \cong \Omega^n_{X^{(1)}}$$

between the de Rham cohomology group $H^n_{DR}(X)$ and the space $\Omega^n_{X^{(1)}}$ of n-forms on the so-called "Frobenius twist" $X^{(1)} = \operatorname{Spec} A^{(1)}$ of the variety X — $A^{(1)}$ coincides with A as a ring, but the k-algebra structure is twisted by the Frobenius automorphism of the field k.

Let us briefly sketch the construction of the inverse isomorphism $C^{-1} : \Omega^n_{X^{(1)}} \to H^n_{DR}(X)$ (this is simpler). Consider the ring $W(k)$ of Witt vectors of the field k — that is, the unramified extension of $\mathbb{Z}_p$ whose residue field is k. Since X is an affine variety, we can lift it to a smooth variety $\widetilde{X}$ over $W(k)$ so that $X = \widetilde{X} \otimes_{W(k)} k$. Moreover, we can lift the Frobenius map $F : X \to X^{(1)}$ to a map $\widetilde{F} : \widetilde{X} \to \widetilde{X}^{(1)}$, where $\widetilde{X}^{(1)}$ means the twist with respect to the Frobenius automorphism of $W(k)$. For any 1-form $fdg \in \Omega^1_{\widetilde{X}}$, we have

$$\widetilde{F}^*(fdg) = f^p dg^p \mod p,$$

so that the pullback map $\widetilde{F}^* : \Omega^1_{\widetilde{X}^{(1)}} \to \Omega^1_{\widetilde{X}}$ is divisible by p, and cosequently, $\widetilde{F}^*$ on $\Omega^n_{\widetilde{X}^{(1)}}$ is divisible by p^n. Let us make this division and consider the map

$$\overline{F} : \widetilde{\Omega}^{\bullet}_{\widetilde{X}^{(1)}} \to \Omega^{\bullet}_{\widetilde{X}}$$

given by $\overline{F} = \frac{1}{p^n}\widetilde{F}^*$ in degree n, where $\widetilde{\Omega}^{\bullet}_{\widetilde{X}^{(1)}}$ is the de Rham complex of the variety $\widetilde{X}^{(1)}$ with differential multiplied by p. Then it is not difficult to check — for instance, by a computation in local coordinates — that the map $\overline{F}$ is a quasiisomorphism. Reducing it modulo p, we obtain a quasiisomorphism

$$\bigoplus_n \Omega^n_{X^{(1)}} \cong \Omega^{\bullet}_X,$$

where the differential in the left-hand side, being divisible by p, reduces to 0. One then checks that the components of this quasiisomorphisms in individual degrees do not depend on our choices — neither of the lifting $\widetilde{X}$, nor on the lifting $\widetilde{F}$. These are the inverse Cartier maps.

We note that the Cartier maps are not easy to write down by an explicit formula even when X is a curve, expect for one especially simple case — and contrary to the expectations, the simple case is not the affine line $X = \operatorname{Spec} k[t]$, but the multiplicative group $X = \operatorname{Spec} k[t, t^{-1}]$. In this case, we have

$$C^{-1}\left(f\frac{dt}{t}\right) = f^p\frac{dt}{t}$$

for any $f \in k[t, t^{-1}]$. Analogously, in dimension n, we have a similar explicit formula for the torus $X = T = \operatorname{Spec} k[L]$, the group algebra of a lattice $L = \mathbb{Z}^n$.

7.2 The categories Λ_p.

To generalize this construction to the non-commutative case, we need one piece of linear algebra which we now describe.

Recall that in the combinatorial description, the cyclic category Λ was obtained as a quotient of the category Λ_∞: for any $[m], [n] \in \Lambda$, we have $\Lambda([n], [m]) = \Lambda_\infty([n], [m])/\sigma$. For any positive integer l, we can define a category Λ_l by a similar procedure: Λ_l has the same objects as Λ, and we set

$$\Lambda_l([n], [m]) = \Lambda_\infty([n], [m])/\sigma^l$$

for any $[n], [m] \in \Lambda_l$. We have an obvious projection $\pi : \Lambda_l \to \Lambda$; just as the projection $\Lambda_\infty \to \Lambda$, this is a connected bifibration whose fiber is the groupoid pt_l with one object and $\mathbb{Z}/l\mathbb{Z}$ as its automorphism group. One the other hand, we also have an embedding $i : \Lambda_l \to \Lambda$ which sends $[n] \in \Lambda_l$ to $[nl] \in \Lambda$. Just as for Λ, the embedding $j : \Delta^{opp} \to \Lambda_\infty$ induces an embedding $j_l : \Delta^{opp} \to \Lambda_l$.

It turns out that most of the facts about the homology of the category Λ immediately generalize to Λ_l, with the same proofs. In particular, for any $E \in \operatorname{Fun}(\Lambda_l, k)$, the homology $H_\bullet(\Lambda_l, E)$ can be computed by a bicomplex

$$\begin{array}{ccccccc}
\dots \longrightarrow & E([1]) & \xrightarrow{\mathsf{id}} & E([1]) & \xrightarrow{\mathsf{id}-\tau} & E([1]) \\
& \uparrow b & & \uparrow b' & & \uparrow b \\
\dots \longrightarrow & E([2]) & \xrightarrow{\mathsf{id}+\dots+\tau^{l-1}} & E([2]) & \xrightarrow{\mathsf{id}-\tau} & E([2]) \\
& \uparrow b & & \uparrow b' & & \uparrow b \\
\dots & \dots & & \dots & & \dots \\
& \uparrow b & & \uparrow b' & & \uparrow b \\
\dots \longrightarrow & E([n]) & \xrightarrow{\mathsf{id}+\tau+\dots+\tau^{ln-1}} & E([n]) & \xrightarrow{\mathsf{id}-\tau} & E([n]), \\
& \uparrow b & & \uparrow b' & & \uparrow b
\end{array} \tag{7.1}$$

we have a periodicity map $H_{\bullet+2}(\Lambda_l, E) \to H_\bullet(\Lambda_l, E)$ which fits into a Connes' exact sequence

$$H_\bullet(\Delta^{opp}, j_l^* E) \longrightarrow H_\bullet(\Lambda_l, E) \xrightarrow{u} H_{\bullet-2}(\Lambda_l, E) \longrightarrow ,$$

and the periodicity map u is induced by the action of the generator u of the cohomology algebra $H^\bullet(\Lambda_l, k) \cong k[u]$. As in Lecture 4, this generator admits an explicit Yoneda representation by a length-2 complex $j_{l*}^{opp} j_l^{opp*} k \to j_{l!} j_l^* k$. Moreover it is easy to check that this complex coincides with the pullback of the analogous complex in $\operatorname{Fun}(\Lambda, k)$ with respect to $i : \Lambda_l \to \Lambda$, so that i induces an isomorphism

$$i^* : H^\bullet(\Lambda, k) \to H^\bullet(\Lambda_l, k)$$

sending the periodicity generator to the periodicity generator. However, there is also one new and slightly surprising fact.

Lemma 7.1. *For any associative unital algebra A over k, the natural map*

$$M : H_\bullet(\Lambda_l, i^* A_\#) \to H_\bullet(\Lambda_l, A_\#)$$

is an isomorphism.

Proof. Since i^* is compatible with the periodicity maps, it suffices to prove that the natural map

$$H_\bullet(\Delta^{opp}, j_l^* i^* A_\#) \to H_\bullet(\Delta^{opp}, j^* A_\#)$$

on Hochschild homology is an isomorphism. By definition, we have

$$j^* A_\# \cong C_\bullet(A) \otimes_{A \otimes A^{opp}} A,$$

where $C_\bullet(A)$ is the bar resolution considered as a simplicial set. Writing down explicitly the definition of $i : \Lambda_l \to \Lambda$, one deduces that

$$j_l^* i^* A_\# \cong (C_\bullet(A) \otimes_A \cdots \otimes_A C_\bullet(A)) \otimes_{A \otimes A^{opp}} A,$$

with l factors $C_\bullet(A)$. But since $C_\bullet(A)$ is a resolution of A, so is the product in the right-hand side. We conclude that $H_\bullet(\Delta^{opp}, j_l^* i^* A_\#)$ is just the Hochschild homology $HH_\bullet(A)$ computed by a diffewrent resolution, and M is indeed an isomorphism. □

Exercise 7.1. *Prove that the map M is an isomorphism for any cyclic vector space $E \in \mathrm{Fun}(\Lambda, k)$, not just for $A_\#$. Hint: use the acyclic models method, and show that* $\mathrm{Fun}(\Lambda, k)$ *has a generator of the form $A_\#$.*

7.3 Frobenius and quasi-Frobenius maps.

Assume now given an associative unital algebra A over k; motivated by comparison theorems of Lecture 2, we want to construct a Cartier isomorphism of the form

$$HH_\bullet(A^{(1)})((u)) \cong HP_\bullet(A). \tag{7.2}$$

Unfortunately, the procedure that we have used in the commutative case breaks down right away: there is no Frobenius map in the non-commutative case. The endomorphism $F : A \to A$ given by $a \mapsto a^p$ is neither additive nor multiplicative for a general non-commutative algebra A.

To analyze the difficulty, split F into the composition

$$A^{(1)} \xrightarrow{\ \varphi\ } A^{\otimes p} \xrightarrow{\ m\ } A$$

of the map φ given by $\varphi(a) = a^{\otimes p}$, and the multiplication map $m : A^{\otimes p} \to A$, $m(a_1 \otimes \cdots \otimes a_p) = a_1 \dots a_p$. The map φ is not additive, nor multiplicative, but this is always so, be A commutative or not. It is the map m that causes the problem: if A is commutative, it is an algebra map, and in general it is not.

This is where the p-cyclic category Λ helps. Although the map m is not an algebra map, so that no Frobenius map acts on A, we still can get an action of this nonexisting Frobenius on Hochschild and cyclic homology by extending m to the isomorphism

$$M : H_\bullet(\Lambda_p, i^* A_\#) \to H_\bullet(\Lambda, A_\#)$$

of Lemma 7.1. As for the map φ, which behaves very badly in all cases, it turns out that it can be replaced by a different map within a certain large class of them. Namely, the only important property of the map φ is the following.

Lemma 7.2. *For any vector space V over k, the map $\varphi : V^{(1)} \to V^{\otimes p}$ induces an isomorphism*

$$H_i(\mathbb{Z}/p\mathbb{Z}, V^{(1)}) \cong H_i(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$$

for any $i \geq 1$, where the cyclic group $\mathbb{Z}/p\mathbb{Z}$ acts trivially on $V^{(1)}$, and by the cyclic permutation on $V^{\otimes p}$.

Proof. The homology of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ with coefficients in some representation M can be computed by the standard periodic complex $M_{\bullet}$ with terms $M_i = M$, $i \geq 0$, and the differentials $d_- = 1 - \sigma$ in odd degrees and $d_+ = 1 + \sigma + \dots + \sigma^{p-1} = (1-\sigma)^{p-1}$ in even degrees, where σ is the generator of $\mathbb{Z}/p\mathbb{Z}$. For the trivial representation $V^{(1)}$, $d_+ = d_- = 0$. The map φ obviously sends $V^{(1)}$ into the σ-invariant subspace in $V^{\otimes p}$, thus into the kernel of both d_+ and d_-. We have to show that (1) φ becomes additive modulo the image of the corresponding differential d_-, d_+, and (2) it actually becomes an isomorphism. Choose a basis in V, so that $V = k[S]$ is the k-vector space generated by a set S. Then $V^{\otimes p} = k[S^p]$. Decompose $S^p = S \coprod (S \setminus S)$, where $S \subset S^p$ is embedded as the diagonal, and consider the corresponding decomposition $V^{\otimes p} = V \oplus V'$, where $V' = k[S^p \setminus S]$. This decomposition is $\mathbb{Z}/p\mathbb{Z}$-invariant, thus compatible with d_+ and d_-; morever, φ obviously becomes an additive isomorphism if we replace $V^{\otimes p}$ with its quotient $V^{\otimes p}/V' = V$. Thus it suffices to prove that the complex which computes $H_{\bullet}(\mathbb{Z}/p\mathbb{Z}, V')$ is acyclic in degrees ≥ 1. This is obvious — the $\mathbb{Z}/p\mathbb{Z}$-action on $S^p \setminus S$ is free. □

For a more natural formulation of Lemma 7.2, one can invert the periodicity endomorphism of the homology functor $H_{\bullet}(\mathbb{Z}/p\mathbb{Z}, -)$ to obtain the so-called *Tate homology* $\check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, -)$ (this is the same procedure that we used in passing from $HC_{\bullet}(-)$ to $HP_{\bullet}(-)$). Then Lemma 7.2 claims that φ induces a canonical isomorphism

$$\check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, V^{(1)}) \cong \check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$$

in all degrees. We will call it *the standard isomorphism.*

Definition 7.3. A *quasi-Frobenius map* for an associative unital algebra A over k is a $\mathbb{Z}/p\mathbb{Z}$-equivariant algebra map $\Phi : A^{(1)} \to A^{\otimes p}$ which induces the standard isomorphism on Tate homology $\check{H}_{\bullet}(-)$.

Given an algebra A with a quasi-Frobenius map Φ, we can construct an inverse Cartier map (7.2) right away. Namely, comparing the bicomplex (7.1) with the usual cyclic bicomplex (3.2), we see that the only difference is that the differential $1 + \tau + \dots + \tau^{n-1}$ is replaced with

$$1 + \tau + \dots + \tau^{np-1} = (1 + \sigma + \circ + \sigma^{p-1}) \circ (1 + \tau + \dots + \tau^{n-1}),$$

where we have used the fact that $\sigma = \tau^n$. But for some $E' \in \operatorname{Fun}(\Lambda_p, k)$ of the form $E' = \pi^* E$, $E \in \operatorname{Fun}(\Lambda, p)$, we have $\sigma = 1$, so that $1 + \sigma + \circ + \sigma^{p-1} = p = 0$. Therefore we have a natural identification

$$HP_{\bullet}(\pi^* A^{(1)}_{\#}) \cong HH_{\bullet}(A^{(1)})((u)), \tag{7.3}$$

where on the left-hand side we have a periodic version of the homology $H_{\bullet}(\Lambda_p, \pi^* A^{(1)}_{\#})$. On the other hand, the quasi-Frobenius map Φ induces a map $\Phi : \pi^* A_{\#} \to i^* A_{\#}$, which induces a map on periodic homology. We define the inverse Cartier map C^{-1} as the map

$$C^{-1} = M \circ \Phi : HH_{\bullet}(A^{(1)})((u)) \cong HP_{\bullet}(\pi^* A^{(1)}_{\#}) \to HP_{\bullet}(i^* A_{\#}) \to HP_{\bullet}(A). \tag{7.4}$$

We must say that this comparatively easy situation is quite rare — in fact, the only situation where I know that a quasi-Frobenius map exists is the case of a group algebra $A = k[G]$ of some group G (one can take, for instance, the map $\Phi : k[G] \to k[G^p] = k[G]^{\otimes p}$ given by $\Phi(g) = g^{\otimes p}$, $g \in G$). This is perhaps not surprising, since in the commutative case, the situation was also explicit and simple only for the torus $A = k[L]$. It remains to do three things.

(i) Prove that the map C^{-1} is an isomorphism.

(ii) Compare it to the usual inverse Cartier isomoprhism in the commutative case.

(iii) Explain what to do when no quasi-Frobenius map is available.

I will give a sketch of (i) next, under an additional assumption that the algebra A has finite homological dimension — it seems that this is a necessary assumption. I will leave (ii) as a not very difficult but tedious exersize. As for (iii), this is unfortunately quite involved, and I cannot really present the procedure here in any detail, however sketchy; let me just mention that the only new thing in the general case is a certain generalization of the notion of a quasi-Frobenius map, while everything that concerns cyclic homology *per se* remains more-or-less the same as in the simple case. I refer the reader to Section 5 of my paper `arXiv.math/0708.1574` for an introductory exposition, with the detailed proofs given in `arXiv.math/0611623`.

7.4 Cartier isomorphism for algebras with a quasi-Frobenius map.

We assume given an associative algebra A/k with a quasi-Frobenius map Φ, and we want to prove that the corresponding inverse Cartier map (7.4) is an isomorphism. We note that the map M induces an isomorphism by Lemma 7.1, so that what we have to prove is that Φ also induces an isomorphism on periodic cyclic homology.

We will need one technical notion. Note that the embedding $j : \Delta^{opp} \to \Lambda_p$ extends to an embedding $\widetilde{j} : \Delta^{opp} \times \mathsf{pt}_p \to \Lambda_p$. Thus every $E \in \operatorname{Fun}(\Lambda_p, k)$ gives by restriction a simplicial $\mathbb{Z}/p\mathbb{Z}$-representation $\widetilde{j}^*E \in \operatorname{Fun}(\Delta^{opp} \times \mathsf{pt}_p, k) \cong \operatorname{Fun}(\Delta^{opp}, \mathbb{Z}/p\mathbb{Z}\text{-}\mathsf{mod})$. By the Dold-Kan equivalence, $\widetilde{j}^*E$ can treated as a complex of $\mathbb{Z}/p\mathbb{Z}$-representations.

Definition 7.4. An object $E \in \operatorname{Fun}(\Lambda_p, k)$ is *small* if $\widetilde{j}^*E$ is chain-homotopic to a complex of $\mathbb{Z}/p\mathbb{Z}$-modules which is of finite length.

Lemma 7.5. *Assume given a small $E \in \operatorname{Fun}(\Lambda_p, k)$ such that $E([n])$ is a free $\mathbb{Z}/p\mathbb{Z}$-module for any $[n] \in \Lambda_p$ (the action of $\mathbb{Z}/p\mathbb{Z}$ is generated by $\sigma = \tau^n$). Then we have*

$$HP_\bullet(E) = 0.$$

Sktech of a proof. We have $H_\bullet(\Lambda_p, E) = H_\bullet(\Lambda, L^\bullet\pi_!E)$, and the Connes' exact sequence for $L^\bullet\pi_!E$ gives an exact triangle

$$H_{\bullet-1}(\Lambda_p, E) \longrightarrow H_\bullet(\Delta^{opp}, \widetilde{j}^*E) \longrightarrow H_\bullet(\Lambda_p, E) \xrightarrow{\pi^*u} ,$$

where the connecting map is induced by the pullback $\pi^*u \in H^2(\Lambda, k)$ of the generator $u \in H^2(\Lambda, k)$. Computing $H^2(\Lambda_p, k)$ by a cohomological version of the bicomplex (7.1), as in Lecture 4, we find that $\pi^*u = 0$ (this is the same computation as in (7.3)). Therefore, to prove that

$$HP_\bullet(E) = \lim_{\overleftarrow{u}} H_\bullet(\Lambda_p, E)$$

vanishes, it suffices to prove the vanishing of

$$\lim_{\overleftarrow{u}} H_\bullet(\Delta^{opp} \times \mathsf{pt}_p, \widetilde{j}^*E),$$

where $\widetilde{j}^*u \in H^2(\Delta^{opp} \times \mathsf{pt}_p, k)$ is the restriction of the periodicity generator $u \in H^2(\Lambda_p, k)$. Using the Yoneda representation of u, we see that with respect to the Künneth isomorphism $H^2(\Delta^{opp} \times$

$\mathsf{pt}_p, k) \cong H^{\bullet}(\mathbb{Z}/p\mathbb{Z}, k)$, the class $\widetilde{j}^*u$ corresponds to the periodicity generator of $H^2(\mathbb{Z}/p\mathbb{Z}, k)$. Therefore in the spectral sequence

$$H_{\bullet}(\Delta^{opp}, \lim_{\substack{\leftarrow \\ u}} H_{\bullet}(\mathbb{Z}/p\mathbb{Z}, \widetilde{j}^*E)) \Rightarrow \lim_{\substack{\leftarrow \\ u}} H_{\bullet}(\Delta^{opp} \times \mathsf{pt}_p, \widetilde{j}^*E),$$

the limit in the left-hand side is the Tate homology $\check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, \widetilde{j}^*E)$. Since E is small, the spectral sequence converges, and since $E([n])$ is a free $\mathbb{Z}/p\mathbb{Z}$-representation for every $[n]$, the Tate homology in question is equal to 0. $\square$

Proposition 7.6. *Assume given an associative algebra A equipped with a quasi-Frobenius map $\Phi : A^{(1)} \to A^{\otimes p}$, and assume that the category A-bimod of A-bimodules has finite homological dimension. Then the Cartier map* (7.4) *for the algebra A is an isomorphism.*

Proof. We first note that the object $i^*A_{\#} \in \mathrm{Fun}(\Lambda_p, k)$ is small in the sense of Definition 7.4. Indeed, by assumption, the diagonal A-bimodule A admits a finite projective resolution $P_{\bullet}$. Therefore the bar resolution $C_{\bullet}(A)$ is chain-homotopic to a finite complex $P_{\bullet}$, its p-th power $C_{\bullet}(A) \otimes_A \cdots \otimes_A C_{\bullet}(A)$ is chain-homotopic to the finite complex $P_{\bullet} \otimes_A \cdots \otimes_A P_{\bullet}$, and the induced chain homotopy equivalence between $i^*A_{\#}$ and the finite complex

$$(P_{\bullet} \otimes_A \cdots \otimes_A P_{\bullet}) \otimes_{A^{opp} \otimes A} A$$

is obviously compatible with the $\mathbb{Z}/p\mathbb{Z}$-action. Moreover, $\pi^*A_{\#}$ is also small. It remains to notice that any quasi-Frobenius map Φ must be injective (otherwise it sends some element $a \in A^{(1)} = \check{H}_0(\mathbb{Z}/p\mathbb{Z}, A^{(1)})$ to 0), and its cokernel $A^{\otimes p}/\Phi(A^{\otimes p})$ by definition has no Tate homology.

Exercise 7.2. *Prove that for some $k[\mathbb{Z}/p\mathbb{Z}]$-module V, $\check{H}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, V) = 0$ if and only if V is free. Hint: identifying $k[\mathbb{Z}/p\mathbb{Z}] = k[t]/t^p$, $\sigma \mapsto 1 + t$, show that V decomposes into a direct sum of modules of the form $k[t]/t^l$, $0 < l \leq p$, and check the statement for all l.*

We conclude that $A^{\otimes p}/\Phi(A^{\otimes p})$ is a free $k[\mathbb{Z}/p\mathbb{Z}]$-module. Therefore for every n, the module $A^{\otimes pn}/\Phi^{\otimes n}(A^{(1)\otimes n})$ is free, and has no Tate homology. This means that the cokernel $i^*A_{\#}/\Phi(\pi^*A_{\#})$ satisfies the assumptions of Lemma 7.5, and Φ indeed induces an isomorphism between $HP_{\bullet}(\pi^*A_{\#})$ and $HP_{\bullet}(i^*A_{\#})$. $\square$

Remark 7.7. In the smooth commutative case, the assumption that A-bimod has finite homological dimension just means that A is of finite type over k. In the general case of the theorem, when no quasi-Frobenius map is available, one actually needs to assume that the homological dimension is less than $2p - 1$. In the commutative case, this reduces to $p > \mathsf{dim}\,\mathrm{Spec}\,A$.

Lecture 8.

Hochschild cohomology of an associative algebra and its Morita-invariance. Hochschild cohomology complex. Multiplication and the Eckman-Hilton argument. Derivations of the tensor algebra and the Gerstenhaber bracket on Hochschild cohomology. Hochschild cohomology and deformations. Quantizations. Kontsevich formality (statements).

8.1 Generalities on Hochschild cohomology.

Up to now, we were studying Hochschild homology of associative algebras and related concepts — cyclic homology, regulator maps, and so on. We will now turn to the other half of the story: Hochschild cohomology.

We recall (see Definition 1.1) that the *Hochschild cohomology* $HH^\bullet(A, M)$ of an associative unital algebra A over a field k with coefficients in an A-bimodule $M \in A\text{-}\mathsf{bimod}$ is given by

$$HH^\bullet(A, M) = \operatorname{Ext}^\bullet_{A\text{-}\mathsf{bimod}}(A, M),$$

where A in the right-hand side is the diagonal bimodule $A \in A\text{-}\mathsf{bimod}$. *Hochschild cohomology of an algebra* A is its cohomology with coefficients in the diagonal bimodule, $HH^\bullet(A) = HH^\bullet(A, A)$.

We note right away that the Hochschild cohomology groups $HH^\bullet(A)$ are Morita-invariant — that is, they only depend on the category $A\text{-}\mathsf{mod}$ of left A-modules. Indeed, all we need to compute $HH^\bullet(A)$ is the tensor abelian category $A\text{-}\mathsf{bimod}$ with its unit object $A \in A\text{-}\mathsf{bimod}$; as we have seen already in Lecture 6, these only depend on $A\text{-}\mathsf{mod}$.

When A is commutative and $X = \operatorname{Spec} A$ is smooth, the Hochschild-Kostant-Rosenberg Theorem (Theorem 1.2) provides a canonical identification

$$HH^\bullet(A) \cong H^0(X, \Lambda^\bullet \mathcal{T}_X),$$

where $\mathcal{T}_X$ is the tangent bundle to X. Roughly speaking, Hochschild cohomology is in the same relation to Hochschild homology as vector fields are to differential forms. We note, however, that to describe $HH^\bullet(A)$, we need not only the tangent bundle $\mathcal{T}_X$, but all its exterior powers $\Lambda^\bullet \mathcal{T}_X$, so that Hochschild cohomology contains not only vector fields, but all the polyvector fields, too. In the non-commutative setting, there is no reasonable way to work only with vector fields, we have to treat all the polyvector fields as a single package.

Just as in the case of Hochschild homology, we can compute Hochschild cohomology $HH^\bullet(A)$ of an algebra A by using the canonical bar resolution $C_\bullet(A)$ of the diagonal bimodule A. This gives the *Hochschild cohomology complex* with terms

$$\operatorname{Hom}(A^{\otimes n}, A), \quad n \geq 0,$$

where Hom means the space of all k-linear maps. Maps $f \in \operatorname{Hom}(A^{\otimes n}, A)$ are called *Hochschild cochains*; we can treat an n-cochain as an n-linear A-valued form on A. The differential δ in the Hochschild cohomology complex is given by

(8.1)

$$\delta(f)(a_0, \dots, a_n) = a_0 f(a_1, \dots, a_n) - \sum_{0 \leq j < n} (-1)^j f(a_0, \dots, a_j a_{j+1}, \dots, a_n) + (-1)^{n+1} f(a_0, \dots, a_{n-1}) a_n.$$

For example, if $f = a \in A$ is a 0-cochain, then $\delta(f)$ is given by $\delta(f)(b) = ab - ba$; if $f : A \to A$ is a 1-cochain, then we have

$$\delta(f)(a, b) = af(b) + f(a)b - f(ab).$$

We conclude that the space $HH^0(A) \subset A$ of Hochschild 0-cocycles is the center of the algebra A; the space of Hochschild 1-cocycles is the space of all *derivations* $f : A \to A$ (that is, maps that satisfy the Leibnitz rule $f(ab) = af(b) + f(a)b$). The Hochschild cohomology group $HH^1(A)$ is the space of all derivations $A \to A$ considered modulo the *inner derivations* given by $b \mapsto ab - ba$.

8.2 Multiplication and the Eckman-Hilton argument.

By definition, Hochschild cohomology $HH^{\bullet}(A) = \operatorname{Ext}^{\bullet}(A, A)$ of an associative unital algebra A is equipped with an additional structure: an associative multiplication, given by the Yoneda product on Ext-groups.

However, the abelian category A-bimod is a tensor category, and the diagonal bimodule $A \in A$-bimod is its unit object. This defines a second multiplication operation on $HH^{\bullet}(A)$: given two elements $\alpha, \beta \in \operatorname{Ext}^{\bullet}(A, A)$, we can consider their tensor product $\alpha \otimes_A \beta \in \operatorname{Ext}^{\bullet}(A, A)$.

Both multiplications are obviously associative, and it seems that this is all we can claim. However, a moment's reflection shows that more is true.

Lemma 8.1. *The two multiplications on Hochschild cohomology $HH^{\bullet}(A)$ are the same, and moreover, this canonical multiplication is (graded)commutative.*

Proof. It is easy to see that the two multiplications we have defined obey the following distribution law:

$$(\alpha_1 \otimes_A \alpha_2) \cdot (\beta_1 \otimes_A \beta_2) = (-1)^{\deg \alpha_2 \deg \beta_1} \alpha_1\beta_1 \otimes_A \alpha_2\beta_2, \tag{8.2}$$

for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in HH^{\bullet}(A)$. This formally implies the claim:

$$\alpha\beta = (\alpha \otimes_A 1) \cdot (1 \otimes_A \beta) = (\alpha \cdot 1) \otimes_A (1 \cdot \beta) = \alpha \otimes_A \beta,$$

and similar for the commutativity, which we leave to the reader. □

This observation is known as the *Eckman-Hilton argument*: two associative multiplications which commute according to (8.2) are commutative and equal. It first appeared in algebraic topology — essentially the same argument shows that the homotopy groups $\pi_i(X)$ of a topological space X are abelian when $i \geq 2$. Although the Eckman-Hilton argument is very elementary, it captures an essential feature of the whole story: in fact, all the results about Hochschild cohomology can be deduced from an elaboration of this semi-trivial observation. A good reference for this is a paper by M. Batanin, `arXiv:math/0207281`. In these lectures, we will not attempt such an extreme treatment and follow a more conventional path, only refering to the Eckman-Hilton argument when it simplifies the exposition.

One example of this is an explicit description of the product in $HH^{\bullet}(A)$ in terms of Hochschild cochains. Writing down the Yoneda product in terms of Ext's computed by an explicit resolution is usually rather cumbersome, and the resulting formulas are not nice. However, the tensor product $f \otimes_A g$ of two Hochschild cochains $f : A^{\otimes n} \to A$, $g : A^{\otimes m} \to A$ is very easy to write down: it is given by

$$(f \otimes_A g)(a_1, \dots, a_{n+m}) = f(a_1, \dots, a_n)g(a_{n+1}, \dots, a_{n+m}). \tag{8.3}$$

By Lemma 8.1, the Yoneda product is given by the same formula.

8.3 The Gerstenhaber bracket.

Recall now that the space of vector fields on a smooth algebraic variety has an additional structure: the Lie bracket. It turns out that such a bracket, known as the *Gerstenhaber bracket*, also exists for an arbitrary associative unital algebra A. To define it, we need to introduce a completely different construction of the Hochschild cohomology complex.

Assume given a k-vector space V, and consider the free graded associative coalgebra $T_{\bullet}(V)$ generated by V placed in degree 1 — explicitly, we have

$$T_nV = V^{\otimes n}, \quad n \geq 0.$$

Consider the graded Lie algebra $DT^\bullet(V)$ of all *coderivations* of the coalgebra $T_\bullet(V)$ — the notion of a coderivation of a coalgebra is dual to that of a derivation of an algebra, and we leave it to the reader to write down a formal definition. Then since the coalgebra $T_\bullet(V)$ is freely generated by V, every $\delta \in DT^\bullet(V)$ is uniquely determined by its composition with the projection $T_\bullet(V) \to V$, so that we have

$$DT^{n+1}(V) \cong \mathrm{Hom}(V^{\otimes n}, V), \quad n \geq 0. \tag{8.4}$$

Lemma 8.2. *Assume that* $\mathsf{char}\, k \neq 2$. *A coderivation* $\mu \in DT^1(V) = \mathrm{Hom}(V^{\otimes 2}, V)$ *satisfies* $\mu^2 = 0$ *if and only if the corresponding binary operation* $V^{\otimes 2} \to V$ *is associative.*

Proof. Since μ is an odd derivation, $\mu^2 = \frac{1}{2}\{\mu, \mu\} : T_{\bullet+2}(V) \to T_\bullet(V)$ is also a derivation; thus it suffices to prove that the map $\mu^2 : V^{\otimes 3} \to V$ is equal to 0 if and only if the map $\mu : V^{\otimes 2} \to V$ is associative. This is obvious: by the Leibnitz rule, we have

$$\mu^2(v_1, v_2, v_3) = \mu(\mu(v_1, v_2), v_3) - \mu(v_1, \mu(v_2, v_3))$$

for any $v_1, v_2, v_3 \in V$. □

Thus if we are given an associative algebra A, the product in A defines an element $\mu \in DT^1(A) = \mathrm{Hom}(A^{\otimes 2}, A)$ such that $\{m, m\} = 0$. Then setting $\delta(a) = \{\mu, a\}$ for any $a \in DT^\bullet(A)$ defines a differential $\delta : DT^\bullet(A) \to DT^{\bullet+1}(A)$ and turns $DT^\bullet(A)$ into a graded Lie algebra. But as we can see from (8.4), the space $DT^n(A)$ is exactly the space of Hochschild $(n+1)$-cochains of the algebra A.

Exercise 8.1. *Check that under the identification* (8.4)*, the differential* δ *in* $DT^\bullet(A)$ *becomes equal to the differential in the Hochschild cohomology complex.*

Thus the Hochschild complex for the algebra A becomes a graded Lie algebra, with a Lie bracket of degree -1, and we get an induced graded Lie bracket on Hochschild cohomology $HH^\bullet(A)$. This is known as the *Gerstenhaber bracket.* Explicitly, the Gerstenhaber bracket $\{f, g\}$ of two cochains $f : A^{\otimes n} \to A$, $g : A^{\otimes m} \to A$ is given by

$$\begin{aligned} \{f, g\}(a_1, \dots, a_{n+m-1}) = &\sum_{1 \leq i < n} (-1)^i f(a_1, \dots, g(a_i, \dots, a_{i+m-1}), \dots, a_{n+m-1}) \\ &- \sum_{1 \leq i < m} (-1)^i g(a_1, \dots, f(a_i, \dots, a_{i+n-1}), \dots, a_{n+m-1}). \end{aligned} \tag{8.5}$$

Exercise 8.2. *Prove this. Hint: use the Leinitz rule.*

We note that if we take $g = \mu$, (8.5) recovers the formula (8.1) for the differential δ in the Hochschild cohomology complex. On the other hand, if both f and g are 1-cochains — that is, k-linear maps from A to itself — then $\{f, g\} : A \to A$ is their commutator, $\{f, g\} = fg - gf$. If f and g are also 1-cocycles, that is, derivations of the algebra A, then so is their commutator $\{f, g\}$: the Gerstenhaber bracket on $HH^1(A)$ is given by the commutator of derivations.

Thus we have two completely different interpretation of the Hochschild complex, and two natural structures on it: the multiplication and the Lie bracket. These days, the corresponding structure on $HH^\bullet(A)$ is usually axiomatized under the name of a *Gerstenhaber algebra.*

Definition 8.3. A *Gerstenhaber algebra* is a graded-commutative algebra $B^\bullet$ equipped with a graded Lie bracket $\{-, -\}$ of degree -1 such that

$$\{a, bc\} = \{a, b\}c + (-1)^{\mathsf{deg}\, b\, \mathsf{deg}\, c}\{a, c\}b \tag{8.6}$$

for any $a, b, c \in B^\bullet$.

Exercise 8.3. *Check that the Hochschild cohomology algebra $HH^\bullet(A)$ equipped with its Gerstenhaber bracket satisfies* (8.6)*, so that $HH^\bullet(A)$ is a Gerstenhaber algebra in the sense of Definition* 8.3.

We note that the definition of a Gerstenhaber algebra is very close to that of a *Poisson algebra* — the difference is that the bracket has degree -1, and (8.6) acquires a sign. We will discuss this analogy in more detail at a later time.

8.4 Hochschild cohomology and deformations.

By far the most common application of Hochschild cohomology is its relation to deformations of associative algebras. We will explain this in the form of the so-called *Maurer-Cartan* formalism popularized by M. Kontsevich.

Assume given an Artin local algebra S with maximal ideal $\mathfrak{m} \in S$ and residue field $k = A/\mathfrak{m}$. By an *S-deformation* $\widetilde{A}$ of an associative unital k-algebra A we will understand a flat S-algebra $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/\mathfrak{m} \cong A$.

Assume given such a deformation $\widetilde{A}$, choose a k-linear splitting $A \to \widetilde{A}$ of the projection $\widetilde{A} \to \widetilde{A}/\mathfrak{m} \cong A$, and extend it to an S-module map $\widetilde{A} \cong A \otimes_k S$ — since $\widetilde{A}$ is flat, this map is an isomorphism. We leave it to the reader to check that Lemma 8.2 extends to flat S-modules, with the same statement and proof. Then the multiplication map $\mu : \widetilde{A} \otimes_S \widetilde{A} \to \widetilde{A}$ can be rewritten as

$$\mu = \mu_0 + \gamma \in \operatorname{Hom}(A^{\otimes 2}, A) \otimes S, \tag{8.7}$$

where μ_0 is the multiplication map in A. If the splitting map $A \to \widetilde{A}$ is compatible with the multiplication, then $\gamma = 0$; but in general, it is a non-trivial correction term with values in $\operatorname{Hom}(A^{\otimes 2}, A) \otimes \mathfrak{m} \subset \operatorname{Hom}(A^{\otimes 2}, A) \otimes S$. All we can say is that, since both μ_0 and μ are associative, by Lemma 8.2 we have $\{\mu, \mu\} = 0$ and $\{\mu_0, \mu_0\} = 0$. This can be rewritten as the *Maurer-Cartan equation*

$$\delta(\gamma) + \frac{1}{2}\{\gamma, \gamma\} = 0, \tag{8.8}$$

where δ is the Hochschild differential of the algebra A. Conversely, every solution γ of the Maurer-Cartan equation defines by (8.7) an associative product structure on the S-module $A \otimes_k S$.

This establishes the correspondence between S-deformations of the algebra A and $\mathfrak{m}$-valued degree-1 solutions of the Maurer-Cartan equation in the differential graded Lie algebra $DT^\bullet(A)$. We denote the set of these solutions by $MC(DT^\bullet(A), \mathfrak{m})$; by definition, it only depends on the differential graded Lie algebra $DT^\bullet(A)$ and the local Artin algebra S with its maximal ideal $\mathfrak{m} \subset S$.

How canonical is this correspondence? There is one choice: that of an S-module identification $\widetilde{A} \cong A \otimes S$. The set of all such identifications is a torsor over the algebraic group $GL_{S,\mathfrak{m}}(A)$ of all S-linear invertible maps $A \otimes S \to A \otimes S$ which are equal to identity modulo $\mathfrak{m}$. Assume now that $\operatorname{char} k = 0$. Then we note that since S is local and Artin, this algebraic group is unipotent, and therefore it is completely determined by its Lie algebra $\operatorname{Hom}(A, A) \otimes \mathfrak{m} \cong DT^0(A) \otimes \mathfrak{m}$. Changing an identification $\widetilde{A} \cong A \otimes S$ changes the solution $\gamma \in MC(DT^\bullet(A), \mathfrak{m})$, so that we have an action of the group $GL_{S,\mathfrak{m}}(A)$ on $MC(DT^\bullet(A), \mathfrak{m})$. The corresponding action of its Lie algebra $DT^0(A) \otimes \mathfrak{m}$ is easy to describe: an element $l \in DT^0(A) \otimes \mathfrak{m}$ sends μ to $\{\mu, l\}$, which in terms of γ is given by

$$\gamma \mapsto \{\mu_0, l\} + \{\gamma, l\} = \delta(l) + \{\gamma, l\},$$

where $\delta : DT^0(A) \to DT^1(A)$ is the differential in $DT^\bullet(A)$.

This is the general pattern of deformation theory in the Maurer-Cartan formalism. To a deformation problem, one associates a differential graded Lie algebra $L^\bullet$, which "controls" the problem in the following sense: isomorphism classes of deformations over a local Artin base $\langle S, \mathfrak{m}\rangle$ are in one-to-one correspondence with solutions of the Maurer-Cartan equation in $L^1 \otimes \mathfrak{m}$, considered modulo the natural action of the unipotent algebraic group corresponding to the nilpotent Lie algebra $L^0 \otimes \mathfrak{m}$ (because of this passage from a Lie algebra to a unipotent group, the formalism only works well in characteristic 0). In the case of deformations of an associative algebra A, we have just shown that the controlling differential graded Lie algebra is the Hochschild cohomology complex $DT^\bullet(A)$.

As an interesting special case, one can consider the so-called first-order deformations — that is, one takes $S = k[h]/h^2$, the algebra of dual numbers. Then $\mathfrak{m} = k$ and $\mathfrak{m}^2 = 0$, so that the Lie algebra $L^0 \otimes \mathfrak{m} \cong L^0$ is abelian, the corresponding unipotent group is simply the vector space $L^0 \otimes \mathfrak{m}$, and its action is given by $\gamma \mapsto \gamma + dl$, $l \in L^0$. On the other hand, the term $\{\gamma, \gamma\}$ in the Maurer-Cartan equation vanishes. Thus the set of isomorphism classes of deformations is naturally identified with the degree-1 cohomology classes of the complex $L^\bullet$. We note that this special case does not require the assumption $\operatorname{char} k = 0$ — indeed, integrating an *abelian* Lie algebra to a unipotent group does not require exponentiation, so that no denomitators occurs.

In particular, the first-order deformations of an associative algebra A are classified, up to an isomorphism, by elements in the second Hochschild cohomology group $HH^2(A)$.

We also note that while we have introduce the Maurer-Cartan formalism in the case of a local Artin base S, it immediately extends to complete deformations over a complete local Noetherian base: the only difference is that the Lie algebra $L^\bullet \otimes \mathfrak{m}$ should be replaced with its $\mathfrak{m}$-adic completion, and its degree-0 term $L^\bullet \otimes \mathfrak{m}$ becomes not nilpotent but pro-nilpotent.

8.5 Example: quantizations.

A useful particular case of the deformation formalism described above is that of a commutative algebra A: assume given a commutative algebra A, and assume that $X = \operatorname{Spec} A$ is a smooth algebraic variety. Under the Hochschild-Kostant-Rosenberg isomorphism

$$HH^\bullet(A) = H^0(X, \Lambda^\bullet \mathcal{T}_X),$$

the group $HH^1(A)$ corresponds to the space of vector fields on X, and the Gerstenhaber bracket is the usual Lie bracket of vector fields. The bracket between $HH^1(A) = H^0(X, \mathcal{T}_X)$ and $HH^0(A) = H^0(X, \mathcal{O}_X)$ is given by the action of a vector field on the space of functions. The bracket on $HH^i(A)$, $i \geq 2$ is uniquely defined by (8.6); it is known as the *Schouten bracket* of polyvector fields.

Deformations of the algebra A are classified by $HH^2(A) = H^0(X, \Lambda^2 \mathcal{T}_X)$, the space of bivector fields on X. Such a field $\Theta \in H^0(X, \Lambda^2 \mathcal{T}_X)$ defines a bracket operation $\{-,-\}$ on $\mathcal{O}_X$ by the rule

$$\{f, g\} = \langle df \wedge dg, \Theta \rangle.$$

This bracket is obviously a derivation with respect to either of the arguments: we have $\{f_1 f_2, g\} = f_2\{f_1, g\} + f_1\{f_2, g\}$. Moreover, it satisfies the Jacobi identity if and only if $[\Theta, \Theta] = 0$ with respect to the Schouten bracket. In this case, Θ is called a *Poisson bivector*, and A acquires a structure of a *Poisson algebra*.

Definition 8.4. A *Poisson algebra* is a commutative algebra A equipped with a Lie bracket $\{-,-\}$ such that $\{f_1 f_2, g\} = f_2\{f_1, g\} + f_1\{f_2, g\}$ for any $f_1, f_2, g \in A$.

A natural source of Poisson algebra structures on A is given by its *quantizations*.

Definition 8.5. A *quantization* $\widetilde{A}$ of the algebra A is a flat complete associative unital $k[[h]]$-algebra $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/h \cong A$.

For any quantization $\widetilde{A}$, there obviously exists a unique bracket $\{-,-\}$ on A such that

$$\widetilde{f}\widetilde{g} - \widetilde{g}\widetilde{f} = h\{f,g\} \mod h^2 \tag{8.9}$$

for any $f, g \in A$ and arbitrary $\widetilde{f}, \widetilde{g} \in \widetilde{A}$ such that $\widetilde{f} = f \mod h$, $\widetilde{g} = g \mod h$. It is easy to check that this bracket defines a Poisson algebra structure on A. On the other hand, $\widetilde{A}$ can be treated as a $k[[h]]$-deformation of A, so that we have a solution $\gamma \in \operatorname{Hom}(A^{\otimes 2}, A)[[h]]$ of the Maurer-Cartan equation. Its leading term $\Theta \in \operatorname{Hom}(A^{\otimes 2}, A)$ is a Hochschild cocycle, thus gives a bivector on X.

Exercise 8.4. *Check that the bracket on A defined by the bivector Θ is equal to the bracket given by* (8.9)*.*

The equation $[\Theta, \Theta] = 0$ also immediately follows from the Maurer-Cartain equation.

8.6 Kontsevich formality: the statement.

For some time, an important open question was whether the above construction can be reversed: given a Poisson algebra structure on a commutative smooth algebra A, can we extend it to a quantization $\widetilde{A}$? Or, equivalently: given an element $\Theta \in HH^2(A)$ such that $\{\Theta, \Theta\} = 0$, can we extend it to a solution of the Maurer-Cartan equation in $DT^\bullet(A)[[h]]$? A positive answer to this was first conjectured and then proved by M. Kontsevich. In fact, he proved the following stronger fact.

Theorem 8.6 (Kontsevich Formality Theorem). *Let $A = k[x_1, \dots, x_n]$ be a polynomial algebra over a field k of characteristic* 0*. Then the DG Lie algebra $DT^\bullet(A)$ is* formal *— that is $DT^\bullet(A)$ is quasiisomorphic to its cohomology $HH^{\bullet+1}(A)$ (the DG Lie algebra formed by the Hochschild cohomology groups of A, with trivial differential).*

Here the precise meaning of "quasiisomorphic" is the following: there exists a chain of DG Lie algebras $L_i^\bullet$ and DG Lie algebra maps $DT^\bullet(A) \leftarrow L_1^\bullet \to L_2^\bullet \leftarrow \cdots \to HH^{\bullet+1}(A)$ such that all the maps induces isomorphisms on cohomology of the complexes. Unfortunately, in general there does not exist a single DG Lie algebra quasiisomorphism $HH^{\bullet+1}(A) \to DT^\bullet(A)$ (in particular, the canonical Hochschild-Kostant-Rosenberg map is not compatible with the bracket). However, this is not important for the deformation theory.

Exercise 8.5. *Check that for any local Artin $\langle S, \mathfrak{m}\rangle$, a DG Lie algebra quasiisorphism $L_1^\bullet \to L_2^\bullet$ between two DG Lie algebras $L_1^\bullet$, $L_2^\bullet$ induces a map between the solution sets $MC(L_1^\bullet, \mathfrak{m})$ and $MC(L_2^\bullet, \mathfrak{m})$ of the Maurer-Cartan equation which identified the sets of equivalence classes of the solutions.*

This together with the Formality Theorem implies that quantizations of the algebra A are in one-two one correspondence with equivalence classes of the solutions of the Maurer-Cartan equations in the DG Lie algebra $HH^{\bullet+1}(A)$. However, since the differential in this algebra is trivial, the Maurer-Cartan equation simply reads $\{\Theta, \Theta\} = 0$. In particular, any Poisson bivector on A canonically gives such a solution.

There are two proofs of the Kontsevich Formality Theorem: the original proof of Kontsevich, which is largely combinatorial, and a second proof by D. Tamarkin — this is more conceptual, but it requires a much more detailed study of the the Hochschild cohomology complex $DT^\bullet(A)$. Roughly speaking, one proves an even stronger theorem: $DT^\bullet(A)$ and $HH^{\bullet+1}(A)$ are quasiisomorphic not only as DG Lie algebras, but as Gerstenhaber algebras. This stronger statement is actually easier; in fact, Tamarkin shows without much difficulty that any Gerstenhaber algebra which has cohomology algebra $HH^\bullet(A)$ must be formal. The real difficulty in the proof is the following: *a priori*, the

Hochschild cohomology complex $DT^\bullet(A)$ *is not a Gerstenhaber algebra* — indeed, while it does have a Lie bracket and a multiplication, the multiplication (8.3) is commutative only on the level of cohomology, not on the nose. What precise structure does exist on $DT^\bullet(A)$ is a subject of the so-called *Deligne Conjecture*. We will return to this later, after introducing some appropriate machinery.

Lecture 9.

The language of operads. Poisson and associative operad. Gerstenhaber operad and small discs. Braided algebras. Deligne Conjecture.

9.1 The language of operads.

These days, it has become common practice to use the language of the so-called *operads* to describe various non-trivial algebraic structures such as that of a Gerstenhabe algebra. It must be mentioned that the notion of an operad has been introduced 35 years ago by P. May essentially as a quick hack; it is not very natural, and in many cases it is not quite what one needs, so that descriptions using operads tend to be somewhat ugly and somewhat artificial. But at least, from the formal point of view, everything is well-defined. We will only sketch most proofs. For a complete exposition which covers much if not all the material in this lecture, I refer the reader, for instance, to the paper `arXiv:0709.1228` by V. Ginzburg and M. Kapranov which is now considered one of the standard references on the subject (the paper was published in 1994, and I am grateful to V. Ginzburg who finally put it on arxiv in 2007). Another reference is the foundational paper `arXiv:hep-th/9403055` by E. Getzler and J.D.S. Jones, but this has to be used with care, since some advanced parts of it were later found to be wrong.

To define an operad, let Γ be the category of finite sets, and let $\Gamma^{[2]}$ be the category of arrows in Γ (objects are morphisms $f : S' \to S$ between $S', S \in \Gamma$, morphisms are commutative squares). Then Γ has a natural embedding into $\Gamma^{[2]}$: every finite set S has a canonical morphism $p^S : S \to \mathsf{pt}$ into the finite set $\mathsf{pt} \in \Gamma$ with a single element. We note that every $f \in \Gamma^{[2]}$, $f : S' \to S$ canonically decomposes into a coproduct

$$f = \coprod_{s \in S} f^s, \tag{9.1}$$

where $f^s \in \Gamma^{[2]}$ is the canonical map $p^{f^{-1}(s)} f^{-1}(s) \to \mathsf{pt}$ corresponding to the preimage $f^{-1}(s) \subset S'$.

Definition 9.1. An *operad* $O_{\bullet}$ of k-vector spaces is a rule which assigns a vector space O_f to any $f \in \Gamma^{[2]}$ together with the following operations:

(i) for any pair $f : S' \to S$, $g : S'' \to S'$ of composable maps, a map $\mu_{f,g} : O_f \otimes O_g \to O_{f \circ g}$,

(ii) for any $f \in \Gamma^{[2]}$, $f : S' \to S$, an isomorphism

$$O_f \cong \bigotimes_s O_{f^s},$$

where $f^s = p^{f^{-1}(s)}$ are as in (9.1).

Moreover, the assignment $f \mapsto O_f$ should be functorial with respect to isomorphisms in $\Gamma^{[2]}$, the maps in (i) and (ii) should be functorial maps, and for any triple $f, g, h \in \Gamma^{[2]}$ of composable maps, the square

$$\begin{array}{ccc} O_f \otimes O_g \otimes O_h & \longrightarrow & O_{f \circ g} \otimes O_h \\ \big\downarrow & & \big\downarrow \\ O_f \otimes O_{g \circ h} & \longrightarrow & O_{f \circ g \circ h} \end{array}$$

should be commutative.

It is useful to require also that $O_{\mathsf{id}} \cong k$ for an identity map $\mathsf{id} : S \to S$, and we shall do so. We note that by virtue of (ii), it is sufficient to specify only the vector spaces O_{p^S} for the canonical maps $p_S : S \to \mathsf{pt}$ (these are usually denoted O_S, or simply O_n, where n is the cardinality of S). However, the way we have formulated the definition makes it slightly more natural, and slightly easier to generalize.

Definition 9.2. An *algebra* A over an operad $O_{\bullet}$ of k-vector spaces is a k-vector space A, together with an action map

$$a_f : O_f \otimes A^{\otimes S_1} \to A^{\otimes S_2}$$

for any $f \in \Gamma^{[2]}$, $f : S_1 \to S_2$, where for any finite set $S \in \Gamma$, we denote by $A^{\otimes S}$ the tensor product of copies of A numbered by elements $s \in S$. The maps a_f should be functorial with respect to isomoprhisms in $\Gamma^{[2]}$ and satisfy the following rules:

(i) For a pair $f, g \in \Gamma^{[2]}$ of composable maps, we should have $a_f \circ a_g = a_{f \circ g} \circ \mu_{f,g}$.

(ii) For any $f \in \Gamma^{[2]}$, $f : S' \to S$, we should have

$$a_f = \bigotimes_{s \in S} a_{f^s}.$$

As in the definition of an operad, (ii) insures that it is sufficient to specify the action maps $a_n = a_S = a_{p^S} : O_{p^S} \otimes A^{\otimes S} \to A$ for all $S \in \Gamma$, but our formulation is slightly more natural. We also note that algebras over a fixed operad O form a category, which has a forgetfull functor into the category of k-vector spaces. The left-adjoint functor associates to a k-vector space V the free O-algebra $F_O V$ generated by V, which is explicitly given by

$$F_O V = \bigoplus_{S \in \Gamma} \left(O_S \otimes V^{\otimes S} \right)_{\mathsf{Aut}(S)}, \tag{9.2}$$

where the sum is over all the isomorphism classes of finite sets — in other words, over all integers — and $\mathsf{Aut}(S)$ is the symmetric group of all automorphisms of a finite set S.

The reasoning behind these definitions is the following. We want to describe algebras of a certain kind — associative algebras, commutative algebras, Lie algebras, Poisson algebras, etc. To do so, one usually says that an algebra is a vector space A equipped with some multilinear structural maps which satisfy some axioms (associativy, the Jacobi identity, and so forth). However, this is not always convenient — just as describing a concrete algebra by its generators and relations is usually too cumbersome. An operad O encodes *all the polylinear operations* we want our algebra to have. More precisely, given some $f : S_1 \to S_2$, we collect in the vector space O_f all the operations from $A^{\otimes S_1}$ to A^{S_2} which can be obtained from the structural maps by composing them and substituting one into the other; and we take the quotient by all the relations our concrete type of algebraic structure imposes on these compositions. Moreover, we only want to consider those algebraic structures which are defined by operations with values in A itself, not its tensor powers. This is the reason for the condition (ii) in Definition 9.1 and Definition 9.2.

9.2 Examples.

Probably the simplest example of an operad is obtained by taking $O_f = k$, the 1-dimensional vector space, for any $f \in \Gamma^{[2]}$. This operad is denoted by Com. A moment's reflection shows that algebras over Com are nothing but *commutative associative unital algebras*. Indeed, by definition, we must have a unique action map

$$a_S : A^{\otimes S} \to A$$

for any $S \in \Gamma$, and moreover, this map should be functorial with respect to isomorphisms in Γ — in other words, a_{p^S} must the equivariant with respect to the natural action of the symmetric group $\mathsf{Aut}(S)$. Thus first, we must have a commutative multiplication $\mu : A^{\otimes 2} \to A$ corresponding to the generator of $\mathsf{Com}_2 = k$, and second, any way to compose this operation to obtain an operation $A^{\otimes n} \to A$ for any n must give the same result — which for $n = 3$ implies associativity,

$$\mu \circ (\mu \otimes \mathsf{id}) = \mu \circ (\mathsf{id} \otimes \mu).$$

One checks easily that conversely, associativity implies the uniqueness for any $n \geq 3$. The free Com-algebra $F_{\mathsf{Com}} V$ generated by a vector space V is given by (9.2) and coincides with the symmetric algebra $S^{\bullet} V$.

Exercise 9.1. *Check that for a* Com*-algebra* A*, the action map* $a_0 : k = A^{\otimes 0} \to A$ *provides a unity in the commutative associative algebra* A*.*

A slightly more difficult example is the operad Ass which encodes the structure of an associative unital algebra: it is usually described by setting

$$\mathsf{Ass}_S = k[\mathsf{Aut}(S)],$$

the regular representation of the symmetric group $\mathsf{Aut}(S)$. To define the operadic composition, one can, for example, consider the so-called *category* Σ *of non-commutative sets*: objects are finite sets, morphisms from S' to S are pairs of a map $f : S' \to S$ of finite sets and a total ordering on every preimage $f^{-1}(s)$, $s \in S$. The composition is obvious, and we obviously have the forgetfull functor $\gamma : \Sigma \to \Gamma$ which forgets the total orders. Then we set

$$\mathsf{Ass}_f = k[\{f' \in \Sigma(S', S) \mid \gamma(f') = f\}] \tag{9.3}$$

for any $f \in \Gamma$, $f : S' \to S$, and the composition in Σ induces the composition maps $\mathsf{Ass}_f \otimes \mathsf{Ass}_g \to \mathsf{Ass}_{f \circ g}$. The free algebra $F_{\mathsf{Ass}} V$ generated by a vector space V is the tensor algebra $T^{\bullet} V$.

Let us assume from now on that the base field k has characteristic 0, $\mathsf{char}\, k = 0$. For any vector space V, the diagonal map $V \to V \oplus V$ induces a coproduct $T^{\bullet} V \to T^{\bullet} V \otimes T^{\bullet} V$ which turns the tensor algebra $T^{\bullet} V$ into a cocommutative Hopf algebra. Since $\mathsf{char}\, k = 0$, this means that $T^{\bullet} V$ is the universal envelopping algebra of some Lie algebra $L^{\bullet} V$. In fact, by the universality property of a universal envelopping algebra, $L^{\bullet} V$ is the free Lie algebra generated by V. The universal envelopping algebra $T^{\bullet} V$ acquires a Poincaré-Birkhoff-Witt increasing filtration $K_{\bullet} T^{\bullet} V$, and the associated graded quotient with respect to this filtration is the symmetric algebra generated by $L^{\bullet} V$ — we have a canonical identification

$$\mathsf{gr}^F_{\bullet} T^{\bullet} V \cong S^{\bullet} L^{\bullet} V.$$

This graded quotient is a Poisson algebra, and it is easy to see by spelling out the universal properties that $P_{\bullet} V = \mathsf{gr}^F_{\bullet} T^{\bullet} V$ is actually the free Poisson algebra generated by V.

Now, both the PBW filtration and the isomorphism $\mathsf{gr}^F_{\bullet} T^{\bullet} V \cong P_{\bullet} V$ are functorial in V; this implies that what we actually have is a decreasing filtration $F^{\bullet}\, \mathsf{Ass}$ on the associative operad Ass, and an identification $\mathsf{gr}^{\bullet}_F \mathsf{Ass} \cong \mathsf{Poi}$ between the associated graded quotient of Ass and an operad Poi which encodes the structure of a Poisson algebra (in particular, the PBW filtration on Ass is compatible with the operadic structure). We see that Poi is in fact an operad of graded vector spaces. This is also obvious from the definition: if we assign degree 0 to multiplication and degree 1 to the Poisson bracket, then all the axioms of a Poisson algebra are compatible with these degrees.

The highest degree term of the PBW filtration on Ass — or equivalently, the highest term in the associated graded quotient $\mathsf{gr}^{\bullet}_F \mathsf{Ass} \cong \mathsf{Poi}$ — is the Lie operad Lie; the natural maps $\mathsf{Lie} \to \mathsf{Ass}$,

$\mathsf{Lie} \to \mathsf{Poi}$ encode the fact that both a Poisson algebra and an associative algebra are Lie algebras in a canonical way (in the associative case, the bracket is given by the commutator, $[a, b] = ab - ba$). We note that it is not trivial to describe Lie explicitly. For example, the dimension of Lie_n is $(n-1)!$. If the base field k is algebraically closed, then Lie_n can be described as the representation of the symmetric group Σ_n induced from the non-trivial character of the cyclic subgroup $\mathbb{Z}/n\mathbb{Z} \subset \Sigma_n$ spanned by the long cycle. It is a pleasant exersize to check that this representation is actually defined over k even when k is not algebraically closed.

Finally, the example that interest us most is that of Gerstenhaber algebras. Since the definition of a Gerstenhaber algebra differs from that of a Poisson algebra only in the degree assigned to the bracket, one might expect that Gerstenhaber algebras are controlled by an operad $\mathsf{Gerst}^{\bullet}$ essentially isomorphic to $\mathsf{Poi}^{\bullet}$. This is true, but there is the following subtlety. Both $\mathsf{Poi}^{\bullet}$ and $\mathsf{Gerst}^{\bullet}$ are operads of graded k-vector spaces, but this can means one of two distinct things: either we define the product of graded vector spaces simply as their product with induced grading, or we treat the degree as a homological degree. The difference is in the symmetry isomorphism $\sigma : V_{\bullet} \otimes W_{\bullet} \to W_{\bullet} \otimes V_{\bullet}$ of the tensor product of graded vector spacee $V_{\bullet}$, $W_{\bullet}$: if the degree is homological, then by convention we introduce the sign and define σ by

$$\sigma(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a.$$

Now, $\mathsf{Gerst}^{\bullet}$ and $\mathsf{Poi}^{\bullet}$ are both operads of graded vector spaces, and the difference between them is the following: the action of the symmetric group $\mathsf{Aut}(S)$ on Gerst_S is twisted by the sign representation — for any n, S, we have

$$\mathsf{Gerst}^n_S \cong \mathsf{Poi}^n_S \otimes \varepsilon^{\otimes n}, \tag{9.4}$$

where ε is the one-dimensional sign representation of $\mathsf{Aut}(S)$. But while $\mathsf{Poi}^{\bullet}$ is a graded operad in the usual naive sense, the degree in $\mathsf{Gerst}^{\bullet}$ is homological, and because of this, the isomorphisms (9.4) are still compatible with the operadic structure.

9.3 Little cubes operad.

It turns out, however, that there is a different, more conceptual construction of the Gerstenhaber operad Gerst.

One immediately notes that in the definition of an operad, one can use any symmetric monoidal category instead of the category of k-vector spaces. Thus we can speak not only about operads of vector spaces, or graded vector spaces, but also abouts operads of sets and operads of topological spaces. And historically, it was the operads of topological spaces which appeared first — specifically, the so-called *operad of little n-cubes.*

Let I be the unit interval $[0, 1]$. Fix a positive integer n, and consider the cube I^n of size 1 of dimension n. For any finite set S, say that an *S-cube configuration in I^n* is an open subset in I^n whose complement is the union connected components numbered by elements of S, each being a subcube in I of smaller size, whose faces are parallel to faces of I^n. Let O^n_S be the set of all such configurations. A configuration is completely determined by the centers and the sizes of all the cubes, so that O^n_S is naturally an open subset in $(I^{(n+1)})^S$. This turns it into a topological space.

We now note that the collection O^n_S with a fixed n naturally defines an operad of topological spaces. The composition is given by the following procedure: take an S_1-cube configuration in I^n, rescale it to a smaller size, and plug it into an S_2-cube configuration by filling in one of the connected components of its complement. When the sizes fit, the result is obviously an $(S_1 \cup S_2 \setminus \{s\})$-cube configuration, where $s \in S_2$ is the point which we used for the operation. We leave it to the reader to check that this procedure indeed gives a well-defined operad, and that all the structure maps of this operad are continuous maps.

Definition 9.3. The operad $O^n_\bullet$ is called the *operad of little n-cubes.*

What one is interested in is not the topological spaces O^n_S but their homotopy types, and these have a simpler description. Forgetting the size of a cube defines a projection $O^n_S \to (I^n)^S \setminus \mathsf{Diag}$, the complement to all the diagonals in the power $(I^n)^S$, and this projection is a homotopy equivalence — in other words, O^n_S is homotopy-equivalent to the *configuration space* of injective maps from S to I^n. Equivalently, one can take $\mathbb{R}^n$ instead of the cube I^n. Unfortunately, the structure of the operad is not visible in this model.

If $n = 1$, we can go even further: the configuration space of injective maps from S to the interval I has $|\operatorname{\mathsf{Aut}}(S)|$ connected components, numbered by the induced total order on the set S, and each connected component is a simplex, thus contractible. We conclude that O^1_S is homotopy-equivalent to the (discrete finite) set of total orders on S.

Now, taking the homology with coefficients in k turns any operad of topological spaces into an operad of graded k-vector spacers. In particular, for any $n \geq 1$ we have an operad formed by $H_\bullet(O^n_S, k)$.

Exercise 9.2. *Check that for $n = 1$, $H_\bullet(O^n_\bullet, k)$ is the operad $\mathsf{Ass}_\bullet$. Hint: use the description* (9.3).

Proposition 9.4. *Algebras over the homology operad $H_\bullet(O^2_S, k)$ of the operad $O^2_\bullet$ of little squares are the same as Gerstenhaber algebras, and $H_\bullet(O^2_S, k)$ is isomorphic to the Gerstenhaber operad $\mathsf{Gerst}^\bullet$.*

Proof. This is an essentially well-known but rather non-trivial fact; for example, it implies that $H_n(O^2_n, k)$ is the n-th space Lie_n of the Lie operad — as far as I know, this was first proved by V. Arnold back in the late 60-es.

Let us first construct a map of operads $a_\bullet : \mathsf{Gerst}^\bullet_\bullet \cong H_\bullet(O^2_\bullet, k)$. The component $\mathsf{Gerst}^\bullet_2$ is spanned by the product and the bracket, and O^2_2 is the complement to the diagonal in the product $I^2 \times I^2$, which is homotopy-equivalent to the circle S^1. We define a_2 by sending the product in Gerst^0_2 to the class of a point in $H_0(S^1, k) \cong k$, and the bracket in Gerst^1_2 to the fundamental class in $H_1(S^1, k) \cong k$.

Exercise 9.3. *Check that this extends to a map of operads. Hint: since all the relations in $\mathsf{Gerst}^\bullet$ invlove only three indeterminates, it is sufficient to consider O^2_3.*

Now assume by induction that a_i is an isomorphism for all $i \leq n$. By definition, $\mathsf{Gerst}^\bullet_{n+1}$ is spanned by all expressions involving the product and the bracket in $n + 1$ indeterminates $x_1, \dots, x_{n+1}$. Substituting the unity instead of x_{n+1} gives a map $\mathsf{Gerst}^\bullet_{n+1} \to \mathsf{Gerst}_n$; this map is obviously surjective. Substituting $\{x_{n+1}, x_i\}$ instead of x_i gives a map $\mathsf{Gerst}^\bullet_n \otimes k[S] \to \mathsf{Gerst}^{\bullet+1}_{n+1}$, where S is the set of indeterminates $x_1, \dots, x_n$. Since $\{1, x_i\}$ is by definition equal to 0, we have a sequence

$$\mathsf{Gerst}^{\bullet-1}_n \otimes k[S] \longrightarrow \mathsf{Gerst}^\bullet_{n+1} \longrightarrow \mathsf{Gerst}^\bullet_n \longrightarrow 0. \tag{9.5}$$

which is exact on the right.

On the geometric side, filling in the $(n + 1)$-st cube in a cube configuration — or equivalently, forgetting the $(n+1)$-st point in a configuration of points in $\mathbb{R}^2$ — defines a projection $O^2_{n+1} \to O^2_n$, and this is a fibration with fiber $E^n = \mathbb{R}^2 \setminus S$, where $S \subset \mathbb{R}^2$ is the configuration of the remaining n distinct points. We have the Leray spectral sequence

$$H_\bullet(O^2_n, H_\bullet(E^2_n, k)) \Rightarrow H_\bullet(O^2_{n+1}, k).$$

The homology $H_\bullet(E^2_n, k)$ is only non-trivial in degrees 0 and 1; the group $H_1(E^2_n, k)$ can be naturally identified with $k[S]$ by sending $s \in S$ to a small circle around its image in $\mathbb{R}^2$. The fundamental group of the base O^2_n is the pure braid group, and it is easy to check that it acts trivially on $H_\bullet(E^2_n, k)$, so that the spectral sequence reads

$$H_\bullet(O^2_n, k) \otimes H_\bullet(E^2_n, k) \Rightarrow H_\bullet(O^2_{n+1}, k).$$

Moreover, replacing $\mathbb{R}^2$ with $\mathbb{C}$, we can treat $O^2_S = \mathbb{C}^S \setminus \mathsf{Diag}$ as a complex algebraic variety whose homology groups have Hodge structures, and in particular, weights. One checks easily that $H_n(O^2_S, k)$ is pure Hodge-Tate of weight $2n$. Therefore the Leray spectral sequence degenerates, so that, taking in account the isomorphism $H_1(E^2_N, k) \cong k[S]$, we have a short exact sequence

$$(9.6) \qquad 0 \longrightarrow H_{\bullet-1}(O^2_n, k) \otimes k[S] \longrightarrow H_\bullet(O^2_{n+1}, k) \longrightarrow H_\bullet(O^2_n, k) \longrightarrow 0.$$

Now, it is obvious from the construction of the map $a_\bullet$ that it is a map between (9.5) and (9.6), so that we have a commutative diagram

$$(9.7) \qquad \begin{array}{ccccccccc} & & \mathsf{Gerst}^{\bullet-1}_n \otimes k[S] & \xrightarrow{f} & \mathsf{Gerst}^\bullet_{n+1} & \longrightarrow & \mathsf{Gerst}^\bullet_n & \longrightarrow & 0 \\ & & \Big\downarrow a_n & & \Big\downarrow a_{n+1} & & \Big\downarrow a_n & & \\ 0 & \longrightarrow & H_{\bullet-1}(O^2_n, k) \otimes k[S] & \longrightarrow & H_\bullet(O^2_{n+1}, k) & \longrightarrow & H_\bullet(O^2_n, k) & \longrightarrow & 0. \end{array}$$

Moreover, we now that a_n is an isomorphism, which implies in particular that the map f in (9.7) is injective. To prove that a_{n+1} is also an isomorphism, it suffices to prove that the top row forms a short exact sequence. But we also have the projection $O^1_{n+1} \to O^1_n$, and it induces a short exact sequence

$$0 \longrightarrow \mathsf{Ass}_n \otimes k[S] \longrightarrow \mathsf{Ass}_{n+1} \longrightarrow \mathsf{Ass}_n \longrightarrow 0$$

which gives (9.5) under taking the associated graded with respect to the Poincaré-Birkhoff-Witt filtration and using the isomorphism $\mathsf{Gerst}^\bullet \cong \mathsf{Poi}^\bullet$. Since this sequence is exact, and its associated graded is exact on the left and on the right, it must also be exact in the middle term for dimension reasons. □

9.4 Braided algebras and Tamarkin's proof.

What we did in Proposition 9.4 was to take two different operads, that of 1-cubes and that of 2-cubes, and identify, up to a sign twist, $H_\bullet(O^2_\bullet, k)$ with a certain associated graded quotient of $H_\bullet(O^1_\bullet, k)$ (which reduces to $H_0(O^1_\bullet, k)$). We now note that $H_\bullet(O^2_\bullet, k)$ can also be treated as an associated graded quotient. Namely, given a topological space X, one can consider its singular chain complex $C_\bullet(X, k)$. Every complex $E_\bullet$ has a "canonical filtration" $F^\bullet E_\bullet$ given by

$$F^i E_j = \begin{cases} 0, & j < i, \\ \mathsf{Ker}\, d, & j = i, \\ E_j, & j > i, \end{cases}$$

where d is the differential. The associated graded quotient $\mathsf{gr}^\bullet_F E_\bullet$ is canonically quasiisomorphic to the sum of homology of the complex $E_\bullet$. In particular, we have

$$\mathsf{gr}^\bullet_F C_\bullet(X, k) \cong H_\bullet(X, k).$$

Thus passing to homology is, up to quasiisomorphism, equivalent to taking the associated graded quotient with respect to the canonical filtration.

Given an operad $X_\bullet$ of topological spaces, we can consider the DG operad formed by $C_\bullet(X_\bullet, k)$. The canonical filtration, being canonical, is automatically compatible with the operadic structure, and the associated graded quotient $\mathsf{gr}^\bullet_F C_\bullet(X_\bullet, k)$ is quasiisomorphic to $H_\bullet(X_\bullet, k)$.

In particular, we can consider the operad $C_\bullet(O^2_\bullet, k)$. Its canonical filtration in fact behaves similarly to the PBW filtration on $\mathsf{Ass} = H_0(O^1_\bullet, k)$, although to define it, we do not need to use the structure of an operad. The associated graded quotient $\mathsf{gr}^\bullet_F C_\bullet(O^2_\bullet, k)$ is quasiisomorphic to the Gerstenhaber operad Gerst.

Definition 9.5. A *braided algebra* is a DG algebra over the DG operad $C_\bullet(O^2_\bullet, k)$.

The term "braided algebra" comes from the relation between O^2_n and the pure braid group B_n of n braids: we have $\pi_1(O^2_n) = B_n$, and one can show that O^2_n has no higher homotopy groups, so that it is homotopy-equivalent to the classifying space of B_n.

We note that as stated, Definition 9.5 is almost useless, since the singular chain complex $C_\bullet(X)$ of a topological space is huge — one cannot expect the DG operad $C_\bullet(O^2_\bullet, k)$ to act on anything reasonable. However, what one can do is to invert quasiisomorphisms and consider DG algebras over some DG operad $O_\bullet$ "up to quasiisomorphism", in the same way as we did for DG Lie algebras. A convenient formalism for this is provided by the so-called *closed model categories* originally introduced by Quillen (a modern reference is the book "Model categories" by M. Hovey). This gives a certain well-defined category $\operatorname{Ho}(O_\bullet)$, and, what is important, it only depends on the defining operad "up to a quasiisomorphism" — a quasiisomorphism $O'_\bullet \to O_\bullet$ between DG operads induces an equivalence $\operatorname{Ho}(O'_\bullet) \cong \operatorname{Ho}(O_\bullet)$. In practice, one is only interested in braided algebras up to a quasiisomorphism, that is, in objects of the category $\operatorname{Ho}(C_\bullet(O^2_\bullet, k))$; and to construct such an algebra, it is sufficient to have a DG algebra over some DG operad quasiisomorphic to $C_\bullet(O^2_\bullet, k)$. It is this structure which one has on the Hochschild cohomology complex of an associative unital algebra A.

Theorem 9.6 (Deligne Conjecture). *For any unital associative k-algebra A, its Hochschild cohomology complex is a DG algebra over a DG operad which is quasiisomorphic to $C_\bullet(O^2_\bullet, k)$.*

This statement has an interesting history. Originally it was a question, not even a conjecture, asked in 1993 by P. Deligne. Almost immediately it was wrongly proved by Getzler and Jones, and independently, also wrongly, by A. Voronov. But in 1998, Tamarkin has discovered his amazingly short proof of the Kontsevich Formality Theorem, which used Deligne conjecture; under close scrutiny, the mistakes were found, and new complete proofs by several groups of people were available by 2000 (among those people I should mention at least Tamarkin, Voronov, J. McClure-J. Smith, and M. Kontsevich-Y. Soibelman). In almost all the proofs, the authors actually construct a single DG operad which works for all associative algebras, but all of them are rather complicated and unnatural. The real reason for this is that what acts naturally on Hochschild cohomology is not an operad but a more complicated object, and this is currently under investigation. However, for practical purposes such as Formality Theorem, any solution is good, since it can used as a black box.

Assuming Deligne Conjecture, Tamarkin's proof of the Formality Theorem is a combination of the following two results.

Theorem 9.7 (Tamarkin,Kontsevich). *The DG operad $C_\bullet(O^2_\bullet, k)$ itself is formal, that is, there exists a chain of quasiisomorphisms connecting it to the Gerstenhaber operad $\mathsf{Gerst}^\bullet = H_\bullet(O^2_\bullet, k)$.*

Theorem 9.8 (Tamarkin). *Let A be the polynomial algebra $k[x_1, \dots, x_n]$ in n variables, equipped with the natural action of the group $GL(n, k)$ which interchanges the variables. Any DG algebra over* Gerst *which is equipped with a $GL(n, k)$-action and whose cohomology is isomorphic to $HH^\bullet(A)$ as a $GL(n, k)$-equivariant Gerstenhaber algebra is formal.*

It is the second result that was the original discovery of Tamarkin, and its proof was very simple. But then the problems with Deligne Conjecture appeared... in the course of fixing them, Kontsevich suggested that the operad $C_{\bullet}(O^2_{\bullet}, k)$ itself should be formal, and Tamarkin promptly proved it (but this proof was combinatorial and not simple at all). Later on, Kontsevich gave a different proof, also combinatorial. There is also a very simple argument in folklore which deduces Theorem 9.7 from Hodge Theorey, similarly to the classic formality result of Deligne-Griffits-Morgan-Sullivan, but this, to the best of my knowledge, has never been written down. In any case, one thing is very important: the quasiisomorphisms in Theorem 9.7, no matter how one produces them, are very non-trivial, and they usually depend on transcedental things like periods of differential forms or the so-called "Drinfeld associator". In addition, there is no canonical choice of these quasiisomorphisms — one expects that the conjectural "motivic Galois group", or even the usual Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, acts on the set of these quasiisomorphisms in a very non-trivial way. On the other hand, the DG operads which appear in the solutions to Deligne Conjecture are quite canonical, and their action on Hochschild cohomology is elementary and defined over $\mathbb{Q}$.

Lecture 10.

Combinatorics of planar trees. Comparison theorem. Brace operad and its action on the Hochschild cohomology complex.

10.1 Planar trees.

The topic of today's lecture is Deligne Conjecture — we want to construct an operad $O_{\bullet}$ quasiisomorphic to the chain complex operad $C_{\bullet}(O^2_{\bullet}, k)$ of the operad of little squares so that $O_{\bullet}$ acts in a natural way on the Hochschild cohomology complex of an associative algebra A.

We start by introducing a certain combinatorial model of the operad of little squares (or rather, it will be more convenient for us to work with little discs).

By a *planar tree* we will understand an unoriented connected graph with no cycles and one distiguished vertex of valency 1 called *the root*, equipped with a cyclic order on the set of edges attached to each vertex. Given such a tree T, we will denote by $V(T)$ the set of all non-root vertices of T, and we will denote by $E(T)$ the set of all edges of T not adjacent to the root.

More generally, by an *n-planar tree* we will understand an unoriented connected graph with no cycles and n distiguished vertices of valency 1, called *external vertices*, one of which is additionally distinguished and called the root; again, the graph should be equipped with a cyclic order on the set of edges attached to each vertex. We note that this automatically induces a cyclic order on the set of external vertices, so that n-planar trees are naturally numbered by an object $[n]$ of the cyclic category Λ. For an n-planar tree T, $V(T)$ denotes the set of all non-external vertices, and $E(T)$ denotes the set of of edges not adjacent to external vertices.

Given a tree T, we denote by $|T|$ its geometric realization, that is, a CW complex with vertices of T as 0-cells and edges of T as 1-cells. We note that for every planar tree T, $|T|$ can be continuously embedded into the unit disc D so that the root of T goes to $1 \in D$, the external vertices, if any, go to points on the boundary $S^1 \subset D$ and split it into a wheel graph, the rest of $|T|$ is mapped into the interior of the disc, and for every vertex $v \in V(T)$, the given cyclic order on the edges adjacent to v is the clockwise order. Moreover, the set of all such embeddings with a natural topology is contractible, so that the embedding is unique up to a homotopy, and the homotopy is also unique up to a homotopy of higher order, and so on.

Given a tree T and an edge $e \in E(T)$, we may contract e to a vertex and obtain a new tree T^e. The contrations of different edges obviously commute, so that for any n edges $e_1, \dots, e_n \in E(T)$, we have a unique tree $T^{e_1,\dots,e_n}$ obtained by contracting $e_1, \dots, e_n$. By construction, we have a natural map $V(T) \to V(T^{e_1,\dots,e_n})$ and a natural map of realizations $|T| \to |T^{e_1,\dots,e_n}|$.

Assume given a finite set S. By a *tree marked by S* we will understand a planar tree or an n-planar tree T together with an injective map $S \to V(T)$. The vertices in the image of this map are called marked, the other ones are unmarked. A marked tree T is *stable* if every unmarked vertex $v \in V(T) \setminus S$ has valency at least 3. Given a marked tree T which is unstable, we can canonically produce a stable tree T' by first recursively removing all unmarked vertices of valency 1 and edges leading to them, and them removing unmarked vertices of valency 2 and gluing together the corresponding edges. We will call this T' the *stabilization* of T.

Given a stable marked tree T and some edges $e_1, \dots, e_n \in E(T)$, we mark the contraction by composing the map $S \to V(T)$ with the natural map $V(T) \to V(T^{e_1,\dots,e_n})$. If the resulting map is injective, then this is again a stable marked tree.

Exercise 10.1. *Check that for any two trees T, T' stably marked by the same set S, there exists at most one subset $\{e_1, \dots, e_n\} \subset E(T)$ such that $T^{e_1,\dots,e_n} \cong T$. Hint: removing an edge splits a tree T into two connected components; first prove that an edge is uniquely defined by the corresponding partition of the set $V(T)$.*

By virtue of this exercize, for every $[n] \in \Lambda$ the collection of all n-planar trees stably marked by the same finite set S acquires a partial order: we say that $T \geq T'$ if and only if T' can be obtained from T by contraction. We will denote this partially ordered set by $\mathsf{T}_S^{[n]}$, or simply by T_S if $[n] = [1]$. This is our combinatorial model for the configuration space.

Theorem 10.1. *For any $[n] \in \Lambda$, the classifying space $|\mathsf{T}_S^{[n]}|$ of the partially ordered set T_S is homotopy equivalent to the configuration space $D^S \setminus \mathsf{Diag}$ of injective maps $S \to D$ to the unit disc D.*

As a first step of proving this, let us construct a map $|\mathsf{T}_S| \to D$.

Let us denote by B_S the fundamental groupoid of the configuration space $D^S \setminus \mathsf{Diag}$: objects are points, that is, injective maps $f : S \to D$, morphisms from f to f' are homotopy classes of paths, that is, homotopy classes of continuous maps $S \times I \to D$, whose restriction to $S \times \{0\}$, resp. $S \times \{1\}$ is equal to f, resp. f' (here $I = [0, 1]$ is the unit interval, and $S \times I$ is equipped with the product topology — it is the disjoint union of S copies of I). Since $D^S \setminus \mathsf{Diag}$ is an Eilenberg-MacLane space of type $K(\pi, 1)$, we have the homotopy equivalence $D^S \setminus \mathsf{Diag} \cong |\mathsf{B}_S|$.

Now consider the following category $\widetilde{\mathsf{T}}_S$. Objects are stable marked trees T together with an embedding $f : |T| \to D$. Maps from $f : |T| \to D$ to $f' : |T'| \to D$ exist only if $T \geq T'$, and they are homotopy classes of continous maps $\gamma : |T| \times I \to D$ such that the restriction $\gamma : |T| \times \{x\} \to D$ is injective for any $x \in [0, 1[$, the restriction $\gamma : |T| \times \{0\} \to D$ is equal to the map f, and the restriction $\gamma : |T| \times \{1\} \to D$ is the composition of the natural map $|T| \to |T'|$ and the map $f' : |T'| \to D$.

Then on one hand, we have a forgetfull functor $\widetilde{\mathsf{T}}_S \to \mathsf{T}_S$ which forgets the embedding, and since the space of embeddings is contractible, this is an equivalence of categories.

On the other hand, we have a comparison functor $\widetilde{\mathsf{T}}_S \to \mathsf{B}_S$ which sends an embedded stable marked tree $|T| \subset D$ to the subset of its marked points $S \subset |T| \subset D$, and forgets the rest. Then Theorem 10.1 for $n = 1$ follows immediately from the following.

Proposition 10.2. *The comparison functor $\widetilde{\mathsf{T}}_S \to \mathsf{B}_S$ induces a homotopy equivalence $|\mathsf{T}_S| \cong |\widetilde{\mathsf{T}}_S| \cong |\mathsf{B}_S|$.*

10.2 Stratified spaces and homology equivalences.

Our strategy of proving Proposition 10.2 is the same as in the study of the Gerstenhaber operad in the last lecture: we want to apply induction on the cardinality of S by forgetting points one-by-one and considering the corresponding projections of the configuration spaces.

Thus we assume given a finite set S' and an element $v \in S'$, and we denote $S = S' \setminus \{v\}$. Then forgetting v defines a projection $\mathsf{B}_{S'} \to \mathsf{B}_S$. On the other hand, unmarking v and applying stabilization defines a projection $\mathsf{T}_{S'} \to \mathsf{T}_S$. This is obviously compatible with the comparison functors, so that we have a commutative diagram

$$\begin{array}{ccc} \mathsf{T}_{S'} & \longrightarrow & \mathsf{B}_{S'} \\ \downarrow & & \downarrow \\ \mathsf{T}_S & \longrightarrow & \mathsf{B}_S. \end{array} \qquad (10.1)$$

Definition 10.3. An *abelian fibration* $\mathcal{C}$ over a small category Γ is a fibration $\mathcal{C}/\Gamma$ such that all fibers $\mathcal{C}_{[a]}$, $[a] \in \Gamma$ are abelian categories, and all the transition functors $f^* : \mathcal{C}_{[b]} \to \mathcal{C}_{[a]}$, $f : [a] \to [b]$ are left-exact.

Just as in Proposition 5.3, one shows easily that the category of sections $\mathrm{Sec}(\mathcal{C})$ of an abelian fibration $\mathcal{C}/\Gamma$ is an abelian category.

Definition 10.4. A functor $\gamma : \Gamma \to \Gamma'$ between small categories is said to be a *homological equivalence* if

(i) for any abelian fibration $\mathcal{C}/\Gamma'$, the pullback functor $\gamma^* :: \mathcal{D}(\mathrm{Sec}(\mathcal{C})) \to \mathcal{D}(\mathrm{Sec}(\gamma^*\mathcal{C}))$ is a fully faithful embedding, and

(ii) the essential image $\gamma^*(\mathrm{Sec}(\mathcal{C})) \subset \mathrm{Sec}(\gamma^*\mathcal{C})$ consists of such $E \in \mathrm{Sec}(\gamma^*\mathcal{C})$ that for any map $f : [a] \to [b]$ in Γ with invertible $\gamma(f)$, the transition map $E_{[a]} \to f^*E_{[b]}$ is invertible.

Here $\gamma^*\mathcal{C} = \mathcal{C} \times_{\Gamma'} \Gamma$ is the pullback of the abelian fibration $\mathcal{C}/\Gamma'$, $\mathcal{D}(-)$ stand for the derived category, $E|_{[a]}$ is the restriction of E to the fiber $(\gamma^*\mathcal{C})_{[a]} \cong \mathcal{C}_{\gamma([a])}$, and similarly for $E_{[b]}$. For example, if $\Gamma' = \mathsf{pt}$, $\gamma : \Gamma \to \mathsf{pt}$ is the projection to the point, and $\mathcal{C} = k$-Vect, the conditions of the definition say that $\mathcal{D}_{lc}(\Gamma, k)$ is equivalent to the derived category $\mathcal{D}(k\text{-Vect})$. As we saw in Corollary 4.4, this implies that the geometric realization $|\Gamma|$ is contractible.

Exercise 10.2. *Prove that if $\gamma : \Gamma \to \Gamma'$ is a homological equivalence, then the induced map $|\gamma| : |\Gamma| \to |\Gamma'|$ is a homotopy equivalence.*

The reason we have put the additional abelian fibration $\mathcal{C}$ in Definition 10.4 is that this way, it becomes recursive: we have the following.

Lemma 10.5. *Assume given cofibrations Γ'_1/Γ_1, Γ'_2/Γ_2, a functor $\gamma : \Gamma_1 \to \Gamma_2$, and a Cartesian functor $\gamma' : \Gamma'_1 \to \Gamma_1 \times_{\Gamma_2} \Gamma'_2 \to \Gamma'_2$. Then if γ is a homological equivalence, and γ' restricts to a homological equivalence on all the fibers, then γ' itself is a homological equivalence.*

Exercise 10.3. *Prove this. Hint: first show that for any cofibration $\pi : \Gamma' \to \Gamma$ and any abelian fibration $\mathcal{C}/\Gamma'$, there exists an abelian fibration $\pi_*\mathcal{C}$ whose fibers are given by*

$$(\pi_*\mathcal{C})_{[a]} = \mathrm{Sec}(\mathcal{C}|_{\Gamma'_{[a]}}), \qquad [a] \in \Gamma,$$

where $\mathcal{C}|_{\Gamma'_{[a]}}$ means the restriction to the fiber $\Gamma'_{[a]} \subset \Gamma'$, and that $\mathrm{Sec}(\mathcal{C}) \cong \mathrm{Sec}(\pi_*\mathcal{C})$.

Exercise 10.4. *Assume given diagrams of categories and functors*

$$\begin{array}{ccc} \Gamma_1 & \longrightarrow & \Gamma_{12} \\ \uparrow & & \uparrow \\ \Gamma_0 & \longrightarrow & \Gamma_2 \end{array} \qquad\qquad \begin{array}{ccc} \Gamma'_1 & \longrightarrow & \Gamma'_{12} \\ \uparrow & & \uparrow \\ \Gamma'_0 & \longrightarrow & \Gamma'_2 \end{array}$$

which are cocartesian in the sense that for any category $\mathcal{C}$, we have

$$\mathrm{Fun}(\Gamma_{12}, \mathcal{C}) \cong \mathrm{Fun}(\Gamma_1, \mathcal{C}) \times_{\mathrm{Fun}(\Gamma_0, \mathcal{C})} \mathrm{Fun}(\Gamma_2, \mathcal{C}),$$

and similarly for Γ'. Assume given a functor $\gamma = \langle \gamma_0, \gamma_1, \gamma_2, \gamma_{12} \rangle$ between them. Prove that if γ_0, γ_1 and γ_2 are homological equivalences, then so is γ_{12}.

Unfortunately, Lemma 10.5 cannot be used to analyze (10.1) directly, since the projection functor $\mathsf{T}_{S'} \to \mathsf{T}_S$ is not a cofibration. To correct this, we have to "compactify" the categories T_S by allowing non-injective markings $S \to V(T)$ — geometrically, this corresponds to adding the diagonals $\mathsf{Diag} \subset D^S$ to the configuration space $D^S \setminus \mathsf{Diag}$.

So, first, for every finite set S we define the category $\overline{\mathsf{T}}_S$ whose objects are trees T equipped with a map $f : S \to V(T)$ such that the induced embedding $\mathsf{Im}(f) \subset V(T)$ is a stable marking, with maps given by contractions of edges.

Second, we consider the topological space D^S as a space stratified by the diagonals, and we define the category $\overline{\mathsf{B}}_S$ as the its "stratified fundamental groupoid" in the following sense.

Definition 10.6. The *stratified fundamental groupoid* of a topological space X stratified by strata $X_i \subset X$ is the category whose objects are points $x \in X$, and whose maps from $x_1 \in X_1$ to $x_2 \in X_2$ exist only when $X_2 \subset X_1$, and are given by homotopy classes of paths $f : I \to X_1$ from x_1 to x_2 such that $f(I) \cap X_2 = f(1) = p_2$, and $f(I) \cap X_3 = \emptyset$ for any proper substratum $X_3 \subset X_2$.

Explicitly, an object in $\overline{\mathsf{B}}_S$ is given by a not necessarily injective map $f : S \to D$, and maps from f_0 to f_1 are given by homotopy classes of maps $\gamma : f_0(S) \times I \to D$ such that $\gamma : f_0(S) \times \{x\} \to D$ is injective for any $x \in [0, 1[$, $\gamma : f_0 \times \{1\} \to D$ is a map onto $f_1(S) \subset D$, and the composition $\gamma \circ f_0 : S \to f_0(S) \to f_1(S)$ is equal to f_1.

Exercise 10.5. *Let $\langle X, X_i \subset X\rangle$ be a stratified topological space, and let $\pi_1(X)$ be its stratified fundamental groupoid. Prove that the category* $\mathrm{Fun}(\pi_1(X)^{opp}, k)$ *is equivalent to the category of constructible sheaves of k-vector spaces on X which are locally constant along the open strata. Hint: consider first the case $X = I$, with a single proper stratum $X_1 = \{1\} \subset I$.*

We leave it to the reader to check that the comparison functor (10.1) extends to a functor $\overline{\mathsf{T}}_S \to \overline{\mathsf{B}}_S$, and we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathsf{T}}_{S'} & \longrightarrow & \overline{\mathsf{B}}_{S'} \\ \downarrow & & \downarrow \\ \overline{\mathsf{T}}_{S} & \longrightarrow & \overline{\mathsf{B}}_{S}. \end{array} \tag{10.2}$$

10.3 The comparison theorem.

We can now prove the comparison theorem between $\overline{\mathsf{T}}_S$ and $\overline{\mathsf{B}}_S$.

Proposition 10.7. *The comparison functor $\overline{\mathsf{T}}_S \to \overline{\mathsf{B}}_S$ is a homological equivalence for any finite set S.*

Proof. One checks easily that the vertical projections in (10.2) are cofibrations; thus by induction, it suffices to check that the comparison functor induces a homological equivalence on all the fibers.

Fix a tree $T \in \overline{\mathsf{T}}_S$, and consider a tree $T' \in \left(\overline{\mathsf{T}}_{S'}\right)_T$. When we remove the mark $v \in S'$ from T', one of the following four things might happen:

(i) The tree remains stable, with the vertex $v \in V(T') = V(T)$ possibly becoming unmarked.

(ii) An unmarked vertex of valency 2 appears; under stabilization, it is removed, and adjacent edges are glued together to give an edge $e \in E(T)$.

(iii) An unmarked vertex of valency 1 appears; under stabilization, we remove this vertex and the adjacent edge.

(iv) An unmarked vertex of valency 1 appears; under stabilization, we remove it with its edge, and then an unmarked vertex of valency 2 appears, which also has to be removed.

In the case (i), T' is completely determined by specifying $v \in V(T)$, and in the case (ii), by specifying $e \in E(T)$. To describe the combinatorial invariants in (iii), it is convenient to embed the tree T into the disc D and draw a small disc around each vertex $v \in V(T)$. The boundary of this disc is a wheel graph $[n] \in \Lambda$ whose vertices correspond to edges adjacent to v. Edges of these graphs are called *angles* of T, and the set of all angles of T is denoted by $A(T)$. Then in the case (iii), to determine T' we need to specify the other vertex $v \in V(T)$ of the removed edge, and the (unique) angle $a \in A(T)$ which this removed edge intersects. Finally, in the case (iv), T' is determined by the new edge $e \in E(T)$ containing the removed vertex of valency 2, and the "side" of this edge at which the removed edge was attached. The set of these sides is denoted by $S(T)$ (it is of course a 2-fold cover of the set $E(T)$). We note that every side defines an angle attached to each of the two vertices of the corresponding edge.

To sum up: the fiber of the projection $\overline{\mathsf{T}}_{S'} \to \overline{\mathsf{T}}_S$ over a tree $T \in \overline{\mathsf{T}}_S$ is the set

$$F_T = V(T) \cup E(T) \cup A(T) \cup S(T),$$

with some partial order.

Exercise 10.6. *Check that F_T has the following order: an edge $e \in E(T)$ is less than either of its vertices, an angle $a \in A(T)$ is less than the vertex where it lives, and a side $s \in S(T)$ is less than the corresponding edge, and less than the two angles it defines.*

On the other hand, the fiber F_p of the projection $\overline{\mathsf{B}}_{S'} \to \overline{\mathsf{B}}_S$ over an object represented by $p : S \to D$ is the stratified fundamental groupoid of the pair $f(S) \subset D$. To finish the proof, it suffices to prove the following.

Lemma 10.8. *Assume given a possibly unstable marked tree T embedded into the disc D, $|T| \subset D$, and let $p : S \to D$ be the corresponding embedding of the set of markings $S \subset V(T)$. Then the comparison functor $F_T \to F_p$ is a homological equivalence.*

Proof. Choose a vertex $v \in T$ of valency 1, let T' be the tree obtained by removing v and the adjacent edge $e \in E(T)$, and let $p' : S' \to D$ be the embedding of its set of markings $S' \subset V(T')$. Then we have a cocartesian diagram

$$\begin{array}{ccc} F_{T'} & \longrightarrow & F_T \\ \uparrow & & \uparrow \\ F_e & \longrightarrow & F_v, \end{array}$$

where $F_e \subset F_T$ is the subset consisting of e and its two sides, and $F_v \subset F_T$ is the subset consisting of v, all its adjacent edges, all its angles, and all their sides. On the other hand, we can shrink D to a small neighborhood of $|T| \subset D$ and then decompose it into the union of a small disc D_v centered at v and a neighborhood $D_{T'}$ of $|T'| \subset D$ so that the intersection $D_v \cap D_{T'}$ is contractible with no stratification. This gives a cocartesian diagram

$$\begin{array}{ccc} F_{p'} & \longrightarrow & F_p \\ \uparrow & & \uparrow \\ \mathsf{pt} & \longrightarrow & F_v, \end{array}$$

where F_v is the fundamental groupoid of D_v if v is unmarked, and the stratified fundamental groupoid of $\{v\} \subset D_v$ if v is marked. By virtue of Exercize 10.4, we can apply induction. Thus it suffices to prove that F_e is homologically equivalent to a point, and the comparison functor $F_v \to F_v$ is a homological equivalence (both if v is marked and if it is not). We leave it as an exercize. Hint: in the marked case, show first that the partially ordered set $F_v \setminus \{v\}$ is homologically equivalent to the fundamental groupoid of a circle S^1. □

Proof of Proposition 10.2. An immediate corollary of Proposition 10.7: every abelian fibration $\mathcal{C}/\mathsf{B}_S$, resp. $\mathcal{C}/\mathsf{T}_S$ can obviously be extended to $\overline{\mathsf{B}}_S$, resp. $\overline{\mathsf{T}}_S$ by setting $\mathcal{C}_{[a]} = 0$ for any $[a] \in \overline{\mathsf{B}}_S \setminus \mathsf{B}_S$, resp. $[a] \in \overline{\mathsf{T}}_S \setminus \mathsf{T}_S$, and this does not change the category of sections; therefore the comparison functor $\mathsf{T}_S \to \mathsf{B}_S$ is also a homological equivalence, and this implies the claim by Exercize 10.2. □

To finish the proof of Theorem 10.1, it remains to consider n-planar trees for $n \geq 2$. Note that for any $[n] \in \Lambda$ and a fixed embedding $f : [n-1] \to [n]$, we have a natural projection $\pi^f : \mathsf{T}_S^{[n]} \to \mathsf{T}_S^{[n-1]}$ obtained by the removing the external vertex not contained in the image of f and applying stabilization.

Exercise 10.7. *Check that π^f is a cofibration whose fiber E_T over a tree $T \in \mathsf{T}_S^{[n-1]}$ is the partially ordered set of cells of a certain cell decomposition of the open interval $]0,1[$, with the order by given adjacency (the decomposition may depend on T). Deduce that π^f is a homological equivalence.*

This Exercize together with Exercize 10.2 finish the proof of Theorem 10.1.

10.4 Regular partially ordered sets.

By virtue of Theorem 10.1, instead of studying the chain complex $C_{\bullet}(D^S \setminus \mathsf{Diag})$ directly, we may study complexes which compute the homology of the partially ordered set $\mathcal{T}_S$ (considered as a small category). This turns out to be easy, since the partially ordered set $\mathcal{T}_S$ is well-behaved.

Assume given a partially ordered set P. For any $p \in P$, denote by $\delta_p \in \mathrm{Fun}(P^{opp}, k)$ the functor given by $\delta_p(p) = k$, $\delta_p(p') = 0$ if $p \neq p'$.

Definition 10.9. A finite partially ordered set P is called *regular* if for any $p \in P$, we have

$$H_i(P^{opp}, \delta_p) \cong \begin{cases} k, & i = n, \\ 0, & \text{otherwise} \end{cases} \tag{10.3}$$

where n is some integer $n \geq 0$ called the *index* of p and denoted $\mathsf{ind}(p)$.

Exercise 10.8. *Prove that the product $P_1 \times P_2$ of two regular partially ordered sets is regular.*

Exercise 10.9. *Prove that P is regular if and only if for any $p \in P$, so the set $U_p = \{p' \in P | p' \leq q\}$.*

Proposition 10.10. *The partially ordered set $\mathsf{T}_S^{[n]}$ is regular for any finite set S and any $[n] \in \Lambda$, and the index of a tree $T \in \mathsf{T}_S^{[n]}$ is equal to $\mathsf{ind}(T) = n - 2 - v(T)$, where $v(T)$ is the cardinality of $V(T)$.*

Proof. For any tree $T \in \mathsf{T}_S^{[n]}$, the partially ordered set U_T of Exercize 10.9 is isomorphic to

$$U_T \cong \prod_{v \in V(T)} \mathsf{T}_{s(v)}^{[n_v]}, \tag{10.4}$$

where $[n_v]$ is the set of edges adjacent to the vertex v with its given cyclic order, and $s(v)$ is pt if v is marked and $\emptyset$ if v is unmarked. Thus by Exercize 10.8, it suffices to consider the cases $S = \mathsf{pt}$ and $S = \emptyset$. In either of these cases, we use induction on n. The sets $\mathsf{T}_\emptyset^{[1]}$ and $\mathsf{T}_\emptyset^{[2]}$ are empty; the sets $\mathsf{T}_{\mathsf{pt}}^{[1]}$ and $\mathsf{T}_\emptyset^{[3]}$ both consist of one point, thus giving the induction base. For the induction step, choose an embedding $[n-1] \to [n]$, and consider the corresponding projection $\mathsf{T}_S^{[n]} \to \mathsf{T}_S^{[n-1]}$. This is a cofibration. Its fibers E_T have been described in Exercize 10.7, and it is easy to check that they are regular. Moreover, for any $T \leq T' \in \mathsf{T}_S^{[n-1]}$, the corresponding transition map $E_T \to E_{T'}$ is obviously a homological equivalence. To finish the proof of the inductive step and the Proposition, it suffices to apply the following to every $\delta_T \in \mathrm{Fun}(\mathsf{T}_S^{[n]opp}, k)$.

Lemma 10.11. *Assume given a fibration $\gamma : \Gamma' \to \Gamma$ of small categories, and assume that the transistion functor f^* is a homological equivalence for any map $f : [a] \to [b]$ in Γ. Then for any $E \in \operatorname{Fun}(\Gamma', k)$ and any $[a] \in \Gamma$, there exists an isomorphism*

$$(L^{\bullet}\gamma_! E)([a]) \cong H_{\bullet}(\Gamma'_{[a]}, E_{[a]}), \tag{10.5}$$

where $E_{[a]} \in \operatorname{Fun}(\Gamma'_{[a]}, k)$ is the restriction of E to the fiber $\Gamma'_{[a]} \subset \Gamma'$.

Proof. Let $i : \mathsf{pt} \to \Gamma$ be the embedding of the object $[a]$, and let $i' : \Gamma'_{[a]} \to \Gamma'$ be the embedding of the fiber. Then we have the adjunction map

$$i_! \circ \gamma_! \circ i'^* \cong \gamma_! \circ i'_! \circ i'^* \to \gamma_!,$$

which by adjunction induces a base change map $\gamma_! \circ i'^* \to i^* \circ \gamma_!$. Taking derived functors, we obtain a map (10.5) functorially for any E. To prove that it is an isomorphism, it suffices to consider the case of a representable E, $E = k_{[b']}$ for some $[b'] \in \Gamma'$. Then the left-hand side of (10.5) is canonically isomorphic to $k[\Gamma([b],[a])]$, where $[b] = \gamma([b']) \in \Gamma$. On the other hand, since γ is a fibration, we have a canonical identification

$$k_{[b']}|_{\Gamma'_{[a]}} \cong \bigoplus_{f \in \Gamma([b],[a])} (f^*)_! k_{[b']}|_{\Gamma'_{[b]}}.$$

But since f^* is a homological equivalence for any $f \in \Gamma([b],[a])$, we have

$$H_{\bullet}(\Gamma'_{[a]}, (f^*)_! k_{[b']}|_{\Gamma'_{[b]}}) \cong H_{\bullet}(\Gamma'_{[b]}, k_{[b']}|_{\Gamma'_{[b]}}) \cong k,$$

which finishes the proof. □

Exercise 10.10. *Prove that the homology $H_{\bullet}(P,k) = H_{\bullet}(P^{opp},k)$ of a finite partially ordered set P can be computed by a complex $C_{\bullet}(P,k)$ with terms $C_i(P,k) = \bigoplus_{\mathsf{ind}(p)=i} k$. Hint: take a maximal element $p \in P$, let $P' = P \setminus \{p\}$, and consider the short exact sequence*

$$0 \longrightarrow j_!^{opp} k^{P'} \longrightarrow k^P \longrightarrow \delta_p \longrightarrow 0,$$

where $k^P \in \operatorname{Fun}(P^{opp},k)$, $k^{P'} \in \operatorname{Fun}(P'^{opp},k)$ are the constant functors, and $j : P' \to P$ is the embedding.

10.5 The brace operad.

Now consider the partially ordered set T_S. It is regular, so its cohomology can be computed by a complex $C_{\bullet}(\mathsf{T}_S,k)$, which we denote by $C_{\bullet}(S)$ to simplify notation. Unfortunately, there is some abguity in the differentials of the complexes $C_{\bullet}(S)$ (for a discussion, see the Kontsevich-Soibelman paper arxiv:math/0001151). As it turns out, with the appropriate choice of the differentials, the complexes $C_{\bullet}(S)$ form a DG operad, called the *brace operad*, and this operad acts naturally on the Hochschild cohomology complex of any associative algebra A.

Namely, assume given an associative unital algebra A, and assume given an m-cochain $f \in \operatorname{Hom}(A^{\otimes m}, A)$ and l other cochains $g_j \in \operatorname{Hom}(A^{\otimes n_j}, A)$, $1 \leq j \leq l$ of degrees $n_1, \dots, n_l$. Then the *brace* $f\{g_1,\dots,g_l\}$ is the cochain of degree $M = m + n_1 + \dots + n_l - l$ given by

$$f\{g_1,\dots,g_l\}(a_1,\dots,a_M) = \sum_I (-1)^{\varepsilon}_I f(a_1,\dots,a_{i_1-1}, g_1(a_{i_1},\dots,a_{i_1+n_1-1}), a_{i_1+n_1},\dots,$$
$$a_{M-m-n+1}, g_l(a_{M-m-n_l+1},\dots,a_{M-m+i_l}), a_{M-m+i_l+1},\dots,a_M),$$

where the sum is taken over all the multiindices $1 \leq i_1 < \dots < i_l \leq m$, and ε_I is given by

$$\varepsilon_I = \sum_{1 \leq j \leq l} n_j(i_j - 1).$$

If $m < l$, then the set of multiindices is empty, and the brace is set to be 0.

In other words, the brace is obtained by substituting $g_1, \dots, g_l$ into f in all possible ways, and taking the alternating sum.

Then every tree $T \in \mathsf{T}_S$ defines an S-linear operation α_T on the Hochschild cohomology complex of A by the following inductive rule.

(i) If T is the tree with exactly one vertex of valency ≥ 1, and this vertex is marked, then

$$\alpha_T(f_1, \dots, f_n) = f_1(f_2, \dots, f_n),$$

where $f_1, \dots, f_n$ are cochains numbered by elements in $S = V(T)$, and f_1 corresponds to the marked vertex.

(ii) If in the situation above the vertex is unmarked,

$$\alpha_T(f_1, \dots, f_n) = f_1 \cdot f_2 \cdot \dots \cdot f_n.$$

(iii) In the general case, split T into two trees T_1, T_2 by cutting an edge $e \in E(T)$, marking one of the resulting new vertices, and declaring the other one the new root vertex, and let

$$\alpha_T(f_1, \dots, f_n) = \alpha_{T_1}(\alpha_{T_2}(f_1, \dots, f_l), f_{l+1}, \dots, f_n),$$

where T_1 is the subtree which contains the original root, and α_{T_2} corresponds to the new marked vertex of T_1.

It is not too difficult to check that the brace operation is associative in the appropriate sense, so that the operation in (iii) does not depend on the choice of the edge $e \in E(T)$. To make (ii) similar to (i), we note that since A is associative, we have a preferred cochain $\mu \in \operatorname{Hom}(A^{\otimes n}, A)$ for any $n \geq 0$ given by the product, and

$$\mu\{f_1, \dots, f_n\} = f_1 \cdot f_2 \cdot \dots \cdot f_n.$$

Moreover, it is clear that the collection of the operations α_T is closed under substitution — more precisely, α_T span a suboperad in the endomorphism operad of the Hochschild cohomology complex of the algebra A. This defines an operad structure on the graded vector spaces $C_{\bullet}(S) = k[\mathsf{T}_S]$.

Theorem 10.12. *With the appropriate choice of the differentials in the complexes $C_{\bullet}(S)$, the operad structure on $C_{\bullet}(S)$ defined by the brace operation and the action of this operad on the Hochschild cohomology complexes is compatible with the differential, so that we have a DG operad, and for any associative unital algebra A, its Hochschild cohomology complex is a DG algebra over the DG operad $C_{\bullet}(S)$.*

I do not give the exact differentials, since I will not prove this result anyway (see the quoted paper of Kontsevich-Soibelman, and also arxiv:math/9910126 of McClure and Smith, where a closely related result is proved). Rather, to finish the lecture, I want to discuss what the result means, and what would a conceptually clear proof look like.

10.6 Discussion.

First of all, we note that Theorem 10.12 *does not prove the Deligne Conjecture.*

Indeed, while we have constructed quasiisomorphisms between the chain complexes of configuration spaces $D^S \setminus \mathsf{Diag}$ and the complexes $C_\bullet(S)$ which act on Hochschild cohomology, we did not prove that they are compatible with the operadic structure. A natural way to do this would be to extend Theorem 10.1 to a comparison theorem between operads; this would also take care of all the signs. But it is completely impossible to do this: while the groupoids $\overline{\mathsf{B}}_S$ do form an operad in an approppriate 2-categorical sense, the partially ordered sets T_S *do not.*

Namely, the operadic structure would allow one to replace a marked vertex v in a tree T with another tree T'. But this is only possible if v has valency 1 — otherwise it is not clear what to do with the extra edges coming into v. We can only replace v with an n-planar tree, where n is the valency of v.

This is why there is a sum in our definition of the brace operation — essentially this is an averaging over all possible ways to take care of the extra edges; and this becomes possible only after we pass to the chain complex. What happens is that we consider the canonical quasiisomorphism $C_\bullet(\mathsf{T}_S^{[n]}, k) \to C_\bullet(\mathsf{T}_S, k) = C_\bullet(S)$ obtained by projection, and forcibly invert it.

Considering all the n-planar trees together does not help much: they do not an operad either, because they can be sustituted one into the other only if the valencies match.

To me, the best way to prove Deligne Conjecture would be not to force the pieces into submission, but rather, to formalize the structure that the partially ordered sets T_S and $\mathsf{T}_S^{[n]}$ do possess; this amounts to generalizing the notion of an operad by replacing the category Γ of finite sets with something else — for example, an appropriately defined category of trees, with (10.4) playing the role of the product decomposition (9.1). However, as far as I know, this has not been done. M. Batanin has realized a similar plan, but a different replacement for Γ — he introduces a notion of a "non-Σ 2-operad" which is encoded by the "category of 2-ordinals"; this category is not directly related to trees, but rather, gives another model of the configuration spaces of points on a disc. Recently D. Tamarkin has shown in `arXiv:math/0606553` how to prove the Deligne Conjecture in this language. The other existing approaches to Deligne Conjecture (for example in the papers by Voronov, McClure-Smith, Kontsevich-Soibelman, in fact also in the original paper by Getzler-Jones) are more indirect. What these authors do is the following: they construct a different and much larger DG operad which maps both onto the brace operad and onto the chain complex operad of small discs, and show that both maps are quasiisomorphisms. The construction usually involve doing some very intricate cellular subdivisions of the configuration spaces and a lot of combinatorics. My feeling is that the "final solution" of the Deligne Conjecture is not yet known.

Finally, some bibliographical notes. I have borrowed the formula for the brace operation from the paper `arXiv:math/9910126` of McClure and Smith, together with the signs. The brace operad also appeared there, or rather, a version of it slightly different from the one presented here (I note that the authors use "formulas" instead of planar trees, but these objects are in fact identical). Exactly the same complex as above appears in `arXiv:math/0001151` by Konstevich and Soibelman, and also in other places in the literature. So does the partially ordered set of planar trees. But our proof of the comparison theorem seems to be new. The usual approach is to take a certain cellular subdivision of the configuration space and quote the general theorem which says that if the subdivision is nice enough, then the geometric realization of the partially ordered set of cells in a space is homotopy equivalent to the space itself. An exact subdivision which corresponds to trees also appears in Kontsevich-Soibelman, but without proof. The other references that I know use different subdivisions which give different partially ordered sets, and then use combinatorics of varying degrees of difficulty to identify the result with trees.

Lecture 11.

Deformations of DG algebras and A_∞ algebras. Deformations in the Poisson and the Gerstenhaber case. Formality and deformations. Tamarkin's Theorem.

11.1 The language of A_∞-maps.

In this last lecture, I will try to sketch the proof of D. Tamarkin's theorem which I have already formulated as Theorem 9.8. I start with a discussion of associative DG algebras.

Assume given an associative unital DG algebra $A^\bullet$ over a field k. To define Hochschild cohomology $HH^\bullet(A^\bullet)$, one can naively write down the Hochschild cohomology complex, just as in the case of usual associative algebras, and obtain a bicomplex; Hochschild cohomology $HH^\bullet(A^\bullet)$ is the cohomology of the total complex of this bicomplex. We note that since the complex $\operatorname{Hom}(A^{\bullet \otimes n}, A^\bullet)$ for every $n \geq 1$ has terms both of positive and of negative degrees, there is an ambiguity in taking the total complex of a bicomplex: one can take either the sum, or the product of the diagonal terms. For the definition of Hochschild cohomology, one needs to take the product: the degree-n term of the resulting total complex is given by

$$\prod_{i \geq 0} \operatorname{Hom}^{n-i}(A^{\bullet \otimes i}, A^\bullet),$$

where $\operatorname{Hom}^n(-)$ stands for the term of degree n in the complex of Hom's. More invariantly, one can consider the category of DG modules over $A^\bullet$, and formally invert quasiisomorphisms. The result is a triangulated category $\mathcal{D}(A^\bullet\text{-}\mathsf{mod})$ known as *derived category of DG-modules over* $A^\bullet$. Analogously, one defines the triangulated category $\mathcal{D}(A^\bullet\text{-}\mathsf{bimod})$ of DG $A^\bullet$-bimodules. Then we have

$$HH^\bullet(A^\bullet) = \operatorname{RHom}^\bullet_{\mathcal{D}(A^\bullet\text{-}\mathsf{bimod})}(A^\bullet, A^\bullet),$$

where $A^\bullet$ in the right-hands side is the diagonal bimodule.

Recall now that for ordinary associative algebras, Hochschild cohomology could be also used to describe deformations. What is the situation with DG algebras? It turns out that a similar theory exists, but it describes deformations of DG algebras "up to a quasiisomorphism", as in Lecture 8.

To explain how this works, we first describe briefly a convenient technical tool — the notion of an *A_∞-map.* For a very good overview of this subject with detailed references, I refer the reader to a paper `arXiv:math/0510508` by B. Keller.

Assume given an associative DG algebra $A^\bullet$, with or without unit, and consider the free coalgebra $T_\bullet(A^\bullet)$ generated by $A^\bullet$. Then by Lemma 8.2, $T_\bullet(A^\bullet)$ has a natural structure of a bicomplex, with one differential induced by the differential in $A^\bullet$, and the other induced by multiplication. Its total complex is then a DG coalgebra with counit. For technical reasons, we need to remove the counit, and we denote the corresponding coalgebra by $\overline{T}_\bullet(A^\bullet)$. Explicitly,

$$\overline{T}_\bullet(A^\bullet) = \bigoplus_{i \geq 1} A^{\bullet \otimes i}[i] \tag{11.1}$$

as a graded vector space.

Exercise 11.1. *Prove that if $A^\bullet$ is a DG algebra with unit, then the complex $\overline{T}_\bullet(A^\bullet)$ is acyclic. Hint: show that $\overline{T}_\bullet(A^\bullet)$ is exactly the acyclic bar complex $C'_\bullet(A^\bullet)$ of Lecture 1 (Lemma* 1.3*).*

Exercise 11.2. *Assume that $A^\bullet$ itself is a free associative DG algebra without unit generated by a complex $V^\bullet$, $A^\bullet = \overline{T}^\bullet(V^\bullet) = \bigoplus_{i \geq 1} V^{\bullet \otimes i}[i]$. Prove that the natural map*

$$V^\bullet[1] \to \overline{T}^\bullet(V^\bullet)[1] = A^\bullet[1] \to \overline{T}_\bullet(A^\bullet)$$

is a quasiisomorphism. Hint: using the previous exercize, first show that the complex $T_\bullet(A^\bullet)$ computes $\operatorname{Tor}^{A^\bullet}_\bullet(k,k)$*, where k is the trivial left, resp. right $A^\bullet$-module.*

Definition 11.1. An A_∞*-map* between associative DG algebras $A_1^\bullet$, $A_2^\bullet$ is a DG coalgebra map $\varphi : \overline{T}_\bullet(A_1^\bullet) \to \overline{T}_\bullet(A_2^\bullet)$.

Since the coalgebra $\overline{T}_\bullet(A_2^\bullet)$ is free, an A_∞-map φ is completely defined by the induced map $\varphi : \overline{T}_\bullet(A_1^\bullet) \to A_2^\bullet$, and this can be decomposed as

$$\varphi = \varphi_0 + \varphi_1 + \dots + \varphi_i + \dots$$

according to (11.1). Here φ_0 is simply a map of complexes $\varphi_0 : A_1^\bullet \to A_2^\bullet$. If all the components φ_i, $i \geq 1$ are equal to zero, then $\varphi_0 : A_1^\bullet \to A_2^\bullet$ is just a map which commutes with multiplication — that is, a DG algebra map in the usual sense. In general, however, φ_0 commutes with multiplication only up to a homotopy, and this homotopy is $\varphi_1 : A_1^{\bullet\otimes 2} \to A_2^\bullet[-1]$. This in turn commutes with multiplication in an appropriate sense, but only up to a homotopy given by φ_2, and so on.

Definition 11.2. An A_∞-map φ is *a quasiisomorphism* if so is its component φ_0.

Of course, a quasiisomorphism $\varphi : A_1^\bullet \to A_2^\bullet$ between two DG algebras is also an A_∞-quasiisomorphism. However, while it is often not invertible in any sense as a DG algebra map, the resulting A_∞-map admits an inverse, in the following sense.

Lemma 11.3. *Assume given an A_∞-quasiisomorphism φ from a DG algebra $A_1^\bullet$ to a DG algebra $A_2^\bullet$. Then there exists an A_∞-quasiisomorphism φ^{-1} from $A_2^\bullet$ to $A_1^\bullet$ such that both $\varphi \circ \varphi^{-1}$ and $\varphi^{-1} \circ \varphi$ induce identity maps on cohomology.*

Proof. Since φ is a quasiisomorphism, there exists a map $\varphi_0^{-1} : A_2^\bullet \to A_1^\bullet$ of the underlying complexes which induces an iverse map on cohomology. We extend it to an A_∞-map by induction. Namely, for any DG algebra $A^\bullet$, denote by $\overline{T}_{<i}(A^\bullet) \subset \overline{T}_\bullet(A^\bullet)$ the subcoalgebra consisting of components $A^{\bullet\otimes j}[j]$ with $j \leq i$, and assume given a DG coalgebra map $\varphi_{<i}^{-1} : \overline{T}_{<i}(A_2^\bullet) \to \overline{T}_{<i}(A_1^\bullet)$ which induces a map on cohomology inverse to that induced by φ. Extend $\varphi_{<i}^{-1}$ to a DG coalgebra map $\varphi_{<i}^{-1} : \overline{T}_{<i+1}(A_2^\bullet) \to \overline{T}_{<i+1}(A_1^\bullet)$. Then this extended map $\varphi_{<i}^{-1}$ no longer necessarily commutes with the differential. However, the commutator is a certain map

$$e : A_2^{\bullet\otimes(i+1)} \to A_1^\bullet[-i+1],$$

and using the inductive assumption, one easily checks that e induces a zero map on cohomology. Therefore it is chain-homotopic to 0 by a certain chain homotopy $\varphi_i : A_2^{\bullet\otimes(i+1)} \to A_1^\bullet[-i]$. We now take $\varphi_{<i+1} = \varphi_{<i} + \varphi_i$. □

Proposition 11.4. *Two DG algebras $A_1^\bullet$, $A_2^\bullet$ are quasiisomorphic if and only if there exists an A_∞-quasiisomorphism $\varphi : \overline{T}_\bullet(A_1^\bullet) \to \overline{T}_\bullet(A^2_\bullet)$.*

Proof. Assume that such a φ exists. Then Lemma 8.2 has an obvious dual statement for coalgebras, so that for any DG coalgebra $B^\bullet$, we have a DG algebra $\overline{T}^\bullet(B^\bullet)$ which is free as an algebra. Applying this to DG coalgebras $\overline{T}_{<i}(A^\bullet)$, $i \geq 1$ corresponding to a DG algebra $A^\bullet$, we obtain a DG algebra

$$\widetilde{T}(A^\bullet) = \lim_{\to i} \overline{T}^\bullet(\overline{T}_{<i}(A^\bullet)).$$

Then φ obviously induces a quasiisomorphism $\widetilde{T}(A_1^\bullet) \to \widetilde{T}(A_2^\bullet)$, so that it suffices to prove that $\widetilde{T}(A^\bullet)$ is quasiisomorphic to $A^\bullet$ for any $A^\bullet$. To construct a DG algebra map $\tau : \widetilde{T}(A^\bullet) \to A^\bullet$, we use induction on i and construct a compatible system of DG algebra maps

$$\tau_i : \overline{T}^\bullet(\overline{T}_{<i}(A^\bullet)) \to A^\bullet.$$

Since the left-hand side is a free algebra, at each stage a map is competely defined by its restriction to the generator $\overline{T}_{<i}(A^\bullet)$. When $i = 1$, we have $\overline{T}_{<i}(A^\bullet) = A^\bullet$; as τ_1, we take the map which is identical on generator. Once the map τ_i is constructed to some i, we first extend it to $\overline{T}_{<i+1}(A^\bullet)$ as a linear map commuting with the differential, in any way we like, and then extend further to a DG algebra map $\tau_{i+1} : \overline{T}^\bullet(\overline{T}_{<i+1}(A^\bullet)) \to A^\bullet$ by multiplicativity. Passing to the limit, we obtain a DG algebra map $\tau : \widetilde{T}(A^\bullet) \to A^\bullet$.

To show that τ is a quasiisomorphism, we consider the increasing filtration $F_\bullet\widetilde{T}(A^\bullet)$ induced by the filtration $F_i\overline{T}_\bullet(A^\bullet) = \overline{T}_{<i}(A^\bullet)$ on the generating graded vector space $\overline{T}_\bullet(A^\bullet)$. It is easy to check that this filtration is compatible with the differential, and it suffices to prove that the induced map

$$\mathsf{gr}^\bullet_F \widetilde{T}(A^\bullet) \cong \overline{T}^\bullet(\mathsf{gr}^\bullet_F \tau_\bullet(A^\bullet)) \to A^\bullet$$

is a quasiisomorphism. But the left-hand side is the DG algebra $\widetilde{T}(\overline{A}^\bullet)$, where $\overline{A}^\bullet$ is $A^\bullet$ with the trivial multiplication. Thus we may assume from the very beginning that the multiplication in $A^\bullet$ is trivial. In this case, the claim is an obvious dualization of Exercize 11.2.

Conversely, assume that $A_1^\bullet$ and $A_2^\bullet$ are quasiisomorphic, that is, there exists a chain $A_1^\bullet \leftarrow A_{1,1}^\bullet \to A_{1,2}^\bullet \leftarrow \cdots \to A_{1,n}^\bullet = A_2^\bullet$ of DG algebras and quasiisomorphisms between them. Then by induction, we may assume that the chain is of length 2, so that we either have a DG quasiisomorphism $\eta : A_1^\bullet \to A_2^\bullet$, or $\eta : A_1^\bullet \to A_2^\bullet$. In the first case, φ is induced by η, and in the second case, we take $\varphi = \eta^{-1}$ provided by Lemma 11.3. □

This Proposition considerable simplifies controlling quasiisomorphism classes of DG algebras. In particular, it allows to describe deformations.

Definition 11.5. Assume given a commutative Artin local k-algebra S with maximal ideal $\mathfrak{m}$, $S/\mathfrak{m} \cong k$. An *S-deformation* $\widetilde{A}^\bullet$ of an associative DG algebra $A^\bullet$ is a DG algebra $\widetilde{A}^\bullet$ which is flat over S and equipped with an isomorphism $\widetilde{A}^\bullet \otimes_S k \cong A^\bullet$. Two such deformations $\widetilde{A}_1^\bullet$, $\widetilde{A}_2^\bullet$ are *equivalent* if there exists an S-linear A_∞-quasiisomorphism $\varphi : \overline{T}_\bullet(\widetilde{A}_1^\bullet) \to \overline{T}_\bullet(\widetilde{A}_2^\bullet)$.

Definition 11.6. The *reduced Hochschild cohomology complex* $\overline{DT}^\bullet(A^\bullet)$ of a DG algebra $A^\bullet$ is the DG Lie algebra of derivations of the DG coalgebra without unit $\overline{T}_\bullet(A^\bullet)$. *Reduced Hochschild cohomology groups* $\overline{HH}^\bullet(A^\bullet)$ are the cohomology groups of the complex $\overline{DT}^\bullet(A^\bullet)$.

Exercise 11.3. *Assume given an associative DG algebra $A^\bullet$. Prove that for any S as in Definition* 11.5*, the set of equivalence classes of S-deformations of $A^\bullet$ is in natural one-to-one correspodence with the set of equivalence classes of $\mathfrak{m}$-valued solutions of the Maurer-Cartan equation* (8.8) *in the reduced Hochschild cohomology complex $\overline{DT}^\bullet(A^\bullet)$. Hint: repeat literally the corresponding statement for associative algebras presented in Lecture 8.*

The reason we have to use DG coalgebras without unit in the definition of an A_∞-map is clear from Lemma 11.3 — otherwise, an A_∞-map φ would also have a component φ_{-1}, and the recursive procedure would fail. Because of this, the relevant deformation theory is controlled by the reduced Hochschild cohomology $\overline{DT}^\bullet(A^\bullet)$, not by the full Hochschild cohomology complex $DT^\bullet(A^\bullet)$. The difference between them is the constant term: we have a natural exact triangle

$$\overline{DT}^\bullet(A) \longrightarrow DT^\bullet(A) \longrightarrow A^\bullet \longrightarrow .$$

We note that strictly speaking, we had to consider the reduced Hochschild cohomology even in the deformation theory of the usual associative algebras. However, there it made no difference: if $A^\bullet = A$ is concentrated in degree 0, we have $HH^i(A) = \overline{HH}^i(A)$ for any $i \geq 2$, and the spaces of the Maurer-Cartan solutions are also isomorphic. For a DG algebra $A^\bullet$ which has non-trivial terms in positive degrees, they might be different.

In the interests of full disclosure, let me mention that in some situations, one can also consider A_∞-maps which have non-trivial (-1)-component; these correspond, roughly speaking, to functors between categories of DG modules which are not induced by a map of DG algebra. One can also develop a deformation theory which is controlled by the full Hochschild cohomology complex $DT^{\bullet}(A^{\bullet})$; this is "the deformation theory of the category of DG modules", in some appropriate sense. Deformations of the category of DG modules which do not come from deformation of a DG algebra do exist, and they are sometimes known as "deformations in the gerby direction". However, this lies outside of the scope of the present course.

11.2 Poisson cohomology.

What we really need to study for Theorem 9.8 is Gerstenhaber algebras, not associative ones; thus we need to extend the above formalism to the Gerstenhaber case. For simplicity, we start with the Poisson case. The reference here is, for instance, the Appendix to my joint paper with V. Ginzburg `arXiv:math/0212279`.

Assume given a vector space V. The free Poisson coalgebra $P_{\bullet}(V)$ generated by V is the associated graded quotient of the free associative coalgebra $T_{\bullet}(V)$ with respect to the Poincaré-Birkhoff-Witt filtration. It turns out that an analog of Lemma 8.2 holds in the Poisson situation, too.

Lemma 11.7. *Poisson algebra structures on V are in one-to-one correspondence with coderivations $\delta : P_{\bullet}(V) \to P_{\bullet-1}(V)$ of degree 1 such that $\{\delta, \delta\} = 0$.*

Sketch of a proof. By definition, we have $P_2(V) = \mathsf{gr}_{PBW} V^{\otimes 2} = S^2(V) \oplus \Lambda^2(V)$, the sum of the symmetric and the exterior square of the vector space V. Thus a coderivation δ consists of two components, $\delta_0 : S^2(V) \to V$ and $\delta_1 : \Lambda^2(V) \to V$. The component δ_0 defines the multiplication, and δ_1 defines the Poisson bracket. The commutator $\{\delta, \delta\}$ has three components, $\{\delta_0, \delta_0\}$, $\{\delta_1, \delta_1\}$ and $2\{\delta_1, \delta_0\}$; their vanishing means, respectively, that the multiplication is associative, the bracket satisfies the Jacobi identity, and that the bracket satisfies the Leibnitz rule with respect to the multiplication. The proof is a direct computation which I leave as an exercize (or see the quoted paper `arXiv:math/0212279`). □

Thus given a Poisson algebra A, we have a canonical differential on the free Poisson coalgebra $P_{\bullet}(A)$, and we can consider the DG Lie algebra $DP^{\bullet}(A)$ of all coderivations of $P_{\bullet}(A)$.

Definition 11.8. *Poisson cohomology $HP^{\bullet}(A)$* of the Poisson algebra A is the cohomology of the complex $DP^{\bullet}(A)$.

As in Lecture 8, we can also consider the DG Lie algebra $DT^{\bullet}(A)$ of coderivations of the tensor coalgebra $T_{\bullet}(A)$, and this is nothing but the Hochschild cohomology complex of the algebra A. The PBW filtration on $T_{\bullet}(A)$ induces a filtration on $DT^{\bullet}(A)$, and we have $\mathsf{gr}^{\bullet}_{PBW} DT^{\bullet}(A) \cong DP^{\bullet}(A)$. The component $DL^{\bullet}(A) = \mathsf{gr}^0_{PBW} DT^{\bullet}(A)$ is particularly important; it depends only on the multiplcation in A, and it coincides with the DG Lie algebra of coderivation of the free Lie coalgebra $L^{\bullet}(A)$ generated by A. This is known as the *tangent complex* of the commutative algebra A, and it computes the so-called *Harrison cohomology* of A. We note that the differential in $DL^{\bullet}(A)$ is A-linear, so that it is a DG Lie algebra of A-modules (in fact, free A-modules). As such, it is quasiisomorphic to the complex

$$\mathrm{RHom}^{\bullet}_A(\Omega_{\bullet}(A), A),$$

where $\Omega_{\bullet}(A)$ is the *cotangent complex* of A first constructed by L. Illusie. The whole Lie algebra $DP^{\bullet}(A)$ also has the structure of an A-module, and coincides with the total complex of the bicomplex

$$\mathsf{gr}^{\bullet}_{PBW} DT^{\bullet}(A) \cong \Lambda^{\bullet}_{A} DL^{\bullet}(A).$$

One differential in this bicomplex is induced by the differential in $DL^{\bullet}(A)$, thus by multiplication in A — explicitly, the multiplication gives a class $\mu \in DL^1(A)$, and the differential is given by $\alpha \mapsto \{\mu, \alpha\}$. The other differential in the bicomplex comes from the Poisson bracket in A — the bracket gives a class

$$\Theta \in \mathrm{Hom}(\Lambda^2 A, A) \subset \Lambda^2_A(DL^0(A)), \tag{11.2}$$

and the differential is given by $\alpha \mapsto \{\Theta, \alpha\}$.

In general, it is very diffucult to compute $DP^{\bullet}(A)$ and the Harrison complex $DL^{\bullet}(A)$. However, the situation becomes much simpler when the algebra A is smooth — that is, in the situaton of the Hochschild-Kostant-Rosenberg Theorem. In this case, the cotangent complex $\Omega_{\bullet}(A)$ reduces to the module $\Omega(A)$ of Kähler differentials of A/k, and this module is flat. Therefore $DL^{\bullet}(A)$ has non-trivial cohomology only in degree 1, and this cohomology is canonically identified with the module $\mathcal{T}(A)$ of derivations of the algebra A (that is, vector fields on $X = \operatorname{Spec} A$). The higher quotients $\mathsf{gr}^{\bullet}_{PBW} DT^{\bullet}(A)$ are then isomorphic to modules of polyvector fields on X, so that the PBW filtration is in fact split — $\mathsf{gr}^{\bullet}_{PBW} DT^{\bullet}(A)$ is quasiisomorphic to the same space of polyvector fields $H^0(X, \Lambda^{\bullet}\mathcal{T}_X)$ as the full Hochschild cohomology complex $DT^{\bullet}(A)$. Under this identification, the class Θ from (11.2) corresponds to the Poisson bivector $\Theta \in H^0(X, \Lambda^2\mathcal{T}_X)$. To sum up:

Proposition 11.9. *Assume given a smooth Poisson algebra A of finite type over a characteristic-0 field k. Then the Poisson cohomology complex $DP^{\bullet}(A)$ is quasiisomorphic to the complex with terms*

$$H^0(X, \Lambda^{\bullet}\mathcal{T}_X)$$

and with differential given by $a \mapsto [\Theta, a]$, where $\Theta \in H^0(X, \Lambda^2\mathcal{T}_X)$ is the Poisson bivector. □

I will not prove this Proposition. Let me just mention that it is rather easy to reduce the statement to the case when $A = S^{\bullet}(V)$ is the symmetric algebra generated by a k-vector space V — in other words, a polynomial algebra — and then the crucial fact is the quasiisomorphism $L^{\bullet}(\overline{S}_{\bullet}(V)) \cong V$, analogous to the quasiisomorphism of Exercize 11.2 (here $\overline{S}_{\bullet}(-)$ means the free commutative coalgebra without unit).

One way to establish this quasiisomorphism uses a more careful analysis of the Hochschild-Kostant-Rosenberg map of Lecture 2, which shows how it interacts with the symmetric group actions on the terms $A^{\otimes n}$ of the Hochschild complex; the reader can find such a proof, for instance, in Loday's book.

Another and slightly more conceptual proof uses the notion of "Koszul duality of operads" introduced in Ginzburg-Kapranov `arXiv:0709.1228`. One of the statements there is that the Lie and the commutative operad are "Koszul dual", and this includes, as a part of the package, canonical quasiisomorphisms $L^{\bullet}(\overline{S}^{\bullet}(V)) \cong V$ and $\overline{S}^{\bullet}(L_{\bullet}(V)) \cong V$. The second quasiisomorphism is semi-obvious, since the left-hand side $\overline{S}^{\bullet}(L_{\bullet}(V))$, with the degree-0 term $S^0(L_{\bullet}(V))$ added, is nothing but the standard Chevalley complex which computes Lie algebra homology $H_{\bullet}(L^{\bullet}(V), k)$. Then the first quasiisomorphism, which we actually need, can be deduced by the general formalism of Koszul duality. I refer the reader to `arXiv:0709.1228` for further details.

Assuming Proposition 11.9, we see that for a smooth algebra A — in particular, for a polynomial algebra $S^{\bullet}(V)$ — the Poisson cohomology can be computed by the very explicit complex whose terms are polyvector fields. This complex was first discovered by J.-L. Brylinski in the early 80es, so that it is sometimes called the *Brylinski complex.* But when the Poisson bivector Θ is non-degenerate, so that the smooth Poisson variety $X = \operatorname{Spec} A$ is actually symplectic, the Poisson cohomology becomes even simpler.

Exercise 11.4. *Prove that for any smooth Poisson variety X, the map $\Omega^1_X \to \mathcal{T}_X$ given by contraction with the Poisson bivector Θ extends to a multiplicative map*

$$\Omega^\bullet_X \to \Lambda^\bullet \mathcal{T}_X$$

from the de Rham complex of X to the Brylinski complex $\langle \Lambda^\bullet \mathcal{T}_X, [-, \Theta] \rangle$.

Applying this in the affine symplectic case $X = \operatorname{Spec} A$, we see that $\Omega^1_X \to \mathcal{T}_X$ is actually an isomorphism, so that the Brylinski complex is quasiisomorphic to the de Rham complex, and the Poisson cohomology $HP^\bullet(A)$ is isomorphic to the de Rham cohomology $H^\bullet_{DR}(X)$ (in particular, it does not depend on the symplectic/Poisson structure at all). When $A = S^\bullet(V)$ is the polynomial algebra generated by a symplectic vector space V, with the Poisson structure induced by the symplectic form on V, we have

$$HP^i(A) = H^i_{DR}(X) = \begin{cases} k, & i = 0, \\ 0, & i \geq 1, \end{cases}$$

where $X = \operatorname{Spec} A$ is the affine space.

The Gerstenhaber case works in exactly the same way, except that we now have to care of the gradings, and use reduced cohomology.

Definition 11.10. The *Gerstenhaber cohomology complex $DG^\bullet(A^\bullet)$*, resp. the *reduced Gerstenhabe cohomology complex $\overline{DG}^\bullet(A^\bullet)$* of a Gerstenhaber DG algebra $A^\bullet$ is the DG Lie algebra of coderivations of the free Gerstenhaber coalgebra with, resp. without unit generated by $A^\bullet[1]$.

There is also a version of the A_∞-formalism for DG Gerstenhaber algebra, and the classification theorem for deformations of DG Gerstenhaber algebras up to a quasiisomorphism; this is completely parallel to the associative case and left to the reader. The end result is that deformations "up to a quasiisomorphims" of a DG Gerstenhaber algebra $A^\bullet$ are controlled by the DG Lie algebra $\overline{DG}^\bullet(A^\bullet)$.

Exercise 11.5. *Let $A^\bullet = S^\bullet(V^\bullet)$ be graded polynomial algebra generated by a graded vector space V, with the Gerstenhaber structure induced by a non-degenerate graded sympletic form $\Lambda^2(V^\bullet) \to k[-1]$. Show that the reduced Gerstenhaber cohomology complex $\overline{DG}^\bullet(A^\bullet)$ is quasiisomorphic to the quotient $A^\bullet/k$, where $k \to A^\bullet$ is the unit map $\lambda \mapsto \lambda \cdot 1$.*

11.3 Tamarkin's Theorem.

We can now explain how to prove Tamarkin's Theorem, ar rather, the following version of it.

Proposition 11.11. *Let $A^\bullet = S^\bullet(V)$ be the polynomial algebra generated by a vector space V, and assume given a DG Gerstenhaber algebra $B^\bullet$ whose cohomology is isomorphic to the Hochschild cohomology Gerstenhaber algebra $HH^\bullet(A)$. Aswsume in addition that $B^\bullet$ admits an action of the group $GL(V)$ such that the isomorphism $H^\bullet(B^\bullet) \cong HH^\bullet(A)$ is $GL(V)$-equivariant. Then the DG Gerstenhaber algebra $B^\bullet$ is formal, that is, quasiisomorphic to $HH^\bullet(A)$.*

Proof. Consider the canonical filtration $F_\bullet B^\bullet$ on the Gerstenhaber algebra $B^\bullet$. Then we have a canonical quasiisomorphism $\operatorname{gr}^F_\bullet B^\bullet \cong HH^\bullet(A)$, and this quasiisomorphism, being canonical, is compatible with the Gerstenhaber algebra structure and with the $GL(V)$-action. There is a standard way to interpret the associated graded quotient $\operatorname{gr}^F_\bullet B^\bullet$ as a special fiber of a certain deformation of the algebra $B^\bullet$ (known as "the deformation to the normal cone"). Namely, consider the Rees algebra

$$\widetilde{B}^\bullet = \bigoplus_i F_i B^\bullet$$

defined by the canonical filtration. This is also a Gerstenhaber algebra which has an additional grading by i. Moreover, the embeddings $F_iB^\bullet \subset F_{i+1}B^\bullet$ give a certain endomorphism of $\widetilde{B}^\bullet$ of degree 1 which we denote by h. Then $\widetilde{B}^\bullet$ is a graded Gerstenhaber algebra over $S = k[h]$. Its generic fiber $\widetilde{B}^\bullet \otimes_S k[h, h^{-1}]$ is isomorphic to $B^\bullet \otimes k[h, h^{-1}]$, while its special fiber $\widetilde{B}^\bullet/h$ is isomorphic to $\mathsf{gr}^\bullet_F B^\bullet$. Thus we have a $GL(V)$-equivariant S-deformation of the Gerstenhaber algebra $\mathsf{gr}^\bullet_F B^\bullet \cong HH^\bullet(A)$, and we have to show that this deformation is trivial up to a quasiisomorphism. But by the Hochschild-Kostant-Rosenberg Theorem, we have $HH^\bullet(A) = S^\bullet(V \oplus V^*[-1])$, and it is easy to check that the Gerstenhaber structure is induced by the natural pairing $V \otimes (V^*[-1]) \to k[-1])$ (it suffices to check this on the generators $V \oplus V^*[-1]$, and this is a trivial exercize). Applying Exercize 11.5, we conclude that

$$\overline{DG}^\bullet(HH^\bullet(A)) \cong HH^\bullet(A)/k.$$

In the right-hand side, the $GL(V)$-invariant part is trivial in degrees ≥ 2, so that *every* $GL(V)$-equivariant deformation of the DG Gerstenhaber algebra $HH^\bullet(A)$ is trivial up to a quasiisomorphism. □

As we have noted already in Lecture 9, this reduces Kontsevich Formality Theorem to Theorem 9.7, the formality of the chain operad of little discs (and the Deligne Conjecture). Indeed, once these both are established, we know that the Hochschild cohomology complex $DT^\bullet(A^\bullet)$ is a DG Gerstenhaber algebra. It is obviously $GL(V)$-equivariant, thus formal by Proposition 11.11.