Monoidal categories and WDVV equation

Nikita Markarian

Witten–**D**ijkgraaf–**V**erlinde–**V**erlinde equation

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\delta \partial t^\mu} = \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\delta \partial t^\mu}$$

$$F_n: \ A^{\otimes n} \to k$$

Hypercommutative operad

Symmetric products

$$(\cdot, \cdots, \cdot): \ A^{\otimes n} \longrightarrow A$$

$$\sum_{S_1 \coprod S_2} \pm((a,b,x_{S_1}),c,x_{S_2}) = \sum_{S_1 \coprod S_2} \pm(a,(b,c,x_{S_1}),x_{S_2})$$

$$(a,b)_t = \sum \frac{1}{k!}(a,b,\underbrace{t,\ldots,t}_{k})$$

+ metric, Euler field (Dubrovin)

geometry	category	cohomology
X (CY or LG or ...)	?	Frobenius manifold

Supermanifolds

Examples:

1. a usual manifold X;

2. Grassman algebra $\Lambda^* V$ — function ring of the supermanifold of dimension $(0, \dim V)$;

3. for a vector bundle E over a usual manifold X the function ring is $\Gamma(\Lambda^* E)$.

Function ring $\mathcal{O}$ is graded and equipped with a (super)derivation Q of degree 1, $Q^2 = 0$.

4. Differential forms $\Omega^*(X)$, Q is deRham differential;

5. for a complex manifold X the function ring is $\Omega^{0,*}(X)$, Q is $\overline{\partial}$.

Batalin–Vilkovisky supermanifold
is the one equipped with a differential operator of the second order Δ without free term and of degree -1 such that

$$\Delta^2 = 0$$

and $[\Delta, Q]$ is a derivation.

Bracket of degree -1

$$[a,b] = \Delta(ab) - (\Delta a)b - a(\Delta b)$$

Main example: X is CY: $\omega_X = \mathcal{O}$.

$\Gamma(\Lambda^* T \otimes \Omega^{0,*})$, $\overline{\partial}$ $\qquad$ $\deg T = 1$

$\Lambda^i T = \Omega^{\dim X - i}$

$\Delta = \partial : \ \Omega^{i,j} \to \Omega^{i+1,j}$ $\qquad$ $\Delta = \frac{\partial}{\partial x_i}\frac{\partial}{\partial p_i}$

$[,]$ is Schouten–Nijenhuys bracket on polyvector fields

Kodaira–Spencer Theory of Gravity
M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, hep-th/9309140.

Suppose that $\Delta = [Q, r]$ up to derivations.

$$\Downarrow$$

$[,]$ is homotopic to zero.

On $H^*(\mathcal{O})$ there is a solution of WDVV equation (trace is needed)

Recipe (for the main example):
First order deformations are given by

$$\overline{\partial}' = \overline{\partial} + tA_1 + \ldots$$
$$A_1 \in ker(\overline{\partial} : T \otimes \Omega^{0,1} \to T \otimes \Omega^{0,2})$$

Maurer–Cartan equation:

$$\overline{\partial}A + 1/2[A, A] = 0$$
$$A \in T \otimes \Omega^{0,1} (\text{or } \Lambda^i T \otimes \Omega^{0,j})$$

$$F = \int_{cycle} \omega$$

Dictionary Lie — commutative
(Quillen duality, Koszul duality, ...)

Lie algebras	Complex manifolds
$\Lambda^*\mathfrak{g}^\vee$, $d\colon \mathfrak{g}^\vee \to \mathfrak{g}^\vee \wedge \mathfrak{g}^\vee$.	$\Omega^{0,i}(X)$, $\overline{\partial}$.
Free module $E \otimes \Lambda^*\mathfrak{g}^\vee$.	Analytic vector bundle E over X.
Action of $\mathfrak{g}$ on E $E \to E \otimes \mathfrak{g}^\vee$	Holomorphic structure on E $E \to E \otimes \Omega^{0,1}$

defines structure of dg–module.

Getzler (Zuckerman) hep-th/9309057:
If $\mathfrak{g}$ is *bialgebra*

$$\Delta : \mathfrak{g}^\vee \wedge \mathfrak{g}^\vee \to \mathfrak{g}^\vee$$

then $\Lambda^*\mathfrak{g}^\vee$ is BV.

$$\boxed{\Delta \text{ deforms tensor structure}}$$

For semisimple algebras $\Delta = [d, r]$.

Main example: X is a Calaby–Yau, function ring is $(\Gamma(\Lambda^* T \otimes \Omega^{0,*}),\ \overline{\partial})$.

Sheaves of locally free DG-modules over $(\Gamma(\Lambda^* T \otimes \Omega^{0,*}),\ \overline{\partial})$	$\to$	Complexes of sheaves on TX finite over X
E	$\mapsto$	$(E \otimes_{\Lambda^* T} K) \otimes \omega[-\dim]$, $K = (\Lambda^* T \otimes S^* T^\vee, \delta)$
Tensor product	$\to$	Convolution along fibers

$\Delta = \partial$ on $\Omega^* = \Lambda^* T$
is cohomologicaly trivial by Hodge theory

Bracket $[,]$ is Schouten–Nijenhuys bracket

Deforms tensor product to convolution on $X \times X$:

$$A * B = p_{14*}(p_{12}^* A \otimes p_{34}^* B), \text{ where}$$
$$p_{ij} :\ X \times X \times X \times X \to X \times X \text{ are projections}$$

Lie algebras	Complex manifolds
$\Lambda^* \mathfrak{g}^\vee$	$\Omega^{0,i}(TX)$
$d\colon \mathfrak{g}^\vee \to \mathfrak{g}^\vee \wedge \mathfrak{g}^\vee$	$\overline{\partial}$
$\Delta : \mathfrak{g}^\vee \wedge \mathfrak{g}^\vee \to \mathfrak{g}^\vee$	$\Delta = \partial$
Category of reps of g, $\otimes$	Subcat. of $\mathbf{D}^\flat(TX)$ gen. by $\mathcal{O}_0$, $*$
$\downarrow$	$\downarrow$

deforms to

Category of reps of quantum group, $\otimes$	Subcat. of $\mathbf{D}^\flat(X \times X)$ gen. by $\mathcal{O}_\Delta$, $*$
$\Delta = [d, r]$	$\partial = [\overline{\partial}, r]$
$\Downarrow$	$\Downarrow$
$A \otimes B \simeq B \otimes A$	$A * B \simeq B * A$

Operad

is a sequence $K(i)$ of topological spaces, vector spaces or complexes of vector spaces with the symmetric group action.

Equivariant maps (*compositions*)

$$K(n) \times K(i_1) \times \cdots \times K(i_n) \to K(i_1 + \cdots + i_n)$$

are given, obeying compatibility conditions.

Operad K acts on V (or V is an algebra over K):

$$K(n) \times \underbrace{V \times \cdots \times V}_{n} \to V,$$

compatible with compositions and the symmetric group action.

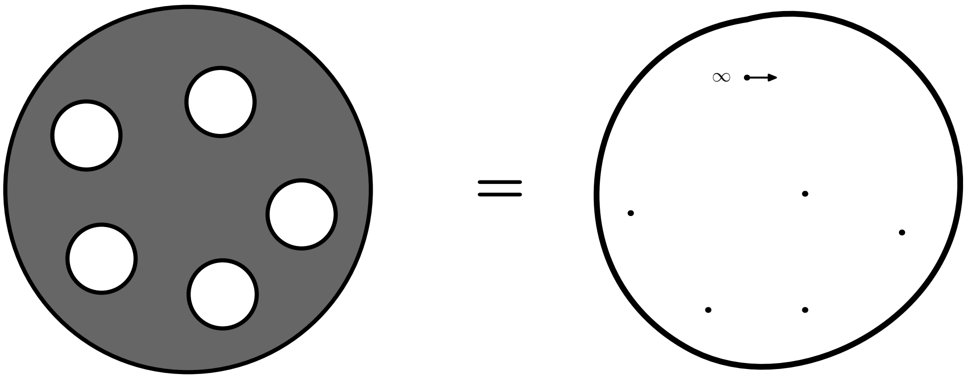
Algebra over a *cyclic* operad is the one equipped with a scalar product such that maps

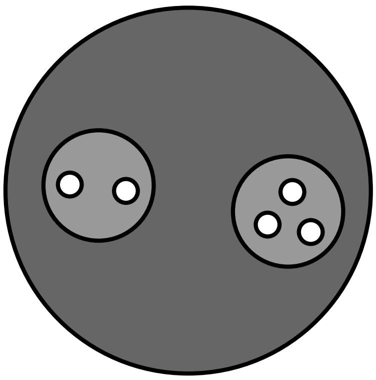
$$K(n) \times \underbrace{V \times \cdots \times V}_{n+1} \to \mathbb{k}$$

are cyclic invariant.

$\mathcal{F}_2$ — little discs operad

$\mathcal{F}_2(5)$





Braid operad

$$\mathbf{Br} = H_\bullet(\mathcal{F}_2)$$

Algebra over $\mathbf{Br}$ = commutative algebra with Lie bracket $[,]$ of degree -1 obeying

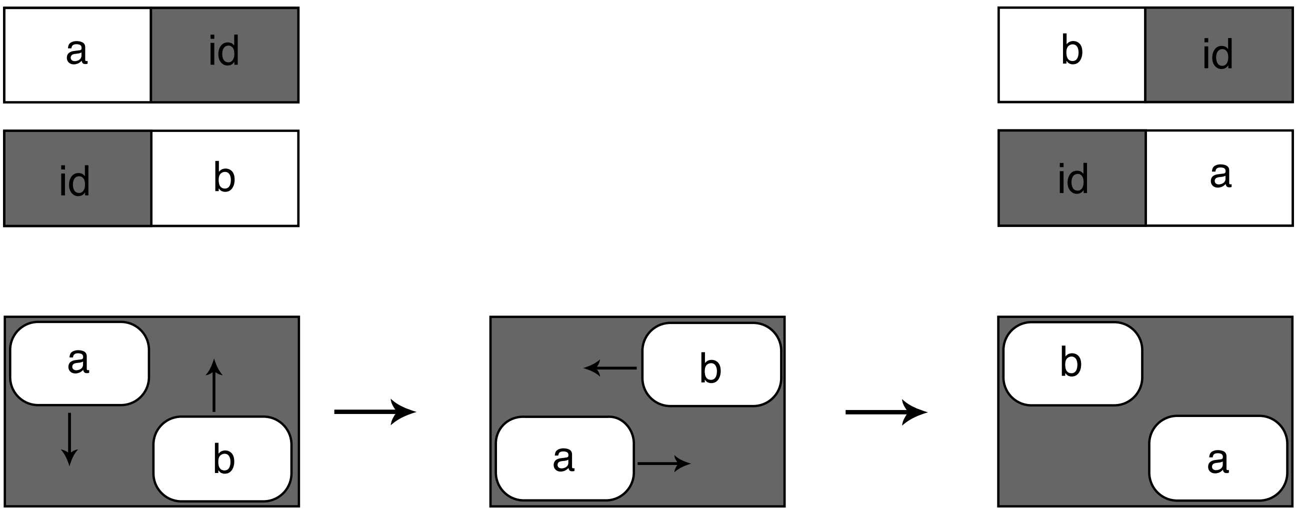
$$[a, bc] = [a, b]c \pm b[a, c]$$

Monoidal category $\mathcal{C}$
DG or enriched over simplicial sets
with homotopical unit $\mathbf{1}$

$* : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ associative up to homotopy

On $Ext^*(\mathbf{1}, \mathbf{1})$ $\mathbf{Br}$ acts:

$$a \circ b = (a \circ \text{id}) * (\text{id} \circ b) = (b \circ \text{id}) * (\text{id} \circ a) = b \circ a$$



$$A(t), B(t) \quad A(0) = B(0) = \mathbf{1}$$
$$\partial A/\partial t = a \quad \partial B/\partial t = b$$
$$\partial^2(ABA^{-1}B^{-1})/\partial t \partial s = [a, b] \in H^1(\mathbf{1}, \mathbf{1})$$

Quasi–symmetric monoidal category
is a monoidal one with a homotopy Σ between functors isomorphic to

$$* : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \text{ and } *^{\sigma} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

given, such that $\Sigma \circ \Sigma^{\sigma}$ is isomorphic to Id.
(More generally , functor $*$ is homotopically $\mathbb{Z}_2$-invariant.)

Fact:
$[,]$ on $Ext^*(\mathbf{1},\mathbf{1})$ for such a category vanishes.

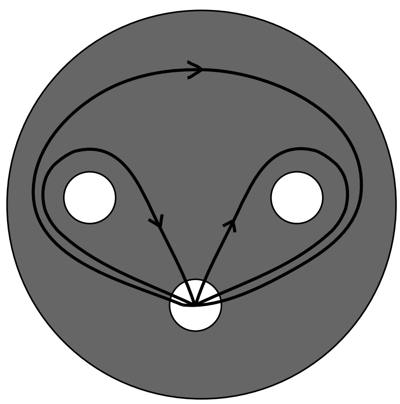
$$\Downarrow$$

On $Ext^*(\mathbf{1},\mathbf{1})$ operad

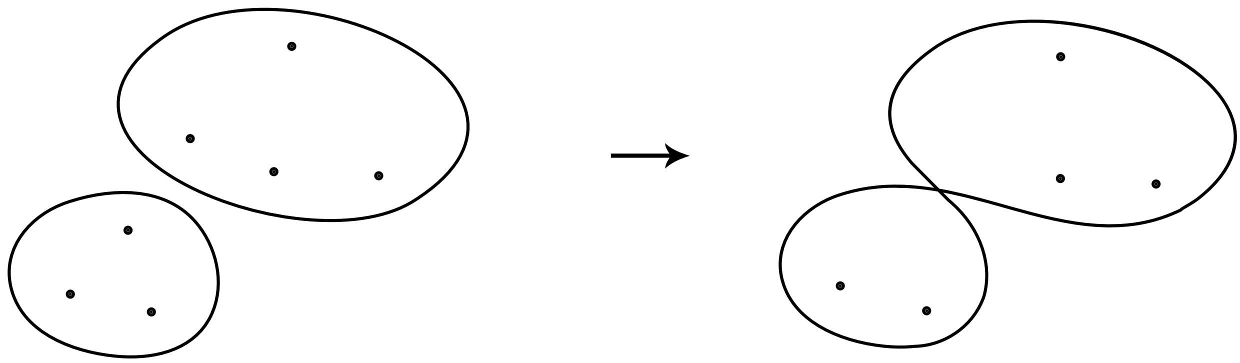
$$(\text{Free resolution of } \mathbf{Br}, \eta)/(d\eta = [,])$$

acts.

Relations in $\mathbf{Br}$ gives new operations in $\mathbf{Br}/[,]$. Relation $[a, bc] = [a, b]c \pm b[a, c]$:



Operad of stable punctured curves of genus 0 $\overline{\mathcal{M}_0}$



Hypercommutative operad

$$\mathbf{Hycomm} = H_\bullet(\overline{\mathcal{M}_0})$$

is generated by $[\overline{\mathcal{M}_{0,n}}]$.

Relations are:

$$\sum_{S_1 \coprod S_2} \pm((a, b, x_{S_1}), c, x_{S_2}) = \sum_{S_1 \coprod S_2} \pm(a, (b, c, x_{S_1}), x_{S_2})$$

(Dijkgraaf–Verlinde–Verlinde, Ginzburg–Kapranov, Getzler, Kontsevich–Manin)

Homotopical digression

Commutative H-space X is the one such that the multiplication map

$$* : X \times X \to X$$

is homotopically $\mathbb{Z}_2$–invariant.

For connected spaces:

X is H-space	$\Longleftrightarrow$	$X = \Omega Y$ (Stasheff, May, ...)
$\mathcal{F}_2$ acts on X	$\Longleftrightarrow$	$X = \Omega\Omega Y$ (Boardman–Vogt, ...)

Conjecture:

$$\overline{\mathcal{M}_0} \text{ acts on } X \iff X = \Omega(\text{comm. H-space})$$

Landau–Ginzburg model

Pair ($X = \mathbb{C}^n$, function W).

Kontsevich: consider $\mathbb{Z}_2$–graded "complexes" with $d^2 = W$. (Matrix factorisation — Eisenbud, 1980.)

Triangulated category $DB_0(W)$.

$Ext^*(\mathbf{1}, \mathbf{1})$ for its endofunctors should be

$$\mathbb{C}[z_1, \dots, z_n]/(dW) = (\Lambda^* T, dW \vdash)$$

Orlov: $DB_0(W)$ equals to the **category of singularities**

$$\mathbf{D}^\flat(X_0)/\begin{pmatrix}\text{subcategory generated} \\ \text{by perfect complexes}\end{pmatrix}$$

Landau–Ginzburg category $\mathbf{D}_{LG}(W)$
is $DB_0(p_1^* W - p_2^* W)$ on $X \times X$,
where $p_i : X \times X \to X$ are projections.

That is sheaves on $X \times X$ annihilated by $p_1^* W - p_2^* W$ modulo the subcategory generated by a universal object.

We will consider only sheaves with support on the diagonal.

Conjecture: for an isolated singularity given by equation $W = 0$, Landau–Ginzburg category is quasi–symmetric monoidal category.

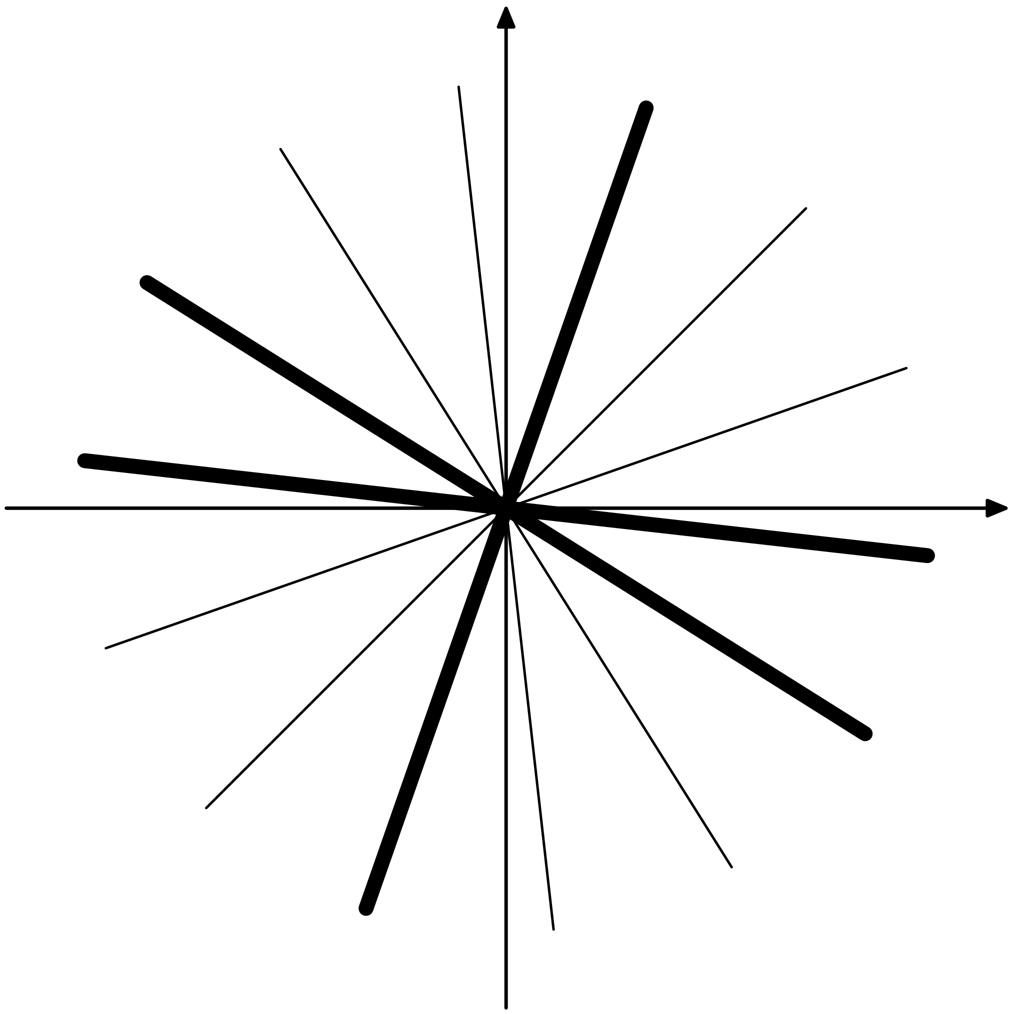
Statement: for an isolated singularity

$$Ext^*_{\mathbf{D}_{LG}(W)}(\mathbf{1}, \mathbf{1}) = \mathbb{C}[z_1, \ldots, z_n]/(dW)$$

Also should works for a non-commutative algebra and an element W: consider bimodules annihilated by $W \otimes 1 - 1 \otimes W$...

Example: $X = \mathbb{C}$, $W = z^n$

$$x^n - y^n = 0$$



An indecomposable object