Topology of locally conformally Kähler manifolds

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LCK manifolds

DEFINITION: A **locally conformally Kähler (LCK)** manifold is a complex Hermitian manifold, with a Hermitian form ω satisfying $d\omega = \omega \wedge \eta$, where η is a closed, non-exact 1-form, called **the Lee form** of M.

REMARK: A compact LCK manifold never admits a Kähler structure.

REMARK: Suppose M is a compact complex manifold. Existence of a Kähler structure gives all kinds of constrains on topology of M (even-dimensionality if $H^{odd}(M)$, strong Lefschetz, homotopy formality).

QUESTION: What can we say about topology of M, if M is LCK?

REMARK: LCK manifolds are not necessarily homotopy formal. A Heisenberg manifold (also known as Kodaira surface) is not homotopy formal (it has non-vanishing Massey products), but it is LCK.

Conjectures about LCK manifolds

REMARK: Izu Vaisman conjectured that **for a compact LCK manifold, $h^1(M)$ odd** (disproven by Oeljeklaus and Toma, 2005)

REMARK: Izu Vaisman also conjectured that no compact LCK manifold can be homotopy equivalent to a Kähler manifold **(still unknown)**.

OBSERVATION: All complex surfaces admitting LCK structures are known, except Kodaira class VII surfaces. For Kodaira class VII with $b_2 = 0$, LCK structures are known to exist on two types of Inoue surfaces (Tricerri), and do not exist on the third type (Belgun). For $b_2 > 0$, all Kodaira class VII surfaces are conjectured to admit a spherical shell (Ma. Kato, I. Nakamura). The known examples minimal Kodaira class VII surfaces are either hyperbolic or parabolic Inoue surfaces. LCK structures on hyperbolic Inoue surfacse were recently constructed by A. Fujiki and M. Pontecorvo (2009).

CONJECTURE: (L. Ornea) **Let M be a compact surface of class VII admitting a spherical shell. Is it LCK?**

LCS manifolds

DEFINITION: A **locally conformally symplectic (LCS)** manifold is a smooth manifold, with a non-degenerate 2-form ω satisfying $d\omega = \omega \wedge \eta$, where η is a closed 1-form, called **the Lee form** of (M, ω).

REMARK: LCK manifolds are obviously LCS.

REMARK: Just like with symplectic and Kähler structures, **existence of an LCS structure gives obvious topological constrains on an LCK manifold**.

REMARK: Just like with symplectic and Kähler structures, **these constrains are quite weak**. There are many symplectic manifolds not admitting any Kähler structure. An LCS structure exists on many nilmanifolds and solvmanifolds, some of which (apparently) do not admit an LCK structure.

Examples LCS manifolds which are not LCK

REMARK: F. Belgun (2000) has proved that an Inoue surface of class $S^+_{n;p,q,u}$, where $u \in \mathbb{C}\backslash\mathbb{R}$ **cannot admit an LCK metric.**

REMARK: F. Tricerri (1982) has constructed **an LCK metric on other Inoue surfaces,** which are diffeomorphic to Belgun's $S^+_{n;p,q,u}$.

CLAIM: An Inoue surface of class S^+ is always admits an LCS structure.

REMARK: This is clear, because the Belgun's $S^+_{n;p,q,u}$ is diffeomorphic to Tricerri's Inoue surface.

DEFINITION: A **nilmanifold** is a quotient of a nilpotent Lie group G by a discrete, cocompact subgroup $\Gamma \subset G$

CONJECTURE: If a nilmanifold admits an LCK structure **then it is a Heisenberg group manifold** (quotient of a Heisenberg group by a cocompact lattice).

REMARK: This is known for homogeneous LCK structures (H. Sawai, 2007).

Morse-Novikov class of an LCK manifold

DEFINITION: Let (M, ω, θ) be an LCK manifold, and

$$d_\theta := d - \theta : \Lambda^i(M) \longrightarrow \Lambda^{i+1}(M)$$

the "Morse-Novikov" differential on differential forms. Its cohomology $H^i_\theta(M)$ are called **the Morse-Novikov cohomology** of M.

DEFINITION: Let (M, ω, θ) be an LCK manifold, and L a trivial line bundle, with flat connection defined as $\nabla := \nabla_0 + \theta$, where ∇_0 is the trivial connection. Then L is called **the weight bundle** of M.

REMARK: The cohomology of the local system (L, ∇) is naturally identified with $H^i_\theta(M)$.

DEFINITION: Clearly, $d_\theta\omega = 0$. Its cohomology class $[\omega] \in H^2_\theta(M)$ is called **the Morse-Novikov class of M**.

REMARK: The Morse-Novikov class is a natural analogue of a Kähler class, and should be studied if we want to understand the topology.

Vaisman manifolds

DEFINITION: An LCK manifold (M, ω, θ) is called **Vaisman** if $\nabla_{LC}\theta = 0$, where ∇_{LC} is the Levi-Civita connection.

THEOREM: (Kamishima-Ornea, 2001) If M is compact, **this is equivalent to M admitting a conformal holomorphic flow,** acting non-isometrically on its Kähler covering.

THEOREM: (Ornea-V., 2006) A compact complex manifold admits Vaisman metric if and only if M **admits a holomorphic embedding into a diagonal Hopf manifold**.

DEFINITION: A **conical Kähler manifold** is a Kähler manifold (C, ω) equipped with a free, proper holomorpic flow $\rho: \mathbb{R} \times C \longrightarrow C$, with ρ acting by homotheties as follows: $\rho(t)^*\omega = e^t\omega$.

THEOREM: (Ornea-V., 2003) A compact Vaisman manifold is conformally equivalent to **a quotient of a conical Kähler manifold by $\mathbb{Z}$ freely acting on (C, ω) by non-isometric homotheties.**

Sasakian manifolds

DEFINITION: A Sasakian manifold is the space of orbits of ρ on a conical Kähler manifold (C, ω, ρ).

STRUCTURE THEOREM: (Ornea-V., 2003) Any compact Vaisman manifold M **admits a smooth Riemannian submersion $\sigma : M \longrightarrow S^1$, with Sasakian fibers.** The weight bundle L is obtained as a pullback from S^1: $L = \sigma^*(L_0)$.

THEOREM: For a Vaisman manifold, $H^i_\theta(M) = 0$.

PROOF: Immediately follows from the Structure Theorem. Let S be the Sasakian fibers of σ. By the Künneth formula,

$$H^*_\theta(M) = H^*(M, L) \cong H^*(S) \otimes H^*(S^1, L_0),$$

and $H^*(S^1, L_0) = 0$.

Kähler potential

DEFINITION: Let (M, I, ω) be a Kähler manifold. Denote by d^c the differential $d^c := -IdI$. A **Kähler potential** is a function satisfying $dd^c\psi = \omega$. Locally, a Kähler potential always exists, and it is unique up to adding real parts of holomorphic functions.

OBSERVATION: Let (M, ω, θ) be a Vaisman manifold, and $(\tilde{M}, \tilde{\omega}) \xrightarrow{\pi} M$ be its Kähler covering, with $\Gamma \cong \mathbb{Z}$ the deck transform group: $M = \tilde{M}/\Gamma$. Then $\pi^*\theta$ is exact on $\tilde{M}$: $\pi^*\theta = d\nu$. Moreover, **the function $\psi := e^{-\nu}$ is a Kähler potential:** $dd^c\psi = \tilde{\omega}$.

OBSERVATION: Let $\gamma \in \Gamma$ be any element. Since Γ preserves θ, we have $\gamma^*\nu = \nu + c_\gamma$, where c_γ is a constant. Then $\gamma^*\psi = e^{-c_\gamma}\psi$ **(automorphic property).**

Automorpic potential

DEFINITION: Let (M,ω,θ) be an LCK manifold, $(\tilde M,\tilde\omega)$ its Kähler covering, Γ the deck transform group, $M = \tilde M/\Gamma$. and $\psi \in C^\infty \tilde M$ a Kähler potential, $\psi > 0$. Assume that for any $\gamma \in \Gamma$, $\gamma^*\psi = c_\gamma \psi$, for some constant c_γ. Then ψ is called **an automorphic potential** of M.

DEFINITION: Let (M,ω,θ) be an LCK manifold, and L its weight bundle. Since L is a local system, its holonomy defines a map $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$. Its image Γ is called **the monodromy group of M**.

REMARK: Let Γ be the smallest quotient of $\pi_1(M)$ such that the corresponding covering $\tilde M$ admits a Kähler metric in the same conformal class as the pullback of ω. Such a metric $\tilde M$ is unique up to constant. We call $(\tilde M,\tilde\omega)$ **the Kähler covering** of M.

PROPOSITION: (Ornea-V., 2007) Let (M,ω,θ) be an LCK manifold with an automorphic potential. **Then there exists another LCK metric on M with automorphic potential and monodromy $\mathbb{Z}$.**

Stein manifolds

DEFINITION: A complex variety M is called **holomorphically convex** if for any infinite discrete subset $S \subset M$, there exists a holomorphic function $f \in \mathcal{O}_M$ which is unbounded on S.

DEFINITION: A complex variety is called **Stein** if it is holomorphically convex and has no compact complex subvarieties.

REMARK: Equivalently, a complex variety is Stein if it admits a closed holomorphic embedding into $\mathbb{C}^n$.

THEOREM: (K. Oka, 1942) **A complex manifold M is Stein** if and only M admits a Kähler metric with a **Kähler potential which is positive and proper** (proper = preimages of compact sets are compact).

THEOREM: (Rossi 1965, Andreotti-Siu 1970) Let M be a complex manifold with a boundary, $\dim_{\mathbb{C}} M > 2$, and φ a proper Kähler potential on M, taking values in $[c, \infty[$, and equal to c in the boundary of M. **Then there exists a Stein variety M_c** with isolated singularities, containing M, and it is unique.

Manifolds with LCK potential

DEFINITION: Let (M,ω,θ) be an LCK manifold with automorphic potential ψ. Assume that its monodromy is $\mathbb{Z}$. Then ψ is called **an LCK potential**.

THEOREM: Let (M,ω,θ) be an LCK manifold with LCK potential, $\dim_{\mathbb{C}} M > 2$, and $\tilde M$ is its Kähler covering. Then $\tilde M$ **can be compactified by adding a single point to its origin, and the resulting variety is Stein.** Moreover, the monodromy Γ acts on $\tilde M$ by holomorphic automorphisms.

PROOF: Follows from Rossi-Andreotti-Siu theorem (we glue in the hole left by excising the set of points where $\psi \leqslant c$).

COROLLARY: An LCK manifold with LCK potential admits a holomorphic embedding into a Hopf manifold.

PROOF: A holomorphic embedding into a Hopf manifold is the same as an automorphic embedding into $\mathbb{C}^n$. Using the Stein property, we find a suitable space $V \subset \mathcal{O}_{\tilde M}$ preserved by Γ. This gives a map $\tilde M/\Gamma \longrightarrow (V\backslash 0)/\Gamma$.

REMARK: Converse is also true: any complex subvariety of a Hopf manifold admits an LCK potential.

Kodaira-type stability of LCK potential

THEOREM: Let (M, I) be a complex manifold admitting an LCK metric with LCK potential. Then **any small deformation of the complex structure I also admits an LCK metric with LCK potential.**

PROOF: Let I_1 be a small deformation of I, $\tilde{M}$ the Kähler covering of M, and $\psi \in C^\infty \tilde{M}$ its potential. Then $dd^c\psi = -dI_1 d\psi$ on $(\tilde{M}, I_1)$ is a small deformation of $dd^c\psi = -dId\psi$. For I_1 close to I, the eigenvalues of $-dI_1 d\psi$ are close to the eigenvalues of $-dId\psi$, hence positive. Therefore, **ψ is an LCK potential on (M, I_1).**

REMARK: Deformations of Vaisman manifolds can be non-Vaisman. Deformations of LCK can be non-LCK (F. Belgun, 2000).

REMARK: It is easy to distinguish Vaisman manifolds from other LCK with potential. Vaisman manifolds are embeddable into diagonal Hopf manifolds, and ones which are non-Vaisman are embeddable into Hopf manifolds which are not diagonal.

Deformations of LCK manifolds with potential

THEOREM: Let (M, ω, θ) be an LCK manifold with potential. Then a small deformation of M admits a Vaisman metric.

PROOF: Step 1. Let $V = \mathbb{C}^n$, $A \in \operatorname{End}(V)$ an invertible linear operator with all eigenvalues $|\alpha_i| < 1$, and $H = (V\backslash 0)/\langle A\rangle$ the corresponding Hopf manifold. **The complex submanifolds of H are identified with complex subvarieties $Z \subset V$, smooth outside of 0 and fixed by A.**

Step 2. We are going to prove that Z **is fixed by the group $G_A := e^{t \log A}$, $t \in \mathbb{R}$, acting on V.** An ideal $I_Z \subset \mathcal{O}_V$ is finitely generated, because $\mathcal{O}_V$ is coherent. Let $\hat{I}_Z$ be the corresponding ideal in the completion of $\mathcal{O}_V$ in 0. To prove that I_Z is fixed by G_A, it suffices to show that $\hat{I}_Z$ is fixed by G_A. However,

$$\hat{I}_Z = \varprojlim I_Z / \left(I_Z \cap \mathfrak{m}^k\right)$$

where $\mathfrak{m}$ is the maximal ideal of 0.

Deformations of Vaisman manifolds (part 2)

To prove that $\hat{I}_Z$ is fixed by G_A it remains to show only that $I_Z/I_Z \cap \mathfrak{m}^k$ is fixed by G_A. However, $I_Z/I_Z \cap \mathfrak{m}^k$ is a plane in a finite-dimensional space

$$\mathcal{O}_V/\mathfrak{m}^k = \mathbb{C} \oplus V \oplus \operatorname{Sym}^2 V \oplus ... \oplus \operatorname{Sym}^{k-1} V$$

and such a plane, if fixed by A, is automatically fixed by G_A.

Step 3. For any linear operator, there is a unique decomposition $A := SU$ onto a product of commuting operators, with S semisimple (diagonal), and U unipotent (this is called **the Jordan-Chevalley decomposition).** For any finite-dimensional representation of $GL(n)$, any vector which is fixed by A, is also fixed by S. By the argument in Step 2, this proves that S **fixes the ideal** $\hat{I}_Z$**, and the subvariety** $Z \subset V$**.**

Step 4. The diagonal Hopf variety $H_S := (V\backslash 0)/\langle S\rangle$ contains a Vaisman submanifold $M_1 := (Z\backslash 0)/\langle S\rangle$. Since S is contained in a closure of a $GL(V)$-orbit of A, we have shown also that M_1 **can be obtained as an arbitrary small deformation of** M**.**

Topology of Vaisman manifolds

OBSERVATION: Any Vaisman manifold is diffeomorphic to a fiber bundle over a circle, with fiber a Sasakian manifold.

OBSERVATION: Any Sasakian manifold is diffeomorphic to a unit circle bundle in a positive line bundle over a compact projective orbifold.

REMARK: This result allows one to obtain information about topology of Vaisman manifolds and LCK manifolds with automorphic potential directly from results about Sasakian manifolds and projective orbifolds.

CONJECTURE: ("$d_\theta d_\theta^c$-lemma for LCK manifolds") **An automorphic potential exists for any LCK manfold with vanishing Morse-Novikov class.**

REMARK: From this conjecture we immediately obtain many topological results about LCK manifolds. In particular, it follows immediately that any nilmanifold with vanishing Morse-Novikov class is diffeomorphic to a Heisenberg group nilmanifold (the argument is essentially the same as the proof of Sawai's theorem)

Harmonic maps in LCK geometry

DEFINITION: Let (M, ω, θ) be an LCK manifold. Then M is called **pluri-canonical** if $(\nabla_{LC}\theta)^{1,1} = 0$, where $()^{1,1}$ denotes the I-invariant part of the tensor.

THEOREM: (Kokarev-Kotschick, 2008) Let M be a compact pluricanonical LCK manifold, such that $\pi_1(M)$ admits a surjective homomorphism to a non-abelian free group. **Then M admits a surjective holomorphic map with connected fibers to a compact Riemannian surface.**

REMARK: For a compact Kähler manifold, this result is known (Siu, Beauville).

REMARK: This theorem is proven by the same harmonic maps argument as in Kähler case. One takes a continuous map from M into a negatively curved Kähler manifold X (e.g. a complex curve), and **shows existence of a harmonic map from M to X in the same homotopy class.** A local argument is used to show that **harmonic implies holomorphic.**

Using this approach, many constrains on topology of pluricanonical LCK manifolds were obtained.

Kokarev's pluricanonical manifolds

DEFINITION: Let (M, ω, θ) be an LCK manifold, $\tilde{M}$ its Kähler covering, and $\tilde{\nabla}$ its Levi-Civita connection. Then $\tilde{\nabla}$ is lifted from a connection ∇_W on M, called **the Weyl connection of M**.

OBSERVATION: The Weyl connection can be expressed through the Levi-Civita connection on M. This gives $\nabla(\theta) - \nabla_W(\theta) = \theta \wedge \theta - g$. Then, the pluricanonical LCK condition $(\nabla\theta)^{1,1} = 0$ is translated into

$$\nabla_W(\theta)^{1,1} = g - (\theta \otimes \theta)^{1,1}.$$

Since ∇_W is torsion-free, this is equivalent to $d\theta^c = \omega - \theta \wedge \theta^c$, where $\theta^c := I(\theta)$.

CLAIM: The condition $d\theta^c = \omega - \theta \wedge \theta^c$ is equivalent to the existence of an automorphic potential.

PROOF: Let $\psi := e^{-\nu}$, where $d\nu = \theta$. Then

$$dd^c\psi = -e^{-\nu}dd^c\nu + e^{-\nu}d\nu \wedge d^c\nu = e^{-\nu}(d^c\theta + \theta \wedge \theta^c) = \psi\omega$$

hence **pluricanonical condition implies that ψ is an automorpic potential.** The converse is true by the same argument.

Harmonic map theory for LCK manifold with potential

From the above argument, a corollary is obtained

COROLLARY: Any pluricanonical LCK manifold is diffeomorphic to a Vaisman manifold.

This allows one to use results on topology of Sasakian manifolds to rediscover results of Kokarev and Kottschik.

The main question (which inspired Kokarev and Kottschik) remains just as hard to solve.

QUESTION: (Vaisman) **Is there any compact LCK manifold homotopy equivalent to a compact Kähler manifold?**