Hypercomplex manifolds with holonomy $SL(n,\mathbb{H})$

Misha Verbitsky

Workshop "Special Geometries in Mathematical Physics",

April 3, 2008, Kühlungsborn

Hypercomplex manifolds

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

Compact hypercomplex manifolds (examples)

0. **Hyperkähler manifolds.**

1. **In dimension 1 (real dimension 4), we have a complete classification,** due to C. P. Boyer (1988)

2. **Many homogeneous examples**, due to D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen (1980-ies, early 1990-ies).

3. **Some nilmanifolds** admit homogeneous hypercomplex structure (M. L. Barberis, I. Dotti, A. Fino) (1990-ies).

4. **Some inhomogeneous examples** are constructed by deformation (G. Grantcharov, H. Pedersen, Y.-S. Poon) or as fiber bundles (V.).

OBATA CONNECTION

Hypercomplex manifolds can be characterized in terms of holonomy

Theorem: (M. Obata, 1955) Let (M,I,J,K) be a hypercomplex manifold. Then M admits a unique torsion-free affine connection preserving I,J,K.

Converse is also true. Suppose that I,J,K are operators defining quaternionic structure on TM, and ∇ a torsion-free, affine connection preserving I, J, K. Then I, J, K are integrable almost complex structures, and (M,I,J,K) is hypercomplex.

Holonomy of Obata connection lies in $GL(n,\mathbb{H})$. *A manifold equipped with an affine, torsion-free connection with holonomy in $GL(n,\mathbb{H})$ is hypercomplex*.

This can be used as a definition of a hypercomplex structure.

Geometry of hypercomplex manifolds

QUESTIONS

1. Given a complex manifold M, **when M admits a hypercomplex structure?** How many?

2. **What are the possible holonomies** of Obata connection, for a compact hypercomplex manifold? What are the special properties of reducible holonomies in this case? Irreducicble holonomies are $GL(n, \mathbb{H})$, $SL(n, \mathbb{H})$ and $Sp(n)$ only (Merkulov, Schwachhöfer).

3. **Describe the structure of automorphism group** of a hypercomplex manifold.

Partial answer to Question 1 is known.

THEOREM: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that the complex manifold (M, I) admits a Kähler structure. Then (M, I) is hyperkähler (V., 2004).

Quaternionic Hermitian structures

DEFINITION: Let (M, I, J, K) be a hypercomplex manifold, and g a Riemannian metric. We say that g is **quaternionic Hermitian** if I, J, K are orthogonal with respect to g.

Given a quaternionic Hermitian metric g on (M, I, J, K), consider its Hermitian forms

$$\omega_I(\cdot,\cdot) = g(\cdot, I\cdot), \quad \omega_J = g(\cdot, J\cdot), \quad \omega_K = g(\cdot, K\cdot)$$

(real, but *not closed*). Then $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is of Hodge type (2,0) with respect to I.

If $d\Omega = 0$, (M, I, J, K, g) **is hyperkähler** (this is one of the definitions).

Consider a weaker condition:

$$\partial\Omega = 0, \quad \partial: \Lambda^{2,0}(M,I) \longrightarrow \Lambda^{3,0}(M,I)$$

DEFINITION: (Howe, Papadopoulos, 1998)
Let (M,I,J,K) be a hypercomplex manifold, g a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ the corresponding $(2,0)$-form. We say that g is **HKT ("weakly hyperkähler with torsion")** if $\partial\Omega = 0$.

HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.

1. **They admit a smooth potential (locally).** There is a notion of an "HKT-class" (similar to Kähler class) in a certain finite-dimensional cohomology group. Two metrics in the same HKT-class differ by a potential, which is a function.

2. When (M,I) has trivial canonical bundle, **a version of Hodge theory is established, giving an $\mathfrak{sl}(2)$-action on cohomology.**

Canonical bundle of a hypercomplex manifold.

0. Quaternionic Hermitian structure always exists.

1. **Complex dimension is even.**

2. **The canonical line bundle $\Lambda^{n,0}(M,I)$ of (M,I) is always trivial topologically.** Indeed, a non-degenerate section of canonical line bundle is provided by top power of a form Ω associated with some quaternionic Hermitian strucure. In particular, $c_1(M,I)=0$.

3. Canonical bundle **is non-trivial holomorphically** in many cases. However, $\Lambda^{n,0}(M,I)$ is trivial and holonomy of Obata connection lies in $SL(n,\mathbb{H})$ **when M is a nilmanifold** (Barberis-Dotti-V., 2007)

4. If $\mathcal{H}\mathrm{ol}(M)$ lies in $SL(n,\mathbb{H})$, canonical bundle is trivial. The converse is true when M is compact and HKT (V., 2004): **an HKT manifold with holomorphically trivial canonical bunlde has $\mathcal{H}\mathrm{ol}(M)\subset SL(n,\mathbb{H})$**.

$SU(2)$-action on $\Lambda^*(M)$

The group $SU(2)$ of unitary quaternions acts on TM, because quaternion algebra acts. By multilinearity, this action is extended to $\Lambda^*(M)$.

1. The Hodge decomposition $\Lambda^*(M) = \bigoplus \Lambda^{p,q}(M)$ is recovered from this $SU(2)$-action. **"Hypercomplex analogue of the Hodge decomposition".**

2. $\langle \omega_I, \omega_J, \omega_K \rangle$ is an irreducible 3-dimensional representation of $SU(2)$, for any quaternionic Hermitian structure ("representation of weight 2").

WEIGHT of a representation.
We say that an irreducible $SU(2)$-representation W **has weight** i if $\dim W = i + 1$. A representation is said to be **pure of weight** i if all its irreducible components have weight i. If all irreducible components of a representation W_1 have weight $\leqslant i$, we say that W_1 **is a representation of weight** $\leqslant i$. In a similar fashion one defines representations of weight $\geqslant i$.

Quaternionic Dolbeault algebra

The weight is multiplicative, in the following sense: a tensor product of representations of weights $\leqslant i$ and $\leqslant j$ has weight $\leqslant i+j$.

Clearly, $\Lambda^1(M)$ has weight 1. Therefore, **$\Lambda^i(M)$ has weight $\leqslant i$.**

Let $V^i \subset \Lambda^i(M)$ be the maximal $SU(2)$-invariant subspace of weight $< i$.

By multiplicativity, **$V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$.** We also have $V^i = \Lambda^i(M)$ for $i > 2n$. Also, $dV^i \subset V^{i+1}$, hence $V^* \subset \Lambda^*(M)$ is a differential ideal in $(\Lambda^i(M), d)$.

Denote by $(\Lambda^*_+(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$.
We call it **the quaternionic Dolbeault algebra (qD-algebra)** of M.

A similar construction was given by Salamon in a more general situation.

The Hodge decomposition of quaternionic Dolbeault algebra.

The Hodge decomposition is induced from the $SU(2)$-action, hence it is **compatible with weights**: $\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M)$.

Let $\sqrt{-1}\,\mathcal{I}$ be an element of the Lie algebra $\mathfrak{su}(2)\otimes\mathbb{C}$ acting as $\sqrt{-1}\,(p-q)$ on $\Lambda^{p,q}(M)$. This vector generates the Cartan algebra of $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$-action induces an isomorphism of $\Lambda^{p,q}_{+,I}(M)$ for all $\{p,q \mid p+q=k,\ p,q\geqslant 0\}$. This gives

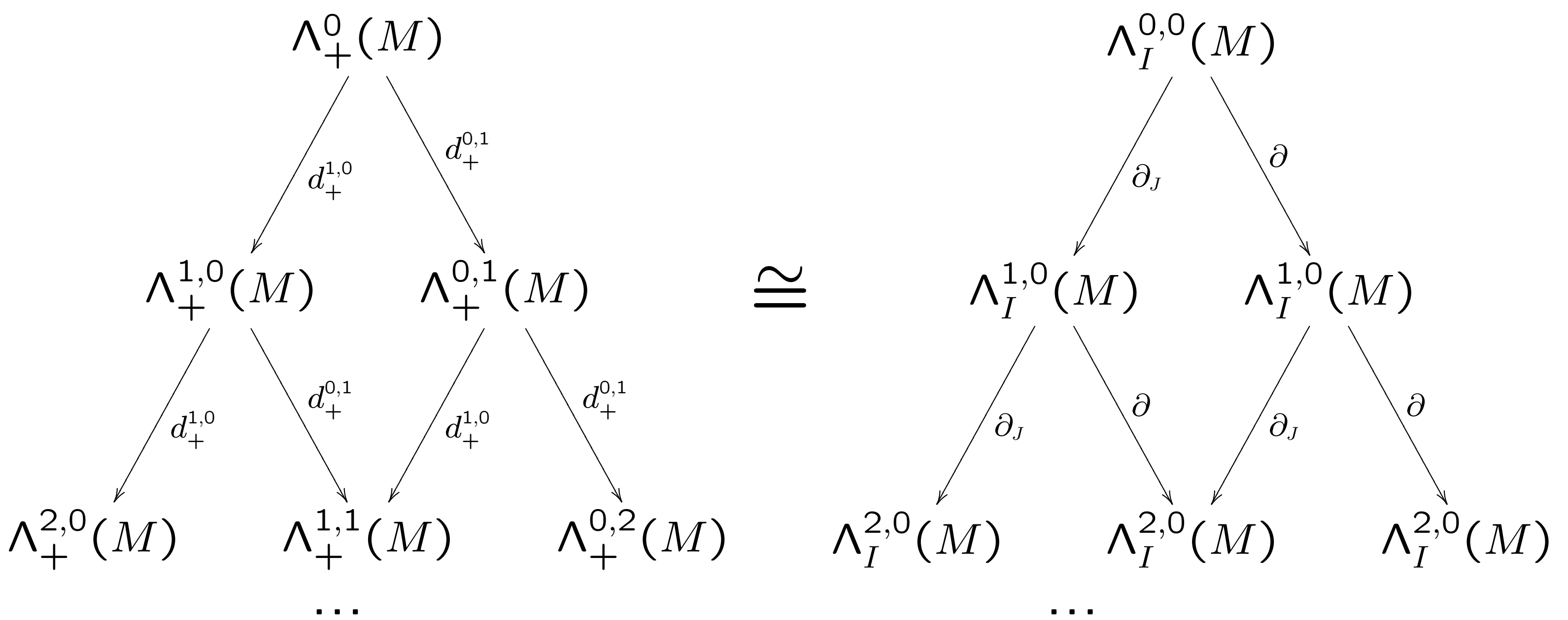
Theorem: $\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{0,p+q}(M,I)$.

This isomorphism is provided by the $\mathfrak{su}(2)\otimes\mathbb{C}$-action.

Differentials in the qD-complex

We extend $J: \Lambda^1(M) \longrightarrow \Lambda^1(M)$ to $\Lambda^*(M)$ by multiplicativity. Since I and J anticommute on $\Lambda^1(M)$, **we have** $J(\Lambda^{p,q}(M, I)) = \Lambda^{q,p}(M, I)$.

Denote by $\partial_J : \Lambda^{p,0}(M, I) \longrightarrow \Lambda^{p,0}(M, I)$ **the operator** $J \circ \overline{\partial} \circ J$, where $\overline{\partial} : \Lambda^{0,p}(M, I) \longrightarrow \Lambda^{0,p}(M, I)$ is the standard Dolbeault differential. Then ∂, ∂_J anticommute. Moreover, **there exists a multiplicative isomorphism of bicomplexes**.



Potentials for HKT-metrics

A quaternionic Hermitian metric **can be recovered from the corresponding (2,0)-form:** $\omega_I(x, \overline{y}) = \frac{1}{2}\Omega(x, J(\overline{y}))$, where $x, y \in T^{1,0}(M)$. The HKT-structures uniquely correspond to (2,0)-forms which are

1. Real: $J(\Omega) = \overline{\Omega}$

2. Closed: $\partial\Omega = 0$.

2. Positive: $\Omega(x, J(\overline{x})) > 0$, for any non-zero $x \in T^{1,0}(M)$

Locally, any HKT-metric is given by a potential: $\Omega = \partial\partial_J\varphi$ where φ is a smooth function.

Any convex, and any strictly plurisubharmonic function is a potential of some HKT-structure. Therefore, HKT-structures locally always exist (Grantcharov, Poon).

HKT-manifolds with holonomy in $SL(n,\mathbb{H})$

Let M be a compact HKT-manifold with holonomy in $SL(n,\mathbb{H})$, and $\Delta_{\overline{\partial}} := \overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial}$ be the **antiholomorphic Laplacian** with

$$\ker \Delta_{\overline{\partial}}\Big|_{\Lambda^{0,*}(M)} = H^*(M, \mathcal{O}_{(M,I)}).$$

Theorem: $\Delta_{\overline{\partial}}$ commutes with the multiplication by the HKT-form $\overline{\Omega}$, and with the operator $\eta \longrightarrow J(\overline{\eta})$. In particular, **there is a Lefschetz-like $\mathfrak{sl}(2)$-action on $H^*(M, \mathcal{O}_{(M,I)})$.**

Theorem ("$\partial\partial_J$-lemma") Let Ω, Ω' be HKT-forms on a compact HKT-manifold with holonomy in $SL(n,\mathbb{H})$. Assume that the cohomology classes of $\overline{\Omega}, \overline{\Omega}'$ in $H^*(M, \mathcal{O}_{(M,I)})$ are equal. Then $\Omega - \Omega' = \partial\partial_J\varphi$ for some smooth function φ on M.

Example: Any hypercomplex nilmanifold has holonomy in $SL(n,\mathbb{H})$ (Barberis, Dotti, V.).

Quaternionic Monge-Ampere equation

Let M be an HKT-manifold with holonomy in $SL(n, \mathbb{H})$. (this is equivalent to having trivial canonical bundle). Then the canonical bundle is trivialized by a form $\Phi_I \in \Lambda^{2n,0}$, non-degenerate, closed and satisfying $J(\Phi_I) = \overline{\Phi}_I$.

Quaternionic Monge-Ampere equation:

$$(\Omega + \partial\partial_J\varphi)^n = A_f e^f \Phi_I \quad (*)$$

where $\Omega + \partial\partial_J\varphi$ is an HKT-form. Here φ is unknown, and A_f is a number determined from

$$\int_M \Omega^n \wedge \overline{\Phi}_I = A_f \int_M e^f \Phi_I \wedge \overline{\Phi}_I$$

Theorem: (Alesker, V., 2008) The solution φ of (*) is unique, if exists. Moreover, any solution of (*) admits a C^0-estimation in terms of f, Φ_I, Ω.

Conjecture: ("hypercomplex Calabi-Yau")
The equation (*) has a solution for all f, Φ_I, Ω.

Uniqueness of solutions of Monge-Ampere equations

Suppose Ω_1, Ω_2 are HKT-forms which are solutions of M-A, $\Omega_1 - \Omega_2 = \partial\partial_J\varphi$. Then $\Omega_1^n - \Omega_2^n = 0$. This gives

$$0 = \Omega_1^n - \Omega_2^n = \partial\partial_J\varphi \wedge \sum_{i=0}^{n-1} \Omega_1^i \wedge \Omega_2^{n-1-i}.$$

Denote by P the form $\sum_{i=0}^{n-1} \Omega_1^i \wedge \Omega_2^{n-1-i}$ and consider the differential operator $D: C^\infty(M) \longrightarrow C^\infty(M)$

$$\varphi \longrightarrow \frac{\partial\partial_J\varphi \wedge P}{\Omega^n}.$$

Then D is a second order operator with positive symbol.

Solutions of $D(f) = 0$ cannot have local maxima ("generalized maximum principle"). Since M is compact, **all solutions of $D(f) = 0$ are constant.**

Calabi-Yau HKT manifolds

Definition: Let (M, I, J, K, Ω) be an HKT-manifold with holonomy in $SL(n, \mathbb{H})$, and $\Phi_I \in \Lambda_I^{n,0}(M)$ the parallel section of the canonical class. We say that M is **a Calabi-Yau HKT manifold** if $\Omega^n = \Phi_I$.

Example: Let G be a nilpotent Lie group with a left-invariant hypercomplex HKT-structure. Then $\mathcal{H}\mathrm{ol}(G) \subset SL(n, \mathbb{H})$ (Barberis, Dotti, V.). Since the forms Φ_I and Ω are G-invariant, the quotient $\frac{\Omega^n}{\Phi_I}$ is constant. Rescaling, we obtain that **all HKT-nilmanifolds are Calabi-Yau HKT** .

Claim: Let (M, I, J, K, g) be a Calabi-Yau HKT-manifold. **Then (M, J, g) is balanced**, that is, $d(\omega_J^{2n-1}) = 0$.

Proof: $\omega_J^{2n-1} = \mathrm{Re}(\Omega^{n-1} \wedge \Omega^n)$. However,

$$d(\Omega^{n-1} \wedge \overline{\Omega}^n) \xlongequal{\text{Hodge decomposition}} \partial(\Omega^{n-1} \wedge \overline{\Omega}^n)$$
$$= (n-1)\partial\Omega \wedge \Omega^{n-2} \wedge \overline{\Omega}^n + \Omega^{n-1} \wedge \partial(\overline{\Omega}^n)$$

The first term vanishes because Ω is HKT, the second because $\overline{\Omega}^n$ is closed.

■