Algebraic geometry over quaternions

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November 26, 2007

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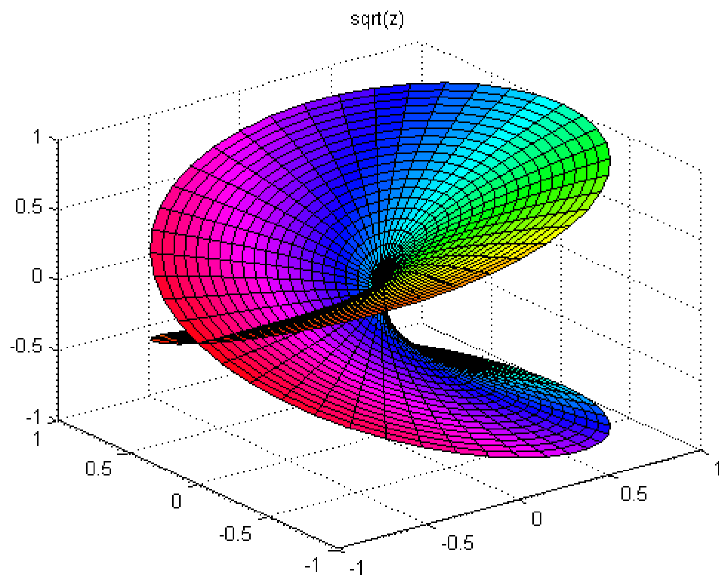
History of algebraic geometry.

1. XIX centrury: Riemann, Klein, Poincaré. Study of elliptic integrals and elliptic functions leads to the notion of a **Riemannian surface** of a holomorphic function. In a modern language, Riemann surface is a smooth 2-dimensional manifold, covered by open disks in $\mathbb{R}^2 = \mathbb{C}$, with transition functions holomorphic.



A Riemann surface for a square root.

History of algebraic geometry.

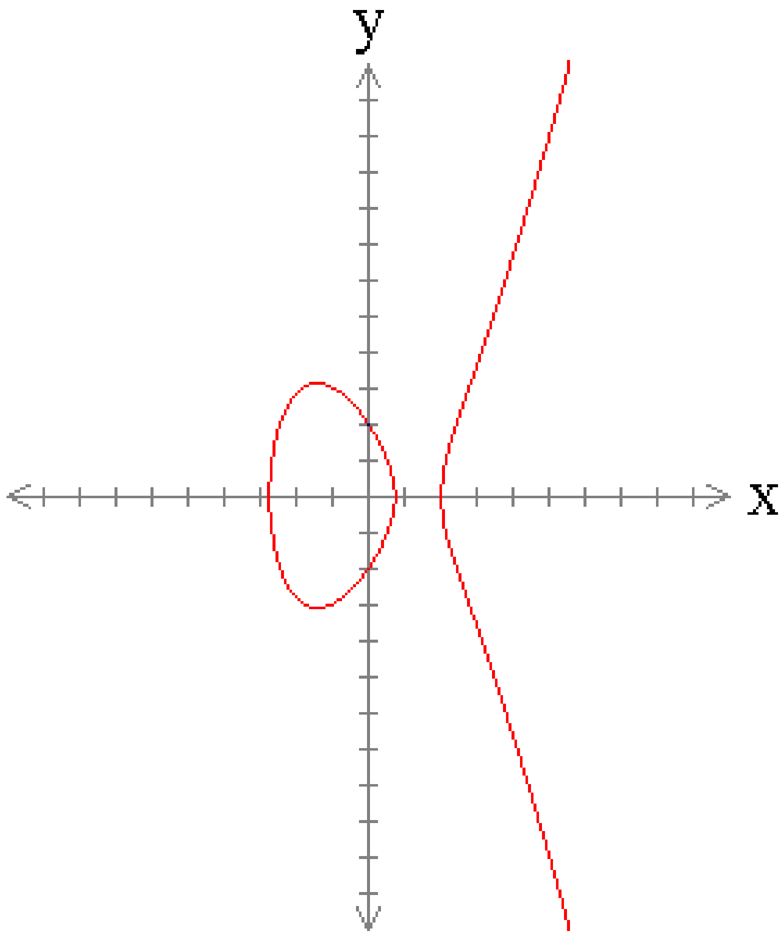
2. Italian school (1885-1935): Segre, Severi, Enriques, Castelnuovo.

An affine algebraic variety is a subset in $\mathbb{C}$ defined as a set of common zeroes of a system of algebraic equations. Two varieties are equivalent, if there exists a polynomial bijection from one to another.

1. Can be defined over any algebraically closed field.

2. If the equations are homogeneous, they define a (compact) subset in a projective space $\mathbb{C}P^n$.

3. Definition is not intrinsic.



$y^2 = x^3 - 6x + 4$

An elliptic curve

Spectrum of a ring: a digression.

1. Marshall Stone's spectra of boolean rings (1936).

A boolean ring A is a commutative ring with an additional axiom: $x^2 = x$ for all x. The **Stone spectrum** Spec(A) of a boolean ring is the set of all homomorphisms from A to the ring $\mathbb{Z}/2\mathbb{Z}$. This space has topology: an base of open sets is given by

$$U_f := \{\varphi : A \longrightarrow \mathbb{Z}/2\mathbb{Z} \mid \varphi(f) \neq 0\}$$

where $f \in A$. This topology is completely disconnected (there exists a base of closed open sets) and Hausdorff. Stone proved that A is isomorphic to the ring of continuous functions from Spec(A) to $\mathbb{Z}/2\mathbb{Z}$, and this is "equivalence of categories" (between completely disconnected Hausdorff spaces and Boolean rings).

2. Grothendieck's notion of a spectrum. A is a commutative ring, Spec(A) is the set of all prime ideals with **Zariski topology**, where the base of open sets is given by

$$U_f := \{I \in A \mid f \notin I\}$$

for some $f \in A$. This is **a ringed space**: with every open set one associates a ring A_f (the localization $A[f^-1]$), and inclusion of open sets correspond to ring homomorphisms, with associativity axiom satisfied. For an open subset U of a ringed space, the corresponding ring is denoted by $\mathcal{O}_U$ (it is called **the structure sheaf**). Examples: the rings of functions (smooth, continuous, complex analytic) on a manifold define a structure of a ringed space.

An exercise:

Check that Stone's spectrum is a special case of Grothendieck's spectrum.

3. Modern approach: Zariski, Weil, Grothendieck, Dieudonné

A **scheme** is a ringed space which is locally isomorphic to a spectrum of a ring (with Zariski topology). Morphisms of schemes are morphisms of ringed spaces: continuous maps $X \xrightarrow{\varphi} Y$, with ring homomorphisms

$$\varphi^* : \mathcal{O}_U \longrightarrow \mathcal{O}_{\varphi^{-1}(U)}$$

defined for any open $U \subset Y$ and commuting with restrictions to subsets.

0. Scheme geometry. All the usual geometric notions (compactness, separability, smoothness...) have their scheme-theoretic versions. Varieties are schemes without nilpotent elements in $\mathcal{O}_X$.

1. Schemes are closed under all natural operations.
(taking products, a graph of a morphism, intersection, union...)

2. The moduli spaces are again schemes (when finite-dimensional).
The **moduli spaces** are the sets parameterizing various algebro-geometric objects (subvarieties, morphisms, fiber bundles) and equipped with a natural algebraic structure. Grothendieck proved that the moduli exist in scheme category, in very general assumptions.

3. Can be used in number theory.
The rings do not need to be defined over $\mathbb{C}$, or any other algebraically closed field. In particular, $\operatorname{Spec}(\mathbb{Z})$ **is a scheme**, which can be studied in geometric terms. This was the original motivation of Grothendieck (at least, one of his motivations).

4. Desingularization (Hironaka).
In characteristic 0, any variety X admits a desingularization, that is, a proper, surjective map $\tilde{X} \longrightarrow X$, with $\tilde{X}$ smooth, and generically one-to-one.

Complex geometry (Grauert, Oka, Cartan, Serre...)

A **complex manifold** is a manifold with an atlas of open subsets in $\mathbb{C}^n$, and translation maps complex analytic.

A **complex analytic subvariety** is a closed subvariety, locally defined as a zero set of a system of complex analytic equations. An **complex analytic variety** is a ringed topological space, locally isomorphic to a closed subvariety of an open ball $B \subset \mathbb{C}^n$. If we allow nilpotents in the structure sheaf, we obtain the notion of a **complex analytic space**.

Complex spaces are just as good as schemes: the products/graphs/moduli spaces of complex spaces are again complex spaces, and Hironaka's desingularization works as well.

Serre's GAGA (Géométrie Algébrique - Géométrie Analitique, 1956):

Some complex varieties can be defined using algebraic equations, they are called **algebraic**. Serre has proved that a complex subvariety of a compact algebraic variety is algebraic, and holomorphic map of compact algebraic varieties is algebraic. The algebraic varieties are special case of complex analytic!

The topology of complex varieties is **infinitely more complicated.**

Kähler manifolds.

A complex manifold is equipped with a natural map $I\, TM \longrightarrow TM$, $I^2 = -\operatorname{Id}$, called **the complex structure map**. A Riemannian metric is called **Hermitian** if $g(Ix, y) = g(x, Iy)$. In this situation $\omega(x, y) = g(x, Iy)$ is a differential form, called Hermitian form. The following conditions are equivalent

1. $d\omega = 0$**.**

2. ω **is parallel** (preserved by the Levi-Civita connection), that is, $\nabla\omega = 0$.

3. Flat approximation. At each point M has complex coordinates, such that g is approximated at this point by a standard (flat) Hermitian structure in this coordinates, up to order 2.

If any of these conditions is satisfied, the metric is called **Kähler** (after Erich Kähler, 1938).

NB: Kähler manifolds are symplectic.

Properties of Kähler manifolds.

1. The $U(n+1)$-invariant metric on $\mathbb{C}P^n$ is called **the Fubini-Study metric** (its uniqueness and existence follows easily from $U(n)$-invariance). Since $\mathbb{C}^n$ does not have $U(n)$-invariant 3-forms, Fubini-Study metric is **Kähler**.

2. A submanifold of a Kähler manifold is again Kähler (restriction of ω is still closed). Therefore, all algebraic manifolds are Kähler.

3. Topology of compact Kähler manifolds is tightly controlled (all rational cohomology operations vanish, etc.) The fundamental group is especially easy to control. It is conjectured that the isomorphism problem for fundamental groups of compact Kähler manifolds has an algorithmic solution.

4. By contast, any finitely-generated, finitely-presented group can be a fundamental group of a compact complex manifold. Therefore the problem of recursively enumerating the fundamental groups cannot be solved.

5. Topology of complex manifolds is infinitely more complicated!

Quaternionic geometry: an introduction

Isometries of $\mathbb{R}^2$ are expressed in terms of complex numbers. This allows one to solve the problems of geometry algebraically.

QUESTION: Can we do that in dimension 3?

ANSWER: Yes!

Sir William Rowan Hamilton
(August 4, 1805 – September 2, 1865)

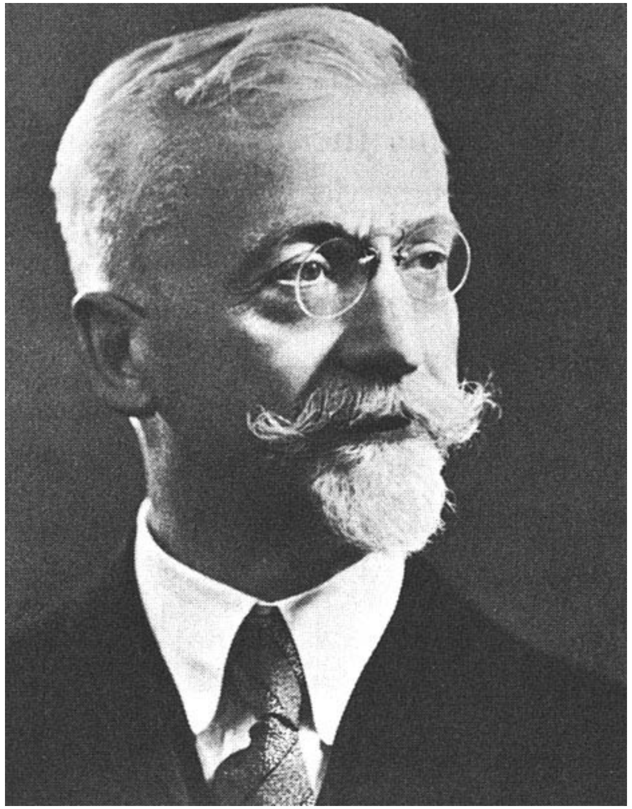
Broom Bridge

"Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$I^2 = J^2 = K^2 = IJK = -1$$

and cut it on a stone of this bridge."

Fast forward 70 years.



Élie Joseph Cartan
(9 April 1869 – 6 May 1951)

"Quaternionic structures" in the sense of Elie Cartan **don't exist.**

THEOREM: Let $f : \mathbb{H}^n \longrightarrow H^m$ be a function, defined locally in some open subset of n-dimensional quaternion space $\mathbb{H}^n$. Suppose that the differential Df is $\mathbb{H}$-linear. Then f is a linear map.

Proof (a modern one): The graph of f is a "hyperkähler submanifold" in $\mathbb{H}^n \times H^m$, hence "geodesically complete", hence linear. ■

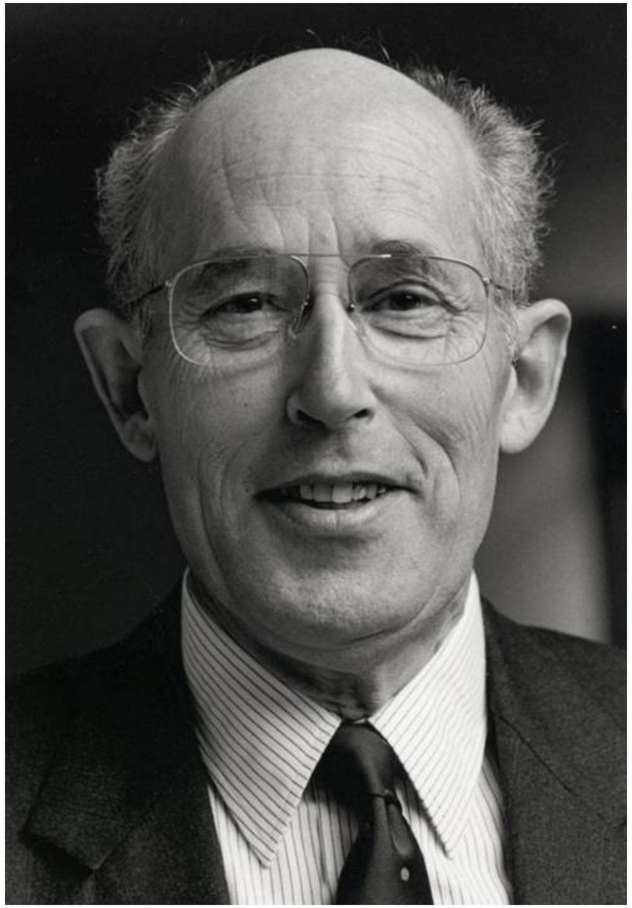
Algebraic geometry over $\mathbb{H}$.

Over $\mathbb{C}$, we have 3 distinct notions of “algebraic geometry”:

1. Schemes over $\mathbb{C}$.
2. Complex manifolds.
3. Kähler manifolds.

The first notion does not work for $\mathbb{H}$, because polynomial functions on $\mathbb{H}^n$ generate all real polynomials on $\mathbb{R}^4$. The second version does not work, because any quaternionic-differentiable function is linear. The third one works!

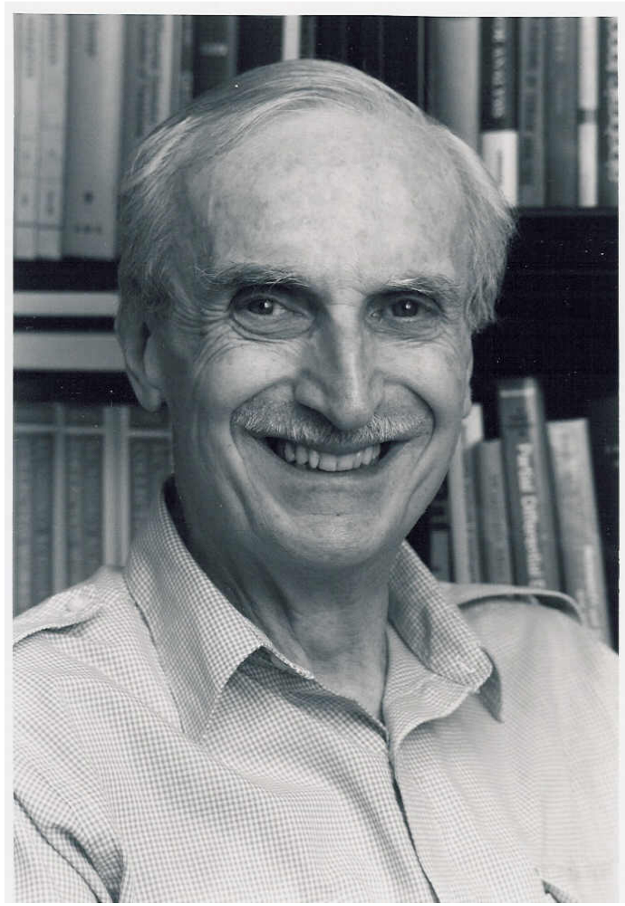
Hyperkähler manifolds.



Marcel Berger

Classification of holonomies.

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds



Eugenio Calabi

Definition: (E. Calabi, 1978)
Let (M, g) be a Riemannian manifold equipped with three complex structure operators $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M, I, J, K, g) is called **hyperkähler**.

Holomorphic symplectic geometry

A hyperkähler manifold (M, I, J, K), considered as a complex manifold (M, I), is holomorphically symplectic (equipped with a holomorphic, non-degenerate 2-form). Recall that, M is equipped with 3 symplectic forms ω_I, ω_J, ω_K.

Lemma: The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic 2-form on (M, I). ■

Converse is also true, as follows from the famous conjecture, made by Calabi in 1952.

Theorem: (S.-T. Yau, 1978) Let M be a compact, holomorphically symplectic Kähler manifold. Then M admits a hyperkähler metric, which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

Induced complex structures are complex structures of form

$$L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.$$

They are non-algebraic (mostly). Indeed, for generic a, b, c, (M, L) has no divisors. The set of induced complex structures is parametrized by $S^2 \cong \mathbb{C}P^1$.

These complex structures can be glued together to form a **"twistor space"**, $\operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$, holomorphically fibered over $\mathbb{C}P^1$. The hyperkähler structure can be defined in terms of a twistor space. You can have "hyperkähler singular spaces", and even schemes.

"Hyperkähler algebraic geometry" is almost as good as the usual one.

Define **trianalytic subvarieties** as closed subsets which are complex analytic with respect to I, J, K.

0. Trianalytic subvarieties are singular hyperkähler.

1. Let L be a generic induced complex structure. Then all complex subvarieties of (M, L) are trianalytic.

2. A normalization of a hyperkähler variety is smooth and hyperkähler. This gives a desingularization ("hyperkähler Hironaka").

3. A complex deformation of a trianalytic subvariety is again trianalytic, the corresponding moduli space is (singularly) hyperkähler.

4. Similar results (also very strong) are true for vector bundles which are holomorphic under I, J, K ("hyperholomorphic bundles")