Hypercomplex Structures on Kähler Manifolds

August 08, 2007, La Falda, Argentina

Hypercomplex geometry: an introduction

Definition: (E. Calabi, 1978)

Let (M, g) be a Riemannian manifold equipped with three complex structure operators

$$I, J, K : \ TM \longrightarrow TM,$$

satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M, I, J, K, g) is called **hyperkähler**.

Holonomy of a hyperkähler manifold is $Sp(n)$.

Indeed, Levi-Civita connection preserves I, J, K, because M is Kähler. The group of matrices preserving quaternionic structure and metric is $Sp(n)$.

Converse is also true. Hyperkähler manifolds are often *defined* as manifolds with affine connection and holonomy in $Sp(n)$.

Holomorphic symplectic geometry

A hyperkähler manifold (M, I, J, K, g), considered as a complex manifold (M, I), is holomorphically symplectic (equipped with a holomorphic, non-degenerate 2-form). Recall that M is equipped with 3 symplectic forms $\omega_I(\cdot, \cdot) = g(\cdot, I\cdot)$, ω_J, ω_K.

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic 2-form on (M, I). ∎

Converse is also true.

THEOREM: (E. Calabi, 1952, S.-T. Yau, 1978) Let M be a compact, holomorphically symplectic Kähler manifold. Then M admits a hyperkähler metric, which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

HYPERCOMPLEX MANIFOLDS

"Hyperkähler manifolds without a metric"

Definition: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are integrable almost complex structures. Then

$$(M, I, J, K)$$

is called **a hypercomplex manifold**.

4

EXAMPLES:

Compact hypercomplex manifolds which are not hyperkähler

1. In dimension 1 (real dimension 4), we have a complete classification, due to C. P. Boyer (1988)

2. Many homogeneous examples, due to D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen (1980-ies, early 1990-ies).

3. Some nilmanifolds admit homogeneous hypercomplex structure (M. L. Barberis, I. Dotti, A. Fino) (1990-ies).

4. Some inhomogeneous examples are constructed by deformation or as fiber bundles.

In dimension > 1, no classification results are known (and no conjectures either).

OBATA CONNECTION

Hypercomplex manifolds can be characterized in terms of holonomy

Theorem: (M. Obata, 1952) Let (M, I, J, K) be a hypercomplex manifold. Then M admits a unique torsion-free affine connection preserving I, J, K.

Converse is also true. Suppose that I, J, K are operators defining quaternionic structure on TM, and ∇ a torsion-free, affine connection preserving I, J, K. Then I, J, K are integrable almost complex structures, and (M, I, J, K) is hypercomplex.

Holonomy of Obata connection lies in $GL(n, \mathbb{H})$. *A manifold equipped with an affine, torsion-free connection with holonomy in $GL(n, \mathbb{H})$ is hypercomplex.*

This can be used as a definition of a hypercomplex structure.

QUESTIONS

1. Given a complex manifold M, when M admits a hypercomplex structure? How many?

2. What are possible holonomies of Obata connection, for a compact hypercomplex manifold? Can $SL(n, \mathbb{H})$ be a holonomy group?

3. Describe the structure of automorphism group of a hypercomplex manifold.

THEOREM: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that the complex manifold (M, I) admits a Kähler structure. Then (M, I) is hyperkähler.

Quaternionic Hermitian structures

DEFINITION: Let (M, I, J, K) be a hypercomplex manifold, and g a Riemannian metric. We say that g is quaternionic Hermitian if I, J, K are orthogonal with respect to g.

Given a quaternionic Hermitian metric g on (M, I, J, K), consider its Hermitian forms

$$\omega_I(\cdot, \cdot) = g(\cdot, I\cdot), \omega_J, \omega_K$$

(real, but *not closed*). Then $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is of Hodge type (2,0) with respect to I.

If $d\Omega = 0$, (M, I, J, K, g) is hyperkähler (this is one of the definitions).

Consider a weaker condition:

$$\partial\Omega = 0, \quad \partial : \Lambda^{2,0}(M, I) \longrightarrow \Lambda^{3,0}(M, I)$$

DEFINITION: (Howe, Papadopoulos, 1998)
Let (M, I, J, K) be a hypercomplex manifold, g a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ the corresponding $(2,0)$-form. We say that g is **HKT ("weakly hyperkähler with torsion")** if

$$\partial \Omega = 0.$$

HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.

1. They admit a smooth potential (locally). There is a notion of an "HKT-class" (similar to Kähler class) in a certain finite-dimensional coholology group. Two metrics in the same HKT-class differ by a potential, which is a function.

2.When (M, I) has trivial canonical class, a version of Lefschetz-type identities can be proven giving an $\mathfrak{sl}(2)$-action on cohomology.

A digression: some obvious topological restrictions on (M, I), when (M, I) admits a hypercomplex structure.

0. Quaternionic Hermitian structure always exists.

1. Complex dimension is even.

2. The canonical line bundle $\Lambda^{n,0}(M, I)$ of (M, I) is always trivial topologically. Indeed, a non-degenerate section of a canonical line bundle is provided by a top power of an form Ω associated with some quaternionic Hermitian strucure. In particular,

$$c_1(M, I) = 0.$$

3. When (M, I) admits a Kähler structure,

$$c_1(M, I) = 0$$

implies that the canonical bundle is trivial holomorphically. This follows from Calabi-Yau theorem.

Lefschetz identities for Kähler manifolds.

The usual Lefschetz $\mathfrak{sl}(2)$-action is constructed as follows. Let M be a Kähler manifold, $\dim_{\mathbb{C}} M = n$, ω its Kähler form, $L: \Lambda^i(M) \longrightarrow \Lambda^{i+2}(M)$ the operator of multiplication by ω, Λ its Hermitian adjoint, and H acts on $\Lambda^i(M)$ as a scalar multiplication by $(n-i)$. **Then (L, Λ, H) is an $\mathfrak{sl}(2)$-triple.** It commutes with the Laplacian, giving an $\mathfrak{sl}(2)$-**action on cohomology.**

Two ingredients of the proof:

1. Use linear algebra to show that (L, Λ, H) is an $\mathfrak{sl}(2)$-triple. This is true for any almost complex Hermitian manifold

2. Show that (L, Λ, H) commutes with the Laplacian. Need Kähler identities for this.

11

Lefschetz identities for HKT-manifolds with trivial canonical bundle.

THEOREM: Let (M, I, J, K, g) be a hypercomplex manifold, $\dim_{\mathbb{H}}(M) = n$. Assume that the canonical bundle of $M_I := (M, I)$, is trivial, as a holomorphic vector bundle. Consider the Dolbeault resolution for the holomorphic cohomology $H^*(M_I, \mathcal{O})$

$$\Lambda^0(M_I) \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M_I) \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M_I) \xrightarrow{\overline{\partial}} \ldots$$

The multiplication map $L(\eta) = \eta \wedge \overline{\Omega}$ commutes with the differential, because $\overline{\Omega}$ is $\overline{\partial}$-closed. Let Λ be its Hermitian adjoint, and $H(\eta) = i - n$, for all $\eta \in \Lambda^{0,i}(M_I)$. **Then (L, Λ, H) is an $\mathfrak{sl}(2)$-triple acting on cohomology.**

12

The proof of HKT Lefschetz identities:

1. Linear algebra (same as in the usual Kähler case).

2. An HKT-analogue of Kähler identities ("supersymmetry for HKT-manifolds").

We immediately obtain that the cohomology class of $\overline{\Omega}$ in $H^2(M_I, \mathcal{O})$ is non-trivial (often false, when canonical bundle is non-trivial). Its top power is non-trivial in $H^{2n}(M_I, \mathcal{O})$ (L^k acts as an isomorphism from $H^{n-k}(M_I, \mathcal{O})$ to $H^{n+k}(M_I, \mathcal{O})$).

Hypercomplex structures on Kähler manifolds

0. Let (M, I) be a manifold admitting Kähler structure and a hypercomplex structure. The canonical class of (M, I) is trivial by Calabi-Yau.

1. From a Kähler form, an HKT form is obtained by averaging with $SU(2)$ (the group of unitary quaternions, acting on TM).

2. The cohomology class of Ω is non-trivial by HKT-Lefschetz.

3. Since (M, I) is Kähler, this class is represented by a holomorphic form $\tilde{\Omega}$. The top power of this form is non-trivial, by HKT-Lefschetz.

4. The top power of $\tilde{\Omega}^n$ is a non-trivial holomorphic section of a canonical class, which is trivial. Therefore, $\tilde{\Omega}^n$ is nowheere vanishing, and (M, I) is holomorphic symplectic.

5. Using Calabi-Yau to obtain that it is hyperkähler.

Lefschetz identities for general HKT-manifolds.

A full strength theorem (don't need it).

THEOREM: Let (M, I, J, K, g) be a hypercomplex manifold, K the canonical bundle of $M_I := (M, I)$, $K^{1/2}$ its square root (considered as a holomorphic vector bundle). Consider the map

$$L: \ H^i(M_I, K^{1/2}) \longrightarrow H^{i+2}(M_I, K^{1/2})$$

mapping a class represented by a form

$$\eta \in \Lambda^{0,p}(M_I) \otimes K^{1/2}$$

to $\eta \wedge \overline{\Omega}$ (this defines a correct operation on cohomology, because $\overline{\Omega}$ is $\overline{\partial}$-closed. Then L is a term in an $\mathfrak{sl}(2)$-triple acting on $H^i(M_I, K^{1/2})$.

It is a theorem about harmonic spinors. When $K^{1/2}$ is non-trivial, the cohomology groups $H^i(M_I, K^{1/2})$ are usually empty.