Hodge Theory On Nearly Kähler Manifolds

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Nearly Kähler Manifolds: an introduction

Classification of holonomies.

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

"Hodge theory on manifolds with special holonomy"

1. $\Lambda^*(M) = \oplus \Lambda^*_\varepsilon(M)$ (ε - weights of representations of holonomy). Then $\mathcal{H}^*(\mathcal{M}) = \oplus \mathcal{H}^*_\varepsilon(\mathcal{M})$ (Chern).

2. dd^c-lemma, implying restrictions on topology ("formality": Deligne-Griffiths-Morgan-Sullivan).

3. Solutions of Maurer-Cartan equation $\bar\partial\gamma = \frac{1}{2}\{\gamma,\gamma\}$ (Kodaira-Spencer, Bogomolov-Tian-Todorov).

Holonomy of Riemannian cones

Definition 1.1: Let (M, g) be a Riemannian manifold. The **Riemannian cone** of M is

$$C(M) := (M \times \mathbb{R}^{>0}, t^2 g + dt^2),$$

where t denotes the coordinate on the half-line $\mathbb{R}^{>0}$.

Conical singularities of manifolds with special holonomy give special holonomy on Riemannian cones.

Suppose $C(M)$ has special holonomy. What can we say about geometry of M?

Riemannian cones with special holonomy		
Holonomy of $C(M)$	*Geometry of $C(M)$*	*Geometry of M*
$SO(n)$	Riemannian	—
$U(n)$	Kähler	Sasakian
$SU(n)$	Calabi-Yau	Sasaki-Einstein
$Sp(n)$	hyperkähler	3-Sasakian
$Sp(n)Sp(1)$	quaternionic-Kähler	—
G_2	G_2-manifolds	nearly Kähler
$Spin(7)$	$Spin(7)$-manifolds	nearly G_2-manifolds

Killing spinors on Riemannian cones

Not essential for understanding of today's talk, because the spinor interpretation will not be used

Recall that we have a "Clifford multiplication map" $TM \otimes \mathfrak{S} \longrightarrow \mathfrak{S}$, where TM is a bundle of tangent vectors on a manifold M, and $\mathfrak{S}$ the bundle of spinors.

"Killing spinor" on M is $\Psi \in \mathfrak{S}$ which satisfies

$$\nabla_X(\Psi) = \lambda X \Psi$$

for all tangent fields $X \in TM$.

Fact 1: Killing spinors on M correspong uniquely to parallel spinors on $C(M)$.

Fact 2: Killing spinors on M exist only if M is an Einstein manifold, with Einstein constant $|\lambda|^2 \geqslant 0$.

Remark: Similarly, if M admits a *parallel* spinor, M is Ricci-Flat (follows from Weitzenböck formula).

Remark: In Berger's list, the following holonomies correspond to Ricci-flat manifolds: $SU(n)$, $Sp(n)$, G_2, Spin(7).

Fact 3: $SU(n)$, $Sp(n)$, G_2, Spin(7) admit parallel spinors.

COROLLARY: Sasaki-Einstein, 3-Sasakian, nearly Kähler and nearly G_2-manifolds admit Killing spinors; hence Einstein.

Proof: Their cones admit a parallel spinor.

Nearly Kähler manifolds

The name is confusing.

The original definition: (Alfred Gray). Let (M, g, I) be a Hermitian almost complex manifold, $\omega \in \Lambda^{1,1}(M)$ its Hermitian form, ∇ the Levi-Civita connection. Then $\nabla\omega$ lies in $\Lambda^1(M) \otimes \Lambda^2(M)$. Gray defined *"nearly Kähler manifolds"* as those that satisfy

$$\nabla\omega \in \Lambda^3(M) \subset \Lambda^1(M) \otimes \Lambda^2(M)$$

($\nabla\omega$ is skew-symmetric).

Trivial remark: In this case $d\omega = \nabla\omega$, because ∇ is torsion-free.

"Strictly nearly Kähler" means that the 3-form $\rho = d\omega$ is non-degenerate, that is, the map

$$TM \xrightarrow{\rho} \Lambda^2 M,$$

defined as $X \longrightarrow \rho(X, \cdot, \cdot)$, is injective. It is *much more restrictive condition* than the Kähler condition $d\omega = 0$. There is a *"splitting theorem"* in nearly Kähler geometry (due to P.-A. Nagy). It follows that strictly nearly Kähler manifolds are products of Einstein ones, hence they are real analytic and have finite-dimensional moduli, if compact.

Some examples

1. 6-manifolds with parallel G_2 cones.

2. Twistor spaces of positive quaternionic-Kähler manifolds with non-standard complex structure due to Eels and Salamon.

Connections with totally antisymmetric torsion

Let $\nabla_0 : \Lambda^1(M) \longrightarrow \Lambda^1(M) \otimes \Lambda^1(M)$ be an orthogonal connection on a Riemannian manifold. Its torsion T lies in

$$T \in \Lambda^1(M)^* \otimes \mathfrak{so}(M) = \Lambda^1(M)^* \otimes \Lambda^2(M).$$

Using the metric, we identify $\Lambda^1(M)^*$ with $\Lambda^1(M)$ and consider T as a 3-form.

Definition: ∇_0 is a *connection with totally anti-symmetric torsion* if T is totaly anti-symmetric.

Connection with totally antisymmetric torsion on a nearly Kähler manifold

Let M be a nearly Kähler manifold (in the sense of Gray), $\rho = d\omega = \nabla\omega$ the corresponding 3-form. Consider the operator

$$\theta : \Lambda^1(M) \longrightarrow \Lambda^2(M) \subset \Lambda^1(M) \otimes \Lambda^1(M).$$

mapping ξ to $\rho(\xi^\sharp, \cdot, \cdot)$, where $\xi^\sharp$ is the dual vector field. Let ∇_T be a new connection.

$$\nabla_T := \nabla + \frac{1}{2}\theta : \Lambda^1(M) \longrightarrow \Lambda^1(M) \otimes \Lambda^1(M).$$

A trivial observation.
∇_T preserves the Hermitian structure.

Another trivial observation.
The torsion of ∇_T is ρ.

This gives...

A simple theorem: Let M be a nearly Kähler manifold (in the sense of Gray). Then ∇_T is a Hermitian connection with totally antisymmetric torsion.

A difficult theorem (V. Kirichenko).
On nearly Kähler manifolds,

$$\nabla_T(T) = 0.$$

The torsion is parallel.

This is used to obtain a splitting theorem.

Manifolds with parallel antisymmetric torsion.

Let M be a Riemannian manifold, ∇_T a connection with totally antisymmetric parallel torsion T. Assume its local holonomy is irreducible.

THEOREM: (R. Cleyton and A. Swann, 2002) Any such manifold is either locally homogeneous, has vanishing torsion, is nearly G_2 (in dimension 7) or nearly Kähler (in dimension 6).

Remark: The last two cases are ones we have seen in the classification of cones with special holonomy.

This is an antisymmetric torsion analogue of Berger's theorem on irreducible holonomies.

This is used to obtain splitting for nearly Kähler manifolds.

COROLLARY: (P.-A. Nagy, 2002)
Let M be a nearly Kähler manifold, in the sense of Gray. Then M is locally a product of the following nearly Kähler types.

1. Homogeneous (classified by J.B. Butruille in 2004)

2. Twistor spaces of positive quaternionic-Kähler manifolds

3. 6-dimensional nearly Kähler

Remark: The positive quaternionic-Kähler manifolds and their twistors are (conjecturally) symmetric. Hence the only interesting example of "nearly Kähler" is 6-dimensional nearly Kähler manifolds.

In modern literature, "nearly Kähler" usually denotes a 6-dimensional Hermitian manifold with $\nabla\omega$ antisymmetric. We shall always assume "6-dimensional".

Nearly Kähler manifolds

The many definitions of NK-manifolds

"A well-known theorem:"

Let (M, I, ω) be a Hermitian almost complex 6-manifold. Then the following conditions are equivalent.

1. *The form $\nabla\omega \in \Lambda^1(M) \otimes \Lambda^2(M)$ is non-zero and totally skew-symmetric (that is, $\nabla\omega$ is a 3-form).*

2. *The structure group of M admits a reduction to $SU(3)$, that is, there is $(3,0)$-form Ω with $|\Omega| = 1$, and*

$$d\omega = 3\lambda \operatorname{Re}\Omega, \quad d\operatorname{Im}\Omega = -2\lambda\omega^2$$

where λ is a non-zero real constant.

Another well-known theorem:

Let M be a Riemannian 6-manifold. Then the following conditions are equivalent.

1. *M admits a nearly Kähler Hermitian structure.*

2. *M admits a Killing spinor.*

3. *The Riemannian cone $C(M)$ has holonomy G_2.*

Remark: Let M be nearly Kähler. Unless $C(M)$ is flat, and M is S^6, the almost complex structure is uniquely determined by the metric (Friedrich). Conversely, the metric is uniquely determined by the almost complex structure.

Examples of nearly Kähler manifolds (all four of them)

1. The sphere S^6. Its cone is $\mathbb{R}^7$.

2 and 3.
$\mathbb{C}P^3$ and the flag variety $F(2,1)$. These are twistor spaces for self-dual Einstein manifolds S^4 and $\mathbb{C}P^2$, we take the Eels-Salamon almost complex structure.

4. $S^3 \times S^3$.

No non-homogeneous compact examples (so far).

Geometry of NK-manifolds

A trivial remark: An NK-manifold is never integrable. Indeed, $d\omega^{1,1} = 3\lambda \operatorname{Re} \Omega^{3,0}$. In fact, the Nijenhuis tensor

$$N: \Lambda^{0,1}(M) \longrightarrow \Lambda^{2,0}(M)$$

is *invertible* (unless $\lambda = 0$).

Another trivial remark: If $\lambda = 0$, the NK-equations *degenerate to equations defining Calabi-Yau.*

Let

$$d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2},$$

be the Hodge decomposition of de Rham differential. Clearly, $d^{2,0}$ *is the Nijenhuis tensor.*

Notation: We use

$$d^{2,-1} =: N, \quad d^{-1,2} =: \overline{N}, \quad d^{1,0} =: \partial, \quad d^{0,1} =: \overline{\partial}.$$

Kähler identities on NK-manifolds

The usual Kähler identities take a form *"a commutator of some Hodge component of de Rham differential with the Hodge operator Λ is proportional to a Hermitian adjoint of some other Hodge component of de Rham differential"*.

On nearly Kähler, we have the same set of identities,

$$[\Lambda_\omega, \partial] = \sqrt{-1}\,\overline{\partial}^*, \quad [\Lambda_\omega, \overline{\partial}] = -\sqrt{-1}\,\partial^*$$

plus

$$[\Lambda_\omega, N] = \sqrt{-1}\,2\overline{N}^*, \quad [\Lambda_\omega, \overline{N}] = -\sqrt{-1}\,2N^*$$

In addition, we have

$$[L_\Omega, \Lambda_\omega] = \lambda N.$$

The proof is similar to the one used to obtain the usual Kähler identities.

Laplacian operators on NK-manifolds

Notation: For an operator v on a Hermitian space, let Δ_v denote $vv^* + v^*v$.

THEOREM:
On a nearly Kähler manifold, we have

$$\Delta_\partial - \Delta_{\overline{\partial}} = R$$

where R is a scalar operator, acting on (p,q)-forms as a multiplication with $\lambda^2(p-q)(3-p-q)$.

THEOREM:
On a nearly Kähler manifold, we have

$$\Delta_d - \Delta_{\partial-\overline{\partial}} = \Delta_N + \Delta_{\overline{N}} = \Delta_{N+\overline{N}}$$

Compare: On a Kähler manifold

$$2\Delta_\partial = 2\Delta_{\overline{\partial}} = \Delta_d = \Delta_{\partial-\overline{\partial}}$$

The relations between Laplacians follow from the Kähler identities in a standard way.

Hodge theory on NK-manifolds

Use the comparison formulas for Laplacians to obtain a Hodge decomposition theorem.

THEOREM: Let M be a compact nearly Kähler manifold, and $\eta = \oplus \eta^{p,q}$ a differential form. Then η is harmonic if and only if all its Hodge components $\eta^{p,q}$ are harmonic and primitive.

Proof: Let $\eta \in \ker \Delta_v$ (for some v). Then

$$0 = (\Delta_v \eta, \eta) = (vv^* \eta + v^* v\eta, \eta) = (v\eta, v\eta) + (v^*\eta, v^*\eta).$$

In other words, *$\eta \in \ker \Delta_v$ if and only if $\eta \in \ker v \cap \ker v^*$, and $(\Delta_v \eta, \eta) > 0$ otherwise.*

From the second comparison formula, the same positivity argument gives

$$0 = (\Delta_d \eta, \eta) = (\Delta_{\partial - \overline{\partial}} \eta, \eta) + (\Delta_N \eta, \eta) + (\Delta_{\overline{N}} \eta, \eta) \geqslant 0$$

hence

$$\eta \in \ker N, \quad \eta \in \ker \overline{N}, \quad \eta \in \ker(\partial - \overline{\partial}).$$

Then

$$0 = d\eta = (N + \partial + \overline{\partial} + \overline{N})\eta = (\partial + \overline{\partial})\eta = 0$$

Adding $(\partial - \overline{\partial})\eta = 0$, obtain $\partial\eta = 0$. Then all Hodge components of d vanish on η:

$$d^{2,-1}\eta = d^{1,0}\eta = d^{0,1}\eta = d^{-1,2}\eta = 0.$$

However, the $(p+i, q+j)$-th component of $d^{i,j}\eta$ is $d^{i,j}\eta^{p,q}$, hence

$$d^{i,j}\eta^{p,q} = 0$$

for all i, j, p, q. Then $d\eta^{p,q} = 0$. The same argument is used to show $d^*\eta^{p,q} = 0$.

Possible applications

"Algebraic geometry of nearly Kähler manifolds."

0. Hitchin's functional on the moduli of almost complex manifolds.

1. Yang-Mills bundles (Hermitian bundles with curvature two-form which satisfies $\Lambda\Theta = 0$).

2. The space of pseudoholomorphic curves in NK-manifold. Its connected components are *compact*.

3. Finding solutions of Maurer-Cartan equation with applications to the moduli of nearly Kähler manifolds.