Quaternionic Monge-Ampere equation

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Generalized Monge-Ampere equation on complex manifold.

**Data:** $M$ - compact complex $n$-manifold, $\Phi$ a closed, positive $(p, p)$-form. A $(k, k)$-form $\nu$ is called **(strictly) $\Phi$-positive** if $\nu \wedge \eta \wedge \Phi$ is (strictly) positive, for any $(i, i)$-form $\eta$ for which $\eta \wedge \Phi$ is (strictly) positive. A $(k, k)$-form $\nu$ is called **$\Phi$-closed** if $\nu \wedge \eta \wedge \Phi$ is closed for for any $(i, i)$-form $\eta$ for which $\eta \wedge \Phi$ is closed. A $(1, 1)$-form $\nu$ is called $\Phi$-Kähler if it is strictly $\Phi$-positive and $\Phi$-closed.

**Example:**

$$\Phi = (-\sqrt{-1})^p \zeta_1 \wedge \bar{\zeta}_1 \wedge \ldots \wedge \zeta_p \wedge \bar{\zeta}_p,$$

where $\zeta_i$ are holomorphic $(1, 0)$-forms. Then $\Phi$-positivity (closedness) means positivity (closedness) on the complex foliation defined by $\bigcap \ker \zeta_i$. 
**Generalized Monge-Ampere equation:**

\[(\omega + \partial \bar{\partial} \varphi)^{n-p} \wedge \Phi = A_f e^f \text{Vol}_M\]

Here \(f\) is an arbitrary function, \(\varphi\) a solution we are looking for, \(A_f\) is a constant, dependent on \(f\), \(\omega\) a \(\Phi\)-Kähler form, \(\text{Vol}_M\) is a fixed volume form on \(M\), \(\omega + \partial \bar{\partial} \varphi\) is \(\Phi\)-Kähler.

**Observation 1: Solutions are unique.** Indeed, suppose that \(\omega\) and \(\omega_1 = \omega + \partial \bar{\partial} \varphi\) are both solutions. Then

\[0 = (\omega + \partial \bar{\partial} \varphi)^{n-p} \wedge \Phi - \omega^{n-p} \wedge \Phi = \partial \bar{\partial} \varphi \wedge \left[\sum_{k=1}^{n-p-1} \omega_k \wedge (\omega_1 \wedge \omega^{n-p-k}) \wedge \Phi\right]\]

The form in brackets is **strictly positive.** When \(\varphi\) reaches maximum, or minimum, \(\partial \bar{\partial} \varphi \wedge \ldots\) cannot vanish ("generalized maximum principle").
Generalized Monge-Ampere equation: more observations

2. $M$ is not required to be Kähler. One obtains interesting structures on non-Kähler manifolds (such as hypercomplex ones).

3. Hessian equation. When $M$ is Kähler, and $\Phi = \omega^n$, we have

$$(\omega + \partial \overline{\partial} \varphi)^{n-p} \wedge \Phi = P_{n-p}(a_1, \ldots a_n)$$

where $a_i$ are eigenvalues of the Hermitian form $\omega + \partial \overline{\partial} \varphi$, and $P_{n-p}$ the fundamental symmetric polynomial. This is called complex Hessian equation (Błocki etc.).

**Conjecture:** Under reasonable assumptions, solution exists.

**Question 1:** Is it known for foliations?

**Question 2:** What is known about existence of solutions for complex Hessian equation?
**Definition:** (E. Calabi, 1978)
Let \((M, g)\) be a Riemannian manifold equipped with three complex structure operators

\[ I, J, K : TM \rightarrow TM, \]

satisfying the quaternionic relation

\[ I^2 = J^2 = K^2 = IJK = -\text{Id}. \]

Suppose that \(I, J, K\) are Kähler. Then \((M, I, J, K, g)\) is called **hyperkähler**.

**Holonomy of a hyperkähler manifold is** \(Sp(n)\).

Indeed, Levi-Civita connection preserves \(I, J, K\), because \(M\) is Kähler. The group of matrices preserving quaternionic structure and metric is \(Sp(n)\).

Converse is also true. Hyperkähler manifolds are often **defined** as manifolds with affine connection and holonomy in \(Sp(n)\).
**Holomorphic symplectic geometry**

A hyperkähler manifold \((M, I, J, K, g)\), considered as a complex manifold \((M, I)\), is holomorphically symplectic (equipped with a holomorphic, non-degenerate 2-form). Recall that \(M\) is equipped with 3 symplectic forms \(\omega_I(\cdot, \cdot) = g(\cdot, I \cdot), \omega_J, \omega_K\).

**Lemma:** The form \(\Omega := \omega_J + \sqrt{-1} \omega_K\) is a holomorphic symplectic 2-form on \((M, I)\). ■

Converse is also true.

**Theorem:** (E. Calabi, 1952, S.-T. Yau, 1978) Let \(M\) be a compact, holomorphically symplectic Kähler manifold. Then \(M\) admits a hyperkähler metric, which is uniquely determined by the cohomology class of its Kähler form \(\omega_I\).

*Hyperkähler geometry is essentially the same as holomorphic symplectic geometry*
HYPERCOMPLEX MANIFOLDS

"Hyperkähler manifolds without a metric"

**Definition:** Let $M$ be a smooth manifold equipped with endomorphisms $I, J, K : TM \to TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\text{Id}.$$ 

Suppose that $I, J, K$ are integrable almost complex structures. Then

$$(M, I, J, K)$$

is called a **hypercomplex manifold**.
EXAMPLES:

Compact hypercomplex manifolds which are not hyperkähler

1. In dimension 1 (real dimension 4), we have a complete classification, due to C. P. Boyer (1988)


4. Some inhomogeneous examples are constructed by deformation or as fiber bundles.

In dimension > 1, no classification results are known (and no conjectures either).
OBATA CONNECTION

Hypercomplex manifolds can be characterized in terms of holonomy

**Theorem:** (M. Obata, 1952) Let \((M, I, J, K)\) be a hypercomplex manifold. Then \(M\) admits a unique torsion-free affine connection preserving \(I, J, K\).

**Converse is also true.** Suppose that \(I, J, K\) are operators defining quaternionic structure on \(TM\), and \(\nabla\) a torsion-free, affine connection preserving \(I, J, K\). Then \(I, J, K\) are integrable almost complex structures, and \((M, I, J, K)\) is hypercomplex.

**Holonomy of Obata connection lies in** \(GL(n, \mathbb{H})\). A manifold equipped with an affine, torsion-free connection with holonomy in \(GL(n, \mathbb{H})\) is hypercomplex.

This can be used as a definition of a hypercomplex structure.
QUESTIONS

1. Given a complex manifold $M$, when $M$ admits a hypercomplex structure? How many?

2. What are possible holonomies of Obata connection, for a compact hypercomplex manifold?

3. Describe the structure of automorphism group of a hypercomplex manifold.

**THEOREM:** Let $(M, I, J, K)$ be a compact hypercomplex manifold. Assume that the complex manifold $(M, I)$ admits a Kähler structure. Then $(M, I)$ is hyperkähler.
**Quaternionic Hermitian structures**

**DEFINITION:** Let \((M, I, J, K)\) be a hypercomplex manifold, and \(g\) a Riemannian metric. We say that \(g\) is quaternionic Hermitian if \(I, J, K\) are orthogonal with respect to \(g\).

Given a quaternionic Hermitian metric \(g\) on \((M, I, J, K)\), consider its Hermitian forms

\[
\omega_I(\cdot, \cdot) = g(\cdot, I\cdot), \omega_J, \omega_K
\]

(real, but not closed). Then \(\Omega = \omega_J + \sqrt{-1} \omega_K\) is of Hodge type \((2,0)\) with respect to \(I\).

If \(d\Omega = 0\), \((M, I, J, K, g)\) is hyperkähler (this is one of the definitions).

Consider a weaker condition:

\[
\partial \Omega = 0, \quad \partial : \Lambda^{2,0}(M, I) \longrightarrow \Lambda^{3,0}(M, I)
\]
**DEFINITION:** (Howe, Papadopoulos, 1998)

Let $(M, I, J, K)$ be a hypercomplex manifold, $g$ a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1} \omega_K$ the corresponding $(2,0)$-form. We say that $g$ is **HKT** ("weakly hyperkähler with torsion") if

$$\partial \Omega = 0.$$

HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.

1. They admit a smooth potential (locally). There is a notion of an "HKT-class" (similar to Kähler class) in a certain finite-dimensional cohomology group. Two metrics in the same HKT-class differ by a potential, which is a function.

2. When $(M, I)$ has trivial canonical bundle, a version of Hodge theory is established giving an $\mathfrak{sl}(2)$-action on cohomology.
Canonical bundle of a hypercomplex manifold.

0. Quaternionic Hermitian structure always exists.

1. Complex dimension is even.

2. The canonical line bundle $\Lambda^{n,0}(M, I)$ of $(M, I)$ is always trivial topologically. Indeed, a non-degenerate section of canonical line bundle is provided by top power of a form $\Omega$ associated with some quaternionic Hermitian structure. In particular,

$$c_1(M, I) = 0.$$ 

It is non-trivial holomorphically in many cases. However, $\Lambda^{n,0}(M, I)$ is trivial and holonomy of Obata connection lies in $SL(n, \mathbb{H})$ when $M$ is a nilmanifold (Barberis-Dotti-V., 2007)
The group $SU(2)$ of unitary quaternions acts on $TM$, because quaternion algebra acts. By multilinearity, this action is extended to $\Lambda^*(M)$.

1. The Hodge decomposition $\Lambda^*(M) = \bigoplus \Lambda^{p,q}(M)$ is recovered from this $SU(2)$-action. “Hypercomplex analogue of the Hodge decomposition”.

2. $\langle \omega_I, \omega_J, \omega_K \rangle$ is an irreducible 3-dimensional representation of $SU(2)$, for any quaternionic Hermitian structure (“representation of weight 2”).

**WEIGHT of a representation.**

We say that an irreducible $SU(2)$-representation $W$ has weight $i$ if $\dim W = i + 1$. A representation is said to be pure of weight $i$ if all its irreducible components have weight $i$. If all irreducible components of a representation $W_1$ have weight $\leq i$, we say that $W_1$ is a representation of weight $\leq i$. In a similar fashion one defines representations of weight $\geq i$. 

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The weight is multiplicative, in the following sense: a tensor product of representations of weights $\leq i$ and $\leq j$ has weight $\leq i + j$.

Clearly, $\Lambda^1(M)$ has weight 1. Therefore, $\Lambda^i(M)$ has weight $\leq i$.

Let $V^i \subset \Lambda^i(M)$ be the maximal $SU(2)$-invariant subspace of weight $< i$.

By multiplicativity, $V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$. We also have $V^i = \Lambda^i(M)$ for $i > 2n$. Also, $dV^i \subset V^{i+1}$, hence $V^* \subset \Lambda^*(M)$ is a differential ideal in $(\Lambda^i(M), d)$.

Denote by $(\Lambda^*_+(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$.
We call it the quaternionic Dolbeault algebra (qD-algebra) of $M$.

A similar construction was given by Salamon in a more general situation.
The quaternionic Dolbeault algebra can be computed explicitly, in terms of the Hodge decomposition.

The Hodge decomposition is induced from the $SU(2)$-action, hence it is **compatible with weights**:

$$\Lambda^i(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_+, I(M).$$

Let $\sqrt{-1} I$ be an element of the Lie algebra $\mathfrak{su}(2) \otimes \mathbb{C}$ acting as $\sqrt{-1} (p - q)$ on $\Lambda^{p,q}(M)$. This vector generates the Cartan algebra of $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$-action induces an isomorphism of $\Lambda^{p,q}_+, I(M)$ for all $\{p, q|p + q = k, p, q \geq 0\}$. This gives

**Theorem:**

$$\Lambda^{p,q}_+, I(M) \cong \Lambda^{0,p+q}(M, I).$$

This isomorphism is provided by the $\mathfrak{su}(2) \otimes \mathbb{C}$-action.

Indeed, the space $\Lambda^{0,p}(M, I) \subset \Lambda^p(M)$ is pure of weight $p$, hence $\Lambda^{0,p}(M, I)$ coincides with $\Lambda^{0,p}_+, I(M)$.
Differentials in the qD-complex

We extend $J : \Lambda^1(M) \longrightarrow \Lambda^1(M)$ to $\Lambda^*(M)$ by multiplicativity. Since $I$ and $J$ anticommute on $\Lambda^1(M)$, we have $J(\Lambda^{p,q}(M, I)) = \Lambda^{q,p}(M, I)$.

Denote by

$$\partial J : \Lambda^{p,0}(M, I) \longrightarrow \Lambda^{p,0}(M, I)$$

the operator $J \circ \overline{\partial} \circ J$, where $\overline{\partial} : \Lambda^{0,p}(M, I) \longrightarrow \Lambda^{0,p}(M, I)$ is the standard Dolbeault differential. Then $\partial, \partial J$ anticommute. Moreover, there exists a multiplicative isomorphism of bicomplexes.
\[
\Lambda_0 + (M) + \bigtriangleup + \Lambda_1 + (M) + \downarrow + \Lambda_2 + (M) + \downarrow + \Lambda_3 + (M) + \downarrow + \cdots \\
\Leftrightarrow \\
\Lambda^0_1 + (M) + \bigtriangleup + \Lambda^1_2 + (M) + \downarrow + \Lambda^2_3 + (M) + \downarrow + \Lambda^3_4 + (M) + \downarrow + \cdots 
\]
Potentials for HKT-metrics

A quaternionic Hermitian metric can be recovered from the corresponding (2, 0)-form: \( \omega_I(x, \bar{y}) = \frac{1}{2} \Omega(x, J(y)) \), where \( x, y \in T^{1,0}(M) \). The HKT-structures uniquely correspond to (2, 0)-forms which are

1. **Real:** \( J(\Omega) = \overline{\Omega} \)

2. **Closed:** \( \partial \Omega = 0 \).

2. **Positive:** \( \Omega(x, J(x)) > 0 \), for any non-zero \( x \in T^{1,0}(M) \)

Locally, any HKT-metric is given by a potential: \( \Omega = \partial \partial_J \varphi \) where \( \varphi \) is a smooth function.

Any convex, and any strictly plurisubharmonic function is a potential of some HKT-structure. Therefore, HKT-structures locally always exist.
Quaternionic Monge-Ampere equation

Let $M$ be an HKT-manifold with holonomy in $SL(n, \mathbb{H})$. (this is equivalent to having trivial canonical bundle). Then the canonical bundle is trivialized by a form $\Phi_I \in \Lambda^{2n,0}$, non-degenerate and satisfying $J(\Phi_I) = \overline{\Phi_I}$.

Quaternionic Monge-Ampere equation:

$$(\Omega + \partial \bar{\partial} j \varphi)^n = A_f e^f \Phi_I$$

This equation can be rewritten as a generalized Monge-Ampere, hence solutions are unique.
Let $R : \Lambda^{2p,0}(M) \rightarrow \Lambda^{p,p+1}(M)$ be the isomorphism provided by $\mathfrak{su}(2)$-action as above.

We call a $(2p,0)$-form $\eta$ **real** if $J(\eta) = \overline{\eta}$, and **positive** if $\eta(x, J(x)) \geq 0$.

**Theorem 1:** Let $M$ be an HKT-manifold with holonomy in $SL(n, \mathbb{H})$, and $\Phi_I \in \Lambda^{2n,0}$ the corresponding real trivialization of a canonical bundle. Then $R(\Phi_I)$ is closed and positive (replace $\Phi_I$ by $-\Phi_I$ if necessary). Moreover, for any $(2p,0)$-form $\eta$, the form $R(\eta) \wedge R(\Phi_I)$ is

1. **Real if and only if** $\eta$ **is real.**

2. **Positive if and only if** $\eta$ **is positive.**

3. **Closed if and only if** $\partial \eta = 0$.

4. $R(\partial \partial J \varphi) \wedge R(\Phi_I) = \partial \overline{\partial} \varphi \wedge R(\Phi_I)$. 

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Quaternionic Monge-Ampere and complex geometry

The quaternionic Monge-Ampere is equivalent to

\[ R(\Phi_I) \wedge R(\Omega + \partial\bar{\partial} J \varphi)^n = A f^f \text{Vol}_M. \]

by Theorem 1 this is the same as

\[ R(\Phi_I) \wedge (\omega_I + \partial\bar{\partial} \varphi)^n = A f^f \text{Vol}_M. \]

This is generalized Monge-Ampere!