Quaternionic Monge-Ampere equation

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Generalized Monge-Ampere equation on complex manifold.

Data: M - compact complex n-manifold, Φ a closed, positive (p,p)-form. A (k,k)-form ν is called **(strictly) Φ-positive** if $\nu \wedge \eta \wedge \Phi$ is (strictly) positive, for any (i,i)-form η for which $\eta \wedge \Phi$ is (strictly) positive. A (k,k)-form ν is called **Φ-closed** if $\nu \wedge \eta \wedge \Phi$ is closed for for any (i,i)-form η for which $\eta \wedge \Phi$ is closed. A $(1,1)$-form ν is called Φ-Kähler if it is strictly Φ-positive and Φ-closed.

Example:

$$\Phi = (-\sqrt{-1}\,)^p \zeta_1 \wedge \overline{\zeta}_1 \wedge ... \wedge \zeta_p \wedge \overline{\zeta}_p,$$

where ζ_i are holomorphic $(1,0)$-forms. Then Φ-positivity (closedness) means positivity (closedness) on the complex foliation defined by $\bigcap \ker \zeta_i$.

Generalized Monge-Ampere equation:

$$(\omega + \partial\overline{\partial}\varphi)^{n-p} \wedge \Phi = A_f e^f \operatorname{Vol}_M$$

Here f is an arbitrary function, φ a solution we are looking for, A_f is a constant, dependent on f, ω a Φ-Kähler form, Vol_M is a fixed volume form on M, $\omega + \partial\overline{\partial}\varphi$ is Φ-Kähler.

Observation 1: Solutions are unique. Indeed, suppose that ω and $\omega_1 = \omega + \partial\overline{\partial}\varphi$ are both solutions. Then

$$0 = (\omega + \partial\overline{\partial}\varphi)^{n-p} \wedge \Phi - \omega^{n-p} \wedge \Phi = \partial\overline{\partial}\varphi \wedge \left[\left(\sum_{k=1}^{n-p-1} \omega_1^k \wedge \omega^{n-p-k}\right) \wedge \Phi\right]$$

The form in brackets is **strictly positive.** When φ reaches maximum, or minimum, $\partial\overline{\partial}\varphi \wedge \left[...\right]$ cannot vanish ("generalized maximum principle").

Generalized Monge-Ampere equation: more observations

2. M is not required to be Kähler. One obtains interesting structures on non-Kähler manifolds (such as hypercomplex ones).

3. Hessian equation. When M is Kähler, and $\Phi = \omega^n$, we have

$$(\omega + \partial\overline{\partial}\varphi)^{n-p} \wedge \Phi = P_{n-p}(a_1, ...a_n)$$

where a_i are eigenvalues of the Hermitian form $\omega + \partial\overline{\partial}\varphi$, and P_{n-p} the fundamental symmetric polynomial. This is called **complex Hessian equation** (Błocki etc.).

Conjecture: Under reasonable assumptions, solution exists.

Question 1: Is it known for foliations?

Question 2: What is known about existence of solutions for complex Hessian equation?

Definition: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators

$$I,J,K: TM \longrightarrow TM,$$

satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **hyperkähler**.

Holonomy of a hyperkähler manifold is $Sp(n)$.

Indeed, Levi-Civita connection preserves I, J, K, because M is Kähler. The group of matrices preserving quaternionic structure and metric is $Sp(n)$.

Converse is also true. Hyperkähler manifolds are often *defined* as manifolds with affine connection and holonomy in $Sp(n)$.

Holomorphic symplectic geometry

A hyperkähler manifold (M, I, J, K, g), considered as a complex manifold (M, I), is holomorphically symplectic (equipped with a holomorphic, non-degenerate 2-form). Recall that M is equipped with 3 symplectic forms $\omega_I(\cdot,\cdot) = g(\cdot, I\cdot)$, ω_J, ω_K.

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic 2-form on (M, I). ■

Converse is also true.

THEOREM: (E. Calabi, 1952, S.-T. Yau, 1978) Let M be a compact, holomorphically symplectic Kähler manifold. Then M admits a hyperkähler metric, which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

HYPERCOMPLEX MANIFOLDS

"Hyperkähler manifolds without a metric"

Definition: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are integrable almost complex structures. Then

$$(M, I, J, K)$$

is called **a hypercomplex manifold**.

EXAMPLES:

Compact hypercomplex manifolds which are not hyperkähler

1. In dimension 1 (real dimension 4), we have a complete classification, due to C. P. Boyer (1988)

2. Many homogeneous examples, due to D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen (1980-ies, early 1990-ies).

3. Some nilmanifolds admit homogeneous hypercomplex structure (M. L. Barberis, I. Dotti, A. Fino) (1990-ies).

4. Some inhomogeneous examples are constructed by deformation or as fiber bundles.

In dimension > 1, no classification results are known (and no conjectures either).

OBATA CONNECTION

Hypercomplex manifolds can be characterized in terms of holonomy

Theorem: (M. Obata, 1952) Let (M, I, J, K) be a hypercomplex manifold. Then M admits a unique torsion-free affine connection preserving I, J, K.

Converse is also true. Suppose that I, J, K are operators defining quaternionic structure on TM, and ∇ a torsion-free, affine connection preserving I, J, K. Then I, J, K are integrable almost complex structures, and (M, I, J, K) is hypercomplex.

Holonomy of Obata connection lies in $GL(n, \mathbb{H})$. *A manifold equipped with an affine, torsion-free connection with holonomy in $GL(n, \mathbb{H})$ is hypercomplex*.

This can be used as a definition of a hypercomplex structure.

QUESTIONS

1. Given a complex manifold M, when M admits a hypercomplex structure? How many?

2. What are possible holonomies of Obata connection, for a compact hypercomplex manifold?

3. Describe the structure of automorphism group of a hypercomplex manifold.

THEOREM: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that the complex manifold (M, I) admits a Kähler structure. Then (M, I) is hyperkähler.

Quaternionic Hermitian structures

DEFINITION: Let (M,I,J,K) be a hypercomplex manifold, and g a Riemannian metric. We say that g is quaternionic Hermitian if I,J,K are orthogonal with respect to g.

Given a quaternionic Hermitian metric g on (M,I,J,K), consider its Hermitian forms

$$\omega_I(\cdot,\cdot) = g(\cdot, I\cdot), \omega_J, \omega_K$$

(real, but *not closed*). Then $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is of Hodge type (2,0) with respect to I.

If $d\Omega = 0$, (M,I,J,K,g) is hyperkähler (this is one of the definitions).

Consider a weaker condition:

$$\partial\Omega = 0, \quad \partial: \Lambda^{2,0}(M,I) \longrightarrow \Lambda^{3,0}(M,I)$$

DEFINITION: (Howe, Papadopoulos, 1998)
Let (M, I, J, K) be a hypercomplex manifold, g a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ the corresponding (2,0)-form. We say that g is **HKT ("weakly hyperk "ahler with torsion")** if

$$\partial\Omega = 0.$$

HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.

1. They admit a smooth potential (locally). There is a notion of an "HKT-class" (similar to Kähler class) in a certain finite-dimensional cohology group. Two metrics in the same HKT-class differ by a potential, which is a function.

2.When (M, I) has trivial canonical bundle, a version of Hodge theory is established giving an $\mathfrak{sl}(2)$-action on cohomology.

Canonical bundle of a hypercomplex manifold.

0. Quaternionic Hermitian structure always exists.

1. Complex dimension is even.

2. The canonical line bundle $\Lambda^{n,0}(M,I)$ of (M,I) is always trivial topologically. Indeed, a non-degenerate section of canonical line bundle is provided by top power of a form Ω associated with some quaternionic Hermitian strucure. In particular,

$$c_1(M,I) = 0.$$

It is non-trivial holomorphically in many cases. However, $\Lambda^{n,0}(M,I)$ is trivial and holonomy of Obata connection lies in $SL(n,\mathbb{H})$ when M is a nilmanifold (Barberis-Dotti-V., 2007)

$SU(2)$-action on $\Lambda^*(M)$

The group $SU(2)$ of unitary quaternions acts on TM, because quaternion algebra acts. By multilinearity, this action is extended to $\Lambda^*(M)$.

1. The Hodge decomposition $\Lambda^*(M) = \bigoplus \Lambda^{p,q}(M)$ is recovered from this $SU(2)$-action. "Hypercomplex analogue of the Hodge decomposition".

2. $\langle \omega_I, \omega_J, \omega_K \rangle$ is an irreducible 3-dimensional representation of $SU(2)$, for any quaternionic Hermitian structure ("representation of weight 2").

WEIGHT of a representation.

We say that an irreducible $SU(2)$-representation W **has weight** i if $\dim W = i+1$. A representation is said to be **pure of weight** i if all its irreducible components have weight i. If all irreducible components of a representation W_1 have weight $\leqslant i$, we say that W_1 **is a representation of weight** $\leqslant i$. In a similar fashion one defines representations of weight $\geqslant i$.

Quaternionic Dolbeault algebra

The weight is multiplicative, in the following sense: a tensor product of representations of weights $\leqslant i$ and $\leqslant j$ has weight $\leqslant i + j$.

Clearly, $\Lambda^1(M)$ has weight 1. Therefore, $\Lambda^i(M)$ **has weight** $\leqslant i$**.**

Let $V^i \subset \Lambda^i(M)$ be the maximal $SU(2)$-invariant subspace of weight $< i$.

By multiplicativity, $V^* = \bigoplus_i V^i$ **is an ideal in** $\Lambda^*(M)$. We also have $V^i = \Lambda^i(M)$ for $i > 2n$. Also, $dV^i \subset V^{i+1}$, hence $V^* \subset \Lambda^*(M)$ is a differential ideal in $(\Lambda^i(M), d)$.

Denote by $(\Lambda^*_+(M), d_+)$ **the quotient algebra** $\Lambda^*(M)/V^*$**.**
We call it **the quaternionic Dolbeault algebra (qD-algebra)** of M.

A similar construction was given by Salamon in a more general situation.

The quaternionic Dolbeault algebra can be computed explicitly, in terms of the Hodge decomposition.

The Hodge decomposition is induced from the $SU(2)$-action, hence it is **compatible with weights**: $\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M)$.

Let $\sqrt{-1}\,\mathcal{I}$ be an element of the Lie algebra $\mathfrak{su}(2)\otimes\mathbb{C}$ acting as $\sqrt{-1}\,(p-q)$ on $\Lambda^{p,q}(M)$. This vector generates the Cartan algebra of $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$-action induces an isomorphism of $\Lambda^{p,q}_{+,I}(M)$ for all $\{p,q | p+q=k,\ \ p,q \geqslant 0\}$. This gives

Theorem:

$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{0,p+q}(M,I).$$

This isomorphism is provided by the $\mathfrak{su}(2)\otimes\mathbb{C}$-action.

Indeed, the space $\Lambda^{0,p}(M,I) \subset \Lambda^p(M)$ is pure of weight p, hence $\Lambda^{0,p}(M,I)$ coincides with $\Lambda^{0,p}_{+,I}(M)$

Differentials in the qD-complex

We extend $J: \Lambda^1(M) \longrightarrow \Lambda^1(M)$ to $\Lambda^*(M)$ by multiplicativity. Since I and J anticommute on $\Lambda^1(M)$, **we have** $J(\Lambda^{p,q}(M,I)) = \Lambda^{q,p}(M,I)$.

Denote by

$$\partial_J: \Lambda^{p,0}(M,I) \longrightarrow \Lambda^{p,0}(M,I)$$

the operator $J \circ \overline{\partial} \circ J$, where $\overline{\partial}: \Lambda^{0,p}(M,I) \longrightarrow \Lambda^{0,p}(M,I)$ is the standard Dolbeault differential. Then ∂, ∂_J anticommute. Moreover, **there exists a multiplicative isomorphism of bicomplexes**.

$$
\begin{array}{ccccccc}
 & & \Lambda^0_+(M) & & \\
 & \swarrow^{d_+^{1,0}} & & \searrow^{d_+^{0,1}} & \\
 & \Lambda^{1,0}_+(M) & & \Lambda^{0,1}_+(M) & \\
 \swarrow^{d_+^{1,0}} & & \searrow^{d_+^{0,1}} \; \swarrow^{d_+^{1,0}} & & \searrow^{d_+^{0,1}} \\
\Lambda^{2,0}_+(M) & & \Lambda^{1,1}_+(M) & & \Lambda^{0,2}_+(M) \\
 & & \dots & &
\end{array}
\quad \cong \quad
\begin{array}{ccccccc}
 & & \Lambda^{0,0}_I(M) & & \\
 & \swarrow^{\partial_J} & & \searrow^{\partial} & \\
 & \Lambda^{1,0}_I(M) & & \Lambda^{1,0}_I(M) & \\
 \swarrow^{\partial_J} & & \searrow^{\partial} \; \swarrow^{\partial_J} & & \searrow^{\partial} \\
\Lambda^{2,0}_I(M) & & \Lambda^{2,0}_I(M) & & \Lambda^{2,0}_I(M) \\
 & & \dots & &
\end{array}
$$

Potentials for HKT-metrics

A quaternionic Hermitian metric **can be recovered from the corresponding (2,0)-form:** $\omega_I(x,\overline{y}) = \frac{1}{2}\Omega(x, J(\overline{y}))$, where $x, y \in T^{1,0}(M)$. The HKT-structures uniquely correspond to (2,0)-forms which are

1. Real: $J(\Omega) = \overline{\Omega}$

2. Closed: $\partial\Omega = 0$.

2. Positive: $\Omega(x, J(\overline{x})) > 0$, for any non-zero $x \in T^{1,0}(M)$

Locally, any HKT-metric is given by a potential: $\Omega = \partial\partial_J\varphi$ where φ is a smooth function.

Any convex, and any strictly plurisubharmonic function is a potential of some HKT-structure. Therefore, HKT-structures locally always exist.

Quaternionic Monge-Ampere equation

Let M be an HKT-manifold with holonomy in $SL(n,\mathbb{H})$. (this is equivalent to having trivial canonical bundle). Then the canonical bundle is trivialized by a form $\Phi_I \in \Lambda^{2n,0}$, non-degenerate and satisfying $J(\Phi_I) = \overline{\Phi}_I$.

Quaternionic Monge-Ampere equation:

$$(\Omega + \partial\partial_J \varphi)^n = A_f e^f \Phi_I$$

This equation can be rewritten as a generalized Monge-Ampere, hence **solutions are unique**.

Let $R: \Lambda^{2p,0}(M) \longrightarrow \Lambda^{p,p}_{+,I}(M)$ be the isomorphism provided by $\mathfrak{su}(2)$-action as above.

We call a $(2p,0)$-form η **real** if $J(\eta) = \overline{\eta}$, and **positive** if $\eta(x, J(\overline{x})) \geqslant 0$.

Theorem 1: Let M be an HKT-manifold with holonomy in $SL(n,\mathbb{H})$, and $\Phi_I \in \Lambda^{2n,0}$ the corresponding real trivialization of a canonical bundle. Then $R(\Phi_I)$ is closed and positive (replace Φ_I by $-\Phi_I$ if necessary). Moreover, for any $(2p,0)$-form η, the form $R(\eta) \wedge R(\Phi_I)$ is

1. Real if and only if η is real.

2. Positive if and only if η is positive.

3. Closed if and only if $\partial\eta = 0$.

4. $R(\partial\partial_J\varphi) \wedge R(\Phi_I) = \partial\overline{\partial}\varphi \wedge R(\Phi_I)$.

Quaternionic Monge-Ampere and complex geometry

The quaternionic Monge-Ampere is equivalent to

$$R(\Phi_I) \wedge R(\Omega + \partial\partial_J\varphi)^n = A_f e^f \operatorname{Vol}_M .$$

by Theorem 1 this is the same as

$$R(\Phi_I) \wedge (\omega_I + \partial\overline{\partial}\varphi)^n = A_f e^f \operatorname{Vol}_M .$$

This is generalized Monge-Ampere!