Principal Toric Fibrations

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Non-linear integral transforms: Fourier-Mukai and Nahm,

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Definition:
A complex principal toric fibration (bundle) M is a complex manifold equipped with a free holomorphic action of a compact complex torus T.

Such a manifold is fibered over M/T, with fiber T.

It is a principal T-bundle: all fibers are identified with T, with T acting on fibers freely.

Can consider this notion in smooth category as well (remove "complex" and "holomorphic" from this definition).

To trivialize a principal group bundle it means to find a section (holomorphic section for complex trivialization, smooth for topological).

Motivation:

(E. Calabi, 1978)
Let (M, g) be a Riemannian manifold equipped with three complex structure operators $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M, I, J, K, g) is called **hyperkähler**.

Have good control on algebraic geometry. (Matsushita, etc).

Definition: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are integrable. Then (M, I, J, K) is called **a hypercomplex manifold**.

Have no control on algebraic geometry of the underlying complex manifold.

Examples:

1. **Hopf manifolds.** Also, locally conformally hyperkähler manifolds.

2. **Compact Lie groups** (D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen).

$$\begin{aligned}&T^4, \quad SU(2l+1), \quad T^1 \times SU(2l), \quad T^l \times SO(2l+1),\\ &T^{2l} \times SO(4l), \quad T^l \times Sp(l), \quad T^2 \times E_6,\\ &T^7 \times E^7, \quad T^8 \times E^8, \quad T^4 \times F_4, \quad T^2 \times G_2\end{aligned}$$

where T^i denotes an i-dimensional compact torus.

3. **Nilmanifolds** (a quotient of a nilpotent Lie group over a cocompact lattice).

Most of known non-hyperkäler hypercomplex manifolds are equipped with a free action of a holomorphic torus.

Topology of principal toric bundles
(not necessarily complex)

A principal T^n-bundle over X is defined topologically by $H^1(X, \mathbb{T})$, where $\mathbb{T}^n$ is a sheaf of smooth T^n-valued functions on X. An exact sequence

$$0 \longrightarrow \Gamma \longrightarrow C^\infty(M)^n \longrightarrow \mathbb{T}^n \longrightarrow 0,$$

gives $H^1(\mathbb{T}^n) = H^2(M, \Gamma)$, where $\Gamma = \pi_1(T)$ If Denote by

$$\tau : \ H^1(T, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{Z})$$

the map which corresponds to the $H^1(X, \mathbb{T}^n)$-class of a fiber bundle.

A principal fiber bundle is determined, up to a topological equivalence, by this invariant. Also, any such τ corresponds to a principal fiber bundle.

Example: A principal S^1-bundle is a determined by its Chern class c_1 in $H^2(M)$.

The transfer map

Using the Leray-Serre spectral sequence, it is easy to express the cohomology of M in terms of $H^*(X)$ and the Chern classes. This gives an exact sequence

$$0 \longrightarrow H^1(X) \longrightarrow H^1(M) \longrightarrow H^1(T) \xrightarrow{d_2} H^2(X) \longrightarrow H^2(M)$$

with d_2 (a differential in Leray-Serre spectral sequence), called **the transfer map**. It is easy to see that $\tau = d_2$.

Examples of principal toric bundles (in smooth category):

1. "Hopf fibration". S^3 fibered over S^2, with fiber S^1, and the Chern class 1. It is a total space of $U(1)$-bundle over $\mathbb{C}P^1$, which is denoted as $\mathcal{O}(-1)$.

2. A generalization of this example. S^{2n+1} is fibered over $\mathbb{C}P^n$, with fiber S^1. Again, it is a total space of $U(1)$-bundle, corresponding to $\mathcal{O}(-1)$.

3. G a Lie group, $T \subset G$ a torus, G fibered over G/T.

4. Nilmanifolds (manifolds with transitive action of a nilpotent Lie group) always admit principal toric fibrations.

Complex principal toric bundles (complex manifolds with a free, holomorphic action of a complex torus T).

1. **The (classical) Hopf surface**. Fix $\alpha \in \mathbb{C}$, $|\alpha| > 1$. Consider the quotient $H = (\mathbb{C}^2 \backslash 0)/\langle\mathbb{Z}\rangle$, with $\mathbb{Z}$ acting on $\mathbb{C}^2$ by $(x,y) \longrightarrow (\alpha x, \alpha y)$. It is called **the Hopf surface**. Topologically the Hopf surface is isomorphic to $S^1 \times S^3$ (hence, non-Kähler). The elliptic curve $T^2 = \mathbb{C}^*/\langle\alpha\rangle$ acts on H by $t, (x,y) \longrightarrow (tx, ty)$. This action is free, and its quotient is $\mathbb{C}P^1$. The Hopf surface is a principal elliptic fibration. Topologically, it's a product of a Hopf fibration $S^3 \longrightarrow S^2$ and a circle.

2. **A generalization of this example.** Let X be a complex manifold, and L a holomorphic line bundle on X. Consider a principal $\mathbb{C}^*$-bundle $\operatorname{Tot}(L^*)$ over X (total space of L without a zero section). Taking a quotient

$$\operatorname{Tot}(L^*)/\langle\mathbb{Z}\rangle, \quad \text{with } \mathbb{Z} \text{ acting as } v \mapsto \alpha v,$$

we obtain, again, a principal elliptic bundle, with fiber $T^2 = \mathbb{C}^*/\langle\alpha\rangle$. When $X = \mathbb{C}P^1$, $L = \mathcal{O}(-1)$, this gives a Hopf surface.

Complex principal toric bundles (cont.)

Using the Leray-Serre spectral sequence

$$0 \longrightarrow H^1(X) \longrightarrow H^1(M) \longrightarrow H^1(T) \xrightarrow{d_2} H^2(X)$$

and the fact that $\operatorname{im} d_2 = \langle c_1(L) \rangle$, we obtain that $H^1(M)$ is odd-dimensional (hence, cannot be Kähler), for any $M = \operatorname{Tot}(L^*)/\langle \mathbb{Z} \rangle$, with $c_1(L)$ nonzero over $\mathbb{Q}$.

3. **Calabi-Eckmann manifolds.**
Fix $\alpha \in \mathbb{C}$, α non-real, $|\alpha| > 1$. Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within $\mathbb{C}^* \times \mathbb{C}^*$. It is clearly co-compact and closed, with $\mathbb{C}^* \times \mathbb{C}^*/G$ being an elliptic curve $\mathbb{C}^*/\langle \alpha \rangle$.

Now, let

$$M = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/G,$$

with $G \subset \mathbb{C}^* \times \mathbb{C}^*$ acting on $(\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)$ by $(t_1, t_2)(x, y) \longrightarrow (t_1 x, t_2 y)$. Clearly, M is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber $\mathbb{C}^* \times \mathbb{C}^*/G$, which is an elliptic curve. The fibration $M \longrightarrow \mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ is called **the Calabi-Eckmann fibration**, its total space M **the Calabi-Eckmann manifold**. It is diffeomorphic to $S^{2n-1} \times S^{2m-1}$.

We obtained a homogeneous complex structure on $S^{2n-1} \times S^{2m-1}$.

It is non-Kähler, because $H^2(M) = 0$.

Borel-Remmert-Tits theorem:
Let M be a compact, complex, simply connected homogeneous manifold ("homogeneous" means that $G = \operatorname{Aut}(M)$ acts on M transitively). Then M is a principal toric fibration, with a base which is a homogeneous, rational projective manifold.

Proof: Let $K^{-1} = \Lambda^{top}(TM)$ be the anticanonical class of M. Since TM is globally generated, the same is true for K^{-1}. This gives a G-invariant morphism

$$M \xrightarrow{\pi} \mathbb{P}H^0(K^{-1}).$$

The fibers F of π are homogeneous with trivial canonical class, and its base is homogeneous and projective (hence, rational). The fundamental group of F is a quotient of $\pi_2(X)$, as follows from the long exact sequence of homotopy groups for a Serre's fibration:

$$\pi_2(X) \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) = 0$$

Therefore, $\pi_1(F)$ is abelian. **It remains to show that it is a torus.**

Lemma: Let F be a compact, complex, homogeneous manifold with $\pi_1(F)$ abelian and a trivial anticanonical class K^{-1}. Then F is a torus.

Proof: The sheaf of holomorphic vector fields on M is globally generated. Taking a vector field v_1 and multiplying it by general vector fields $v_2, ...v_n$, we obtain a section of K^{-1}, which is non-zero for general v_i, and therefore non-degenerate. We obtain that v_i are linearly independent everywhere. Taking the corresponding flows of diffeomorphisms, we obtain that F is a quotient of a holomorphic Lie group G by a cocompact lattice. Since $\pi_1(F)$ is abelian, G is commutative, and T is a torus.

Examples of homogeneous complex manifolds

1. Calabi-Eckmann and Hopf manifolds.

2. Tori.

3. Let G be a compact, even-dimensional Lie group. Then G admits a left-invariant complex structure, constructed by H. Samelson in 1953. Important in physics (being also hypercomplex, due to Joyce and Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen. Samelson's manifolds admit (4,4)-supersymmetry, the only examples known).

Positive line bundles.

Let X be a complex manifold, and L a holomorphic line bundle. L is called **positive**, or **ample** if for sufficiently big N, $L^{\otimes N}$ is globally generated, and, moreover, the natural map

$$X \longrightarrow \mathbb{P}(H^0(L^{\otimes N}))$$

is an embedding. In this case $L^{\otimes N}$ is called **very ample**.

Theorem (Kodaira-Nakano):
A holomorphic line bundle is ample if and only if it admits a Hermitian metric, with curvature Θ which satisfies $-\sqrt{-1}\,\Theta(z,\overline{z}) > 0$ for any non-zero vector $z \in T^{1,0}(M)$.

This means that $-\sqrt{-1}\,\Theta(\cdot, I\cdot)$ is a Kähler metric on X.

Positive elliptic fibrations.

Definition: Let $M \xrightarrow{\pi} X$ be an elliptic fibration, M compact. We say that M is **positive elliptic fibration**, if for some Kähler class ω on X, $\pi^*\omega$ is exact.
"Kähler class" is a cohomology class of a Kähler form.

Examples:

1. Hopf manifold, $H^2(M) = 0$, hence positive

2. Calabi-Eckmann manifold (same)

3. $SU(3)$ is elliptically fibered over the flag manifold $F(2,3)$, also $H^2(M) = 0$.

4. $\operatorname{Tot}(L^*)/\langle\mathbb{Z}\rangle$, where L is an ample line bundle. Such manifold is called **a regular Vaisman manifold**. It is positive, because $\pi^*(c_1(L)) = 0$, and $c_1(L)$ is a Kähler class.

It is possible to interpret τ as a "curvature class" of a fibration, and when it is Kähler, we can say that a fibration is positive. This happens precisely when the image of τ contains a Kähler class.

Subvarieties of positive elliptic fibrations

Theorem: Let $M \xrightarrow{\pi} X$ be a positive elliptic T fibration, and $Z \subset M$ be a subvariety, of positive dimension m. Then Z is T-invariant.

Proof: Let $\omega_0 = \pi^*\omega$ be a pullback of a Kähler form which is exact. Then

$$\int_Z \omega_0^m = 0.$$

On the other hand, all eigenvalues of $\omega_0\big|_Z$ are non-negative, and all are positive, unless Z is tangent to the action of T. In a point where Z is not tangent to T, the form ω_0^m is positive, and in this case the integral $\int_Z \omega_0^m$ is also positive.

A similar result is true for stable coherent sheaves.

Theorem: Let $M \xrightarrow{\pi} X$, $\dim_{\mathbb{C}} X > 1$, be a positive elliptic T fibration, and F a stable reflexive sheaf on M. Then $F \cong L \otimes \pi^* F_0$, where L is a line bundle, and F_0 a stable coherent sheaf on X.

Positive toric fibrations

Definition: Let $M \xrightarrow{\pi} X$ be a complex principal toric fibration, M compact, with fiber T. Assume that the image of $\tau : H^1(T, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C})$ contains a Kähler form. Then the fibration $M \xrightarrow{\pi} X$ is called **convex**. *NB:* Can define convexity for arbitrary fiber bundles.

Consider a holomorphic quotient $T_1 = T/T_2$ of T. Taking the quotient space M/T_2, we obtain a complex principal toric fibration, with fiber T_1.

Assume that for all $T_1 = T/T_2$, $\dim T_1 > 0$, the induced fibration $M/T_2 \longrightarrow X$ is also convex. Then $M \xrightarrow{\pi} X$ is called **positive**.

Example. Let M be a complex, compact homogeneous manifold with $H^2(M) = 0$ (e.g. a Lie group), and $M \xrightarrow{\pi} X$ the Borel-Remmert-Tits toric fibration. Assume that the fibers of π have no proper subtori (easy to insure by taking a generic invariant complex structure). Then M is positive.

Theorem. Consider an irreducible complex subvariety $Z \subset M$ of a positive principal toric fibration $M \xrightarrow{\pi} X$, with fiber T. Then Z is T-invariant, or is contained in a fiber of π.

Proof: 1. For any positive-dimensional subvariety $Z_0 \subset X$, the restriction of π to Z_0 has no multisections (because $\int_Z \omega_0^m$ must vanish).

2. Given a space A (of Fujiki class C) with an action of T, consider an associated fiber bundle $M \times_T A$ over X. Unless T acts on A trivially, $M \times_T A$ is also convex, hence admits no multisections.

3. If $Z \subset M$ is not T-invariant, it provides us with a multisection from X to $M \times_T A$, where A is the space of deformations of the fiber $Z \cap \pi^{-1}(t_0)$. It is of Fujiki class C, hence convex. Cannot have multisections! Contradiction.