Algebraic structures on hyperkähler manifolds.

Misha Verbitsky,

verbit@thelema.dnttm.rssi.ru

Let M be a compact hyperkähler manifold. The hyperkähler structure equips M with a set $\mathcal{R}$ of complex structures parametrized by $\mathbb{C}P^1$, called **the set of induced complex structures.** It was known previously that induced complex structures are *non-algebraic*, except may be a countable set. We prove that a *countable* set of induced complex structures is *algebraic*, and this set is dense in $\mathcal{R}$. A more general version of this theorem was proven by A. Fujiki.

The structure of this paper is following. In Subsection 1.1 we define hyperkähler manifolds and induced complex structures. In Subsection 1.2, we define induced complex structures of general type. The main result of this paper and its proof are given in Section 2.

1 Introduction.

We give the basic definitions and cite the results relevant to this paper.

1.1 Hyperkähler manifolds

Definition 1.1: ([Bes]) A **hyperkähler manifold** is a Riemannian manifold M endowed with three complex structures I, J and K, such that the following holds.

(i) the metric on M is Kähler with respect to these complex structures and

(ii) I, J and K, considered as endomorphisms of a real tangent bundle, satisfy the relation $I\circ J = -J\circ I = K$.

The notion of a hyperkähler manifold was introduced by E. Calabi ([C]).

Clearly, a hyperkähler manifold has the natural action of quaternion algebra $\mathbb{H}$ in its real tangent bundle TM. Therefore its complex dimension is even. For each quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding automorphism of TM is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

Definition 1.2: Let M be a hyperkähler manifold, L a quaternion satisfying $L^2 = -1$. The corresponding complex structure on M is called **an induced complex structure**. The M considered as a complex manifold is denoted by (M, L).

Let M be a hyperkähler manifold. We identify the group $SU(2)$ with the group of unitary quaternions. This gives a canonical action of $SU(2)$ on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of $SU(2)$ on the bundle of differential forms.

Lemma 1.3: The action of $SU(2)$ on differential forms commutes with the Laplacian.

Proof: This is Proposition 1.1 of [V]. ∎

Thus, for compact M, we may speak of the natural action of $SU(2)$ in cohomology.

1.2 Appendix: Induced complex structures of general type.

We cite results useful for understanding of behaviour of induced complex structures. The function of this appendix is illustrative. We do not refer to this subsection in the body of the article. This appendix is perfectly safe to skip.

Definition 1.4: Let M be a hyperkähler manifold, $\mathcal{R}$ the set of all induced complex structures. With each $I \in \mathcal{R}$, we associate the Hodge decomposition $H^*(M) = \oplus H^{p,q}_I(M)$ on the cohomology of M. We say that I is **of general type** when all elements of the group

$$H^{p,p}_I(M) \cap H^{2p}(M, \mathbb{Z})$$

are $SU(2)$-invariant.

As Remark 1.10 below implies, the manifolds (M, I) have no Weil divisors when I is of general type. In particular, induced complex structures of general type are never algebraic.

Proposition 1.5: Let M be a hyperkähler manifold and $\mathcal{R}$ be the set of induced complex structures over M. Let $\mathcal{R}_{ng} \subset \mathcal{R}$ be the set of all induced complex structures **not** of general type. Then $\mathcal{R}_{ng}$ is no more than countable.

Proof: This is Proposition 2.2 from [V] ∎

Let M be a compact hyperkähler manifold, $dim_{\mathbb{R}} M = 2m$.

Definition 1.6: Let $N \subset M$ be a closed subset of M. Then N is called **trianalytic** if N is an analytic subset of (M, L) for any induced complex structure L.

Let I be an induced complex structure on M, and $N \subset (M, I)$ be a closed analytic subvariety of (M, I), $dim_{\mathbb{C}} N = n$. Denote by $[N] \in H_{2n}(M)$ the homology class represented by N. Let $\langle N \rangle \in H^{2m-2n}(M)$ denote the Poincare dual cohomology class. Recall that the hyperkähler structure induces the action of the group $SU(2)$ on the space $H^{2m-2n}(M)$.

Theorem 1.7: Assume that $\langle N \rangle \in H^{2m-2n}(M)$ is invariant with respect to the action of $SU(2)$ on $H^{2m-2n}(M)$. Then N is trianalytic.

Proof: This is Theorem 4.1 of [V]. ∎

Remark 1.8: Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4.

Theorem 1.7 has the following immediate corollary, also proven by a Fujiki ([F], Theorem 4.8 (1)):

Corollary 1.9: Let M be a compact hyperkähler manifold, I induced complex structure of general type, and $S \subset (M, I)$ its complex analytic subvariety. Then S is trianalytic.

∎

Remark 1.10: From Corollary 1.9 and Remark 1.8, it follows that a holomorphically symplectic manifold of general type has no closed complex analytic subvarieties of odd dimension; in particular, such a manifold has no divisors.

2 Induced complex structures which are algebraic

Definition 2.1: Let M be a compact hyperkähler manifold. Then M is called **simple** if M is simply connected and cannot be non-trivially represented as a direct product of hyperkähler manifolds.

The general version of the following theorem was proven by A. Fujiki ([F], Theorem 4.8 (2)). Let $\pi : \mathcal{M} \longrightarrow S$ be a deformation of a simple holomorphically simplectic manifold, with arbitrary base of positive dimension. Assume that $\mathcal{M} \longrightarrow S$ is not isotrivial (not trivial on periods). Fujiki proves that for a dense subset $S_a \subset S$, the fibers $\pi^{-1}(s_a)$ are algebraic for all $s_a \in S_a$. I am grateful to Daniel Huybrechts, who provided me with this reference. Also, a similar (but weaker) result was proven by F. Campana ([Ca]).

Theorem 2.2: Let M be a compact simple hyperkähler manifold and $\mathcal{R}$ be the set of induced complex structures $\mathcal{R} \cong \mathbb{C}P^1$. Let $\mathcal{R}_{alg} \subset \mathcal{R}$ be the set of all $L \in \mathcal{R}$ such that the complex manifold (M, L) is algebraic. Then $\mathcal{R}_{alg}$ is countable and dense in $\mathcal{R}$.

Proof: For each $L \in R$, consider the Kähler cone of (M, L), denoted by $K(L) \subset H^2(M, \mathbb{R})$. By definition, $K(L)$ is the set of all cohomology classes $\omega \subset H^2(M, \mathbb{R})$ which are Kähler classes with respect to some metric on (M, L). Let

$$K := \bigcup_{L \in \mathcal{R}} K(L).$$

By [V2], Lemma 5.6, K is open in $H^2(M, \mathbb{R})$. Therefore, the intersection $H^2(M, \mathbb{Q}) \cap K$ is dense in K. By Kodaira, a compact Kähler manifold is algebraic if and only if there exist a rational Kähler class on M ([GH]). By Lemma 2.3 below, every cohomology class $\omega \in K$ corresponds to a unique induced complex structure $I(\omega) \in \mathcal{R}$ such that ω is Kähler with respect to $I(\omega)$. We also prove that thus obtained map $\pi : K \longrightarrow \mathcal{R}$ is continuous. Thus, $\pi\left(H^2(M, \mathbb{Q} \cap K)\right)$ is dense in $\mathcal{R}$. On the other hand, by Kodaira, $\pi\left(H^2(M, \mathbb{Q} \cap K)\right)$ coinsides with $\mathcal{R}_{alg}$. This proves Theorem 2.2. ∎

The following lemma is implicit from [V2]. We decided to spell out its proof, for clarity; for missing details the reader is referred to [V2].

Lemma 2.3: Let $\omega \in K$ be a cohomology class which is Kähler with respect to $I \in \mathcal{R}$. Then

(i) such I is unique,

(ii) and the obtained map $\pi: K \longrightarrow \mathcal{R}$ is continuous.

Proof: Consider the positively defined scalar product on the cohomology space $H^2(M)$ induced by the standard scalar product on harmonic forms. This scalar product is clearly $SU(2)$-invariant. For an induced complex structure L, denote by $\omega_L \in H^2(M, \mathbb{R})$ the Kähler class of the Kähler structure associated with L and the hyperkähler structure. The corresponding harmonic form can be expressed as $\omega_L(x,y) = (x, L(y))$, where $(\cdot,\cdot)$ is the Riemannian form on M. Let $V \subset H^2(M,\mathbb{R})$ be the 3-dimensional subspace generated by ω_L, for all $L \in \mathcal{R}$ (see [V2], Section 4), and $p: H^2(M) \longrightarrow V$ be the orthogonal projection to V. For a Kähler class ω on (M, I), the product (ω, ω_I) is positive, by Hodge–Riemann relations ([GH]). Thus, for all $\omega \in K$, the vector $p(\omega) \in V$ is non-zero. Now, for each $I \in \mathcal{R}$, the intersection $H^{1,1}_I(M) \cap V$ is one-dimensional, because in notation of Definition 1.1, the cohomology class $\omega_J + \sqrt{-1}\,\omega_K \in V$ belongs to $H^{2,0}_I(M)$ and $\omega_J - \sqrt{-1}\,\omega_K \in V$ belongs to $H^{0,2}_I(M)$. Therefore, ω and ω_I are collinear. Clearly, a complex structure L is uniquely defined by the form ω_L, and for all $L \in \mathcal{R}$, the vectors $\omega_L \in V$ all have the same length. Thus, for each line $l \in V$, there exist no more than two induced complex structures $L \in \mathcal{R}$ satisfying $l \in H^{1,1}_L(M)$. It is easy to check that these induced complex structures are opposite: we have $H^{1,1}_L(M) = H^{1,1}_{-L}(M)$. This implies that in $\mathcal{R}$, only $L = I$ and $L = -I$ satisfy $\omega \in H^{1,1}_L(M)$. On the other hand, $\omega_L = -\omega_{-L}$. Thus, the numbers (ω, ω_I) and (ω, ω_{-I}) cannot be positive simultaneously. This implies that ω cannot be a Kähler class for I and $-I$ at the same time. We proved Lemma 2.3 (i).

To prove Lemma 2.3 (ii), consider the composition s of $p: K \longrightarrow V\backslash 0$ and the natural projection map from $V\backslash 0$ to the sphere $S^2 \subset V \cong \mathbb{R}^3$. If we identify S^2 with $\mathcal{R} \cong \mathbb{C}P^1$, we find that π is equal to s (see, for instance, [V2], the proof of Sublemma 5.6). On the other hand, s is continous by construction. This proves Lemma 2.3 (ii). ∎

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